01006 MATH 1. SOLUTION TO HOMEWORK 4

Problem 3 (Essay Question).
We consider $\mathbf{a}_1 = (0,1,2,2,0)$, $\mathbf{a}_2 = (1,1,4,0,0)$, $\mathbf{a}_3 = (1,2,6,2,1)$, $\mathbf{a}_4 = (-1,2,2,6,-1)$, $\mathbf{a}_5 = (2,-2,0,1,0)$, $\mathbf{a}_6 = (-2,-2,1,0,0)$,

(a) To show that $U := \mathrm{Span}\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}$ has dimension 3, it's enough to show that the three vectors are linearly independent, since the dimension is the number of elements in a linearly indepedent spanning set (i.e. a basis). The dimension of the span is the rank of the matrix whose rows or columns consists of these vectors. Using Maple, we find the reduced row echelon form:

$$RREF([\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has three pivots, so this matrix has rank 3.
To write $\mathbf{a}_4$ as a linear combinatation, we solve: $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = \mathbf{a}_4$ via row operations on the augmented matrix:

$$RREF([\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4]) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

the solution is $(c_1,c_2,c_3) = (4,0,-1)$, so $\mathbf{a}_4 = 4\mathbf{a}_1 - \mathbf{a}_3$.

(b) We have $\mathbf{a}_1 \cdot (\mathbf{a}_2 - \mathbf{a}_1) = 9 - 9 = 0$, so these two vectors are already orthogonal. Since they span the same space as $\mathbf{a}_1$ and $\mathbf{a}_2$, the three vectors

$$\mathbf{u}_1 = \mathbf{a}_1, \quad \mathbf{u}_2 = \mathbf{a}_2 - \mathbf{a}_1, \quad \mathbf{u}_3 = \mathbf{a}_3,$$

are still a basis for U, and we can apply Gram-Schmidt to them to get an orthonormal basis, with $\mathbf{q}_1$ proportional to $\mathbf{a}_1$ and $\mathbf{q}_2$ proportional to $\mathbf{a}_2 - \mathbf{a}_1$.

Since the first two vectors are already orthogonal, we only need to normalize them:

$$\mathbf{q}_1 = \frac{\mathbf{u}_1}{|\mathbf{u}_1|} = \frac{1}{3}(0,1,2,2,0), \qquad \mathbf{q}_2 = \frac{\mathbf{u}_2}{|\mathbf{u}_2|} = \frac{1}{3}(1,0,2,-2,0),$$

then finally:

$$\begin{aligned} \mathbf{w}_3 &= \mathbf{u}_3 - (\mathbf{u}_3 \cdot \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{u}_3 \cdot \mathbf{q}_2)\mathbf{q}_2 \\ &= (0,0,0,0,1), \end{aligned}$$

and:

$$\mathbf{q}_3 = \frac{\mathbf{w}_3}{|\mathbf{w}_3|} = (0,0,0,0,1).$$

As linear combinations of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, we have:

$$\mathbf{q}_1 = \frac{1}{3}\mathbf{a}_1, \qquad \mathbf{q}_2 = \frac{1}{3}(\mathbf{a}_2 - \mathbf{a}_1)$$

and to find the coeffients of $\mathbf{q}_3$ we can solve the equation: $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = \mathbf{q}_3$, either by row reducing the matrix $[\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3|\mathbf{q}_3]$, or using Maples LinearSolve, to get

$$\mathbf{q}_3 = \mathbf{a}_3 - \mathbf{a}_1 - \mathbf{a}_2.$$

(c) The dot products of $\mathbf{a}_5$ and $\mathbf{a}_6$ with any of $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ is zero, and so will it be with any linear combination of these vectors, so $\mathbf{a}_5$, $\mathbf{a}_6 \in U^\perp$.
The vectors $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{a}_5, \mathbf{a}_6)$ are all orthogonal to each other (since $\mathbf{a}_5 \cdot \mathbf{a}_6 = 0$ too). If we just divide the last two by their lengths, then they are an orthonormal set of 5 vectors, hence an orthonormal basis. So we set

$$\mathbf{q}_4 = \frac{\mathbf{a}_5}{|\mathbf{a}_5|} = \frac{1}{3}(2,-2,0,1,0) \quad \mathbf{q}_5 = \frac{\mathbf{a}_6}{|\mathbf{a}_6|} = \frac{1}{3}(-2,-2,1,0,0)$$

(d) To find the matrix of f, we need to find what it does to each of the basis vectors. Since $\mathbf{q}_1$ and $\mathbf{q}_2$ are respectively proportional to $\mathbf{a}_1$ and $\mathbf{a}_2 - \mathbf{a}_1$, the eigenvalue information says:

$$f(\mathbf{q}_1) = \mathbf{q}_1, \qquad f(\mathbf{q}_2) = -\mathbf{q}_2.$$

We have $U^\perp = \mathrm{Span}\{\mathbf{q}_4, \mathbf{q}_5\}$, so if this is the kernel of f we have:

$$f(\mathbf{q}_4) = f(\mathbf{q}_5) = \mathbf{0}.$$

Finally, we showed that $\mathbf{q}_3 = \mathbf{a}_3 - \mathbf{a}_1 - \mathbf{a}_2$, so:

$$\begin{aligned} f(\mathbf{q}_3) = f(\mathbf{a}_3 - \mathbf{a}_1 - \mathbf{a}_3) &= f(\mathbf{a}_3) - f(\mathbf{a}_1) - f(\mathbf{a}_2) \\ &= \mathbf{a}_4 - 2f(\mathbf{a}_1) - f(\mathbf{a}_2 - \mathbf{a}_1) \\ &= (4\mathbf{a}_1 - \mathbf{a}_3) - 2\mathbf{a}_1 + (\mathbf{a}_2 - \mathbf{a}_1) \\ &= \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 \\ &= -\mathbf{q}_3. \end{aligned}$$

Hence the mapping matrix is:

$$_qf_q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we restrict f to $U = \mathrm{Span}\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$, we get the matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

which is an orthogonal matrix and therefore preserves the salar product (hence preserves lengths and angles of vectors). However, on the whole of $\mathbb{R}^4$, f does not preserve lengths and angles: for instance $f(\mathbf{q}_4) = \mathbf{0}$, which does not have the same length as $\mathbf{q}_4$.

(e) The change of basis matrix is:

$$_eM_q = [\mathbf{q}_1, \dots \mathbf{q}_5] = \begin{bmatrix} 0 & 1/3 & 0 & 2/3 & -2/3 \\ 1/3 & 0 & 0 & -2/3 & -2/3 \\ 2/3 & 2/3 & 0 & 0 & 1/3 \\ 2/3 & -2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Writing $_qM_e = (_eM_q)^T$. the mapping matrix for f in the standard basis is:

$$_ef_e = {}_eM_q\,{}_qf_q\,{}_qM_e = \frac{1}{9}\begin{bmatrix} -1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ -2 & 2 & 0 & 8 & 0 \\ 2 & 2 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -9 \end{bmatrix}.$$

This matrix is symmetric, since it is equal to its transpose.