Homework set 4

Eigenvalue Problems

For the evaluation of this homework set the focus will be on your ability to
- *solve systems of differential equations using the diagonalization method*
- *solve initial condition problems*
- *operate with mapping matrices and change-of-basis matrices*
- *understand the geometric meaning of eigenvalues and eigenvectors*
- *use the GramSchmidt algorithm*
- *understand the meaning of the dot product in regards to lengths and angles*
- *use Maple for investigations and visualization*
- *write coherently an precisely and can perform simple mathematical reasoning*

Deadline for upload: Sunday November 26 at 23:55. Remember to hand in your assignment as .pdf.

Problem 1 Initial Condition Problem for System of Differential Equations

For $a \in \mathbb{R}$ a system of differentiaal equations is given in matrix form by

$$\mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -\frac{17}{4} & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

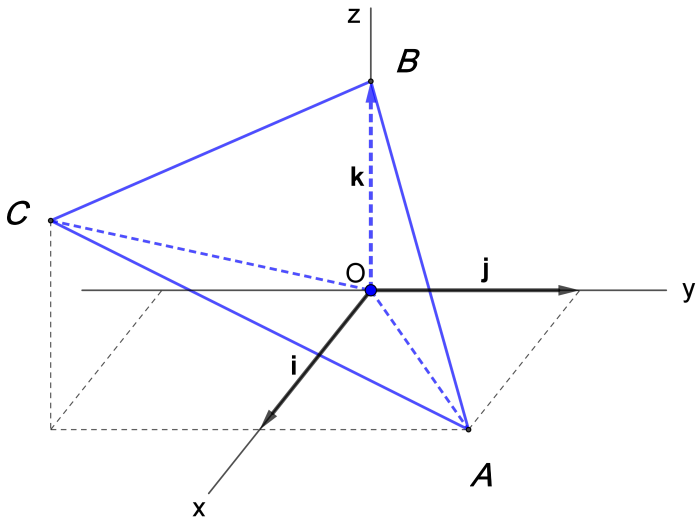
a) State in vector form the complete complex solution to the differential equation system and determine those solutions, expressed via real functions, that fulfill the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 16 \\ 0 \end{bmatrix}.$$

b) The solution found in the previous question is to be illustrated in two ways. First, x_1 and x_2 must be drawn as two functions of $t \in [0, 2\pi]$ in the same coordinate system. Next, the trajectory curve that the point (x_1, x_2) has covered after time $t \in [0, 2\pi]$ must be drawn. Comment