# Writen 2-hours exam, December 2018, Problem 3

In $\mathbb{R}^3$ equipped with the ordinary dot product we consider the set of vectors

$$v = (\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}) = ((1,1,1),(1,0,-1),(-1,1,0)).$$

## 1.

We wish to explain that $v$ is a basis for $\mathbb{R}^3$. A basis for $\mathbb{R}^3$ needs 3 linearly independent vectors. Since $v$ consists of 3 vectors, we check whether they are linearly independent. The vectors are stated as columns in a $3 \times 3$-matrix $V$:

$$V = (\boldsymbol{v_1}\ \boldsymbol{v_2}\ \boldsymbol{v_3}) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

The determinant of $V$ are determined using `Maple` (see Appendix 1,Eq. (1.2)) as

$$\det(V) = 3.$$

Since the determinant is different from 0, $V$ has full rank, and thus it is shown that $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ are linearly independent. Therefore $v$ constitutes a basis for $\mathbb{R}^3$.

Now we consider the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(\boldsymbol{u}) = 5\boldsymbol{u},\ \boldsymbol{u} \in \operatorname{span}\{\boldsymbol{v_1}\},$$
$$f(\boldsymbol{u}) = -4\boldsymbol{u},\ \boldsymbol{u} \in \operatorname{span}\{\boldsymbol{v_2}, \boldsymbol{v_3}\}.$$

## 2.

We are interested in determining the tranformation matrix ${}_vF_v$ for $f$ with respect to basis $v$. From the definition of $f$ it is seen that 5 and $-4$ are eigenvalues for $f$ with corresponding eigenvector spaces $E_5 = \operatorname{span}\{\boldsymbol{v_1}\}$ og $E_{-4} = \operatorname{span}\{\boldsymbol{v_2}, \boldsymbol{v_3}\}$. Since we have already shown that $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ are linearly independent, $v$ is an eigenbasis for $f$. Thus the transformation matrix ${}_vF_v$ is a diagonal matrix with the eigenvalues in the diagonal:

$${}_vF_v = ({}_vf(\boldsymbol{v_1})\ \ {}_vf(\boldsymbol{v_2})\ \ {}_vf(\boldsymbol{v_3})) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Also we wish to be able to change to standard coordinates and therefore we will state the transformation matrix ${}_eF_e$ for $f$ with respect to the standard basis. In order to change coordinates we shall use a change of base matrix. We already know the change of base matrix ${}_eM_v$, that shifts from $e$- to $v$-coordinates. This is exactly the matrix $V$, where the vectors from $v$ is stated with respect to the standard basis $e$:

$${}_eM_v = V = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Now the change of base matrix ${}_vM_e$ that shifts from $v$- to $e$-coordinates can be determined as the inverse of ${}_eM_v$. This is done using `Maple` (see Appendix 1,Eq. (1.5)):

$$ {}_vM_e = ({}_eM_v)^{-1} = \frac{1}{3} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ -1 & 2 & -1 \end{pmatrix}. $$

The transformation matrix ${}_eF_e$ can now be determined as the matrix product ${}_eM_v \cdot {}_vF_v \cdot {}_vM_e$. The matrix product is determined using `Maple` (see Appendix 1, Eq. (1.6)) and the transformation matrix can be written as:

$$ {}_eF_e = {}_eM_v \cdot {}_vF_v \cdot {}_vM_e = \begin{pmatrix} -1 & 3 & 3 \\ 3 & -1 & 3 \\ 3 & 3 & -1 \end{pmatrix} $$

We notice that ${}_eF_e$ is a symmetric matrix, that can be diagonalized using an eigenbasis as e.g. $v$.

