Homework set 3

Vector Spaces and Linear Maps

Deadline is 5/11, 23:00. Your solution of the three problems is to be delivered as one pdf file to your group's module on DTU Learn. Explanations and reasoning is expected to an appropriate amount in regards to methods and calculations. Remember name and student number in the top of your assignment.

For the evaluation of this set the focus will be on your ability to
- *determine whether a subset of a vector space is a sub space*
- *determine a basis of a sub space*
- *determine whether a differential equation is linear*
- *use the guessing method and the solution formula for linear 1st-order differential equations*
- *solve differential equations with initial conditions*
- *operate with mapping matrices with respect to different bases*
- *solve linear vector equations*
- *write coherently and precisely and can perform simple mathematical reasoning*

Problem 1 A 2-Dimensional Subspace in $\mathbb{R}^6$

In $\mathbb{R}^6$ we are given the vectors

$$\mathbf{a}_1 = (1,0,1,0,1,0)\,,\ \mathbf{a}_2 = (1,1,2,1,2,-1)\ \text{ and }\ \mathbf{a}_3 = (0,1,1,1,1,-1)\,.$$

a) Show that the vector set $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ spans a two-dimensional subspace U in $\mathbb{R}^6$.

b) We now consider

$$\mathbf{b}_1 = (4,-4,0,-4,0,4)\ \text{ and }\ \mathbf{b}_2 = (-3,2,-1,2,-1,-2)\,.$$

Show that the set $(\mathbf{b}_1\,, \mathbf{b}_2)$ is a basis of the subspace U from question a) above.

The homework set continues $\longmapsto$

|||| Problem 2 Initial Conditions

For $a \in \mathbb{R}$ we consider the first-order differential equation

$$x'(t) + 2tx(t) = 1 + 2t^2\,,\ t \in \mathbb{R}\,. \tag{1}$$

a) Justify that (1) is linear.

b) Find using the guessing method a first-degree polynomial that is a solution to (1). Then determine the solution to the homogeneous equation that corresponds to (1). Finally use the structural theorem to state the general solution to (1).

c) Now solve (1) using the solution formula for linear 1st-order differential equations. (Hint: First find the derivative of te^{t^2}.)

d) Plot the solutions to (1) that fulfill the initial conditions:

$$x(-1) = 0,\ x(0) = 0,\ x(0) = \mathrm{e}\ \text{ og }\ x(1) = 2\,,$$

and comment on your graph.

|||| Problem 3 Linear Map in the Number Space $\mathbb{R}^3$

Let $e = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denote the standard basis of $\mathbb{R}^3$. A linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$f(\mathbf{e}_1) = (1, 3, -1),\ f(\mathbf{e}_2) = (-2, -2, 2)\ \text{ and }\ f(\mathbf{e}_3) = (3, 3, -3)\,.$$

a) State the mapping matrix of f with respect to basis e.

b) Determine a basis for the kernel of f, and state the dimension of the image space $f(\mathbb{R}^3)$.

c) We are given the vectors $\mathbf{a}_1 = (2, -2, -1)$, $\mathbf{a}_2 = (-1, 2, 1)$ and $\mathbf{a}_3 = (2, -3, -2)$. Justify that the set $a = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is a basis of $\mathbb{R}^3$.

d) State the mapping matrix of f with respect to basis a.

e) Determine three vectors in $\mathbb{R}^3$ whose image vectors from f are given by the coordinate vector $(0, 2, 0)$ with respect to basis a.

End of problem sheet