### 3.

We wish to determine an orthonormal basis $q$ for $\mathbb{R}^3$, consisting of eigenvectors r for $f$, where one vector in $q$ is aligned with $\boldsymbol{v_3}$. We already know the eigenbasis $v$. Since ${}_eF_e$ is symmetric, we also know that the eigen-spaces $E_5$ and $E_{-4}$ are orthogonal. Therefore we shall find an orthonomal basis for $E_5$ and $E_{-4}$ seperately in order to put together an orthonomal basis for $\mathbb{R}^3$ consisting of eigenvektorer for $f$. First we normalize $\boldsymbol{v_1}$ and thus a new vector $\boldsymbol{q_1}$ is formed:

$$ \boldsymbol{q_1} = \frac{\boldsymbol{v_1}}{\|\boldsymbol{v_1}\|} = \frac{\sqrt{3}}{3}(1, 1, 1). $$

An orthonormal basis for $E_5$ now consists of $\boldsymbol{q_1}$.

Now we shall find an orthonormal basis for $E_{-4}$, where one vector is alligned with $\boldsymbol{v_3}$. Therefore we first normalize $\boldsymbol{v_3}$ to get the vector $\boldsymbol{q_3}$:

$$ \boldsymbol{q_3} = \frac{\boldsymbol{v_3}}{\|\boldsymbol{v_3}\|} = \frac{\sqrt{2}}{2}(-1, 1, 0). $$

Finally we form $\boldsymbol{q_2}$ as the cross product between $\boldsymbol{q_1}$ and $\boldsymbol{q_3}$ (see Appendix 1, Eq. (1.9)):

$$ \boldsymbol{q_2} = \boldsymbol{q_1} \times \boldsymbol{q_3} = \frac{\sqrt{6}}{6}(-1, -1, 2). $$

An orthonomal basis for $E_{-4}$ is now $(\boldsymbol{q_2}, \boldsymbol{q_3})$, where $\boldsymbol{q_3}$ is aligned with $\boldsymbol{v_3}$. Therefore the wished for orthonomal basis $q$ for $\mathbb{R}^3$ is $q = (\boldsymbol{q_1}, \boldsymbol{q_2}, \boldsymbol{q_3})$.

# Appendix 1

```
> restart; with(LinearAlgebra):
```

The set of vectors is defined:

```
> v1:=<1,1,1>:
  v2:=<1,0,-1>:
  v3:=<-1,1,0>:
```

## 1)

The three vectors form columns in the matrix $V$:

```
> V:=<v1|v2|v3>;
```

$$V := \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \tag{1.1}$$

The determinant of $V$ is determined:

```
> Determinant(V);
```

$$3 \tag{1.2}$$

## 2)

The transformation matrix ${}_vF_v$ is a diagonal matrix with eigenvalues in the diagonal:

```
> vFv:=DiagonalMatrix(<5,-4,-4>);
```

$$vFv := \begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \tag{1.3}$$

The change of basis matrices *eMv* and *vMe* are stated:

```
> eMv:=V;
```

$$eMv := \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \tag{1.4}$$

```
> vMe:=eMv^(-1);
```

$$vMe := \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \tag{1.5}$$

The transformation matrix *eFe* is found by a change of base:

```
> eFe:=eMv.vFv.vMe;
```

**(1.6)**

$$eFe := \begin{bmatrix} -1 & 3 & 3 \\ 3 & -1 & 3 \\ 3 & 3 & -1 \end{bmatrix} \quad (1.6)$$

# 3)

$v_1$ is normaliized:

```
> q1:=v1/norm(v1,2);
```

$$q1 := \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix} \quad (1.7)$$

$v_3$ is normalized:

```
> q3:=v3/norm(v3,2);
```

$$q3 := \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \quad (1.8)$$

$q_2$ is degtermined as the cross product between $q_1$ and $q_3$:

```
> q2:=CrossProduct(q1,q3);
```

$$q2 := \begin{bmatrix} -\frac{\sqrt{3}\,\sqrt{2}}{6} \\ -\frac{\sqrt{3}\,\sqrt{2}}{6} \\ \frac{\sqrt{3}\,\sqrt{2}}{3} \end{bmatrix} \quad (1.9)$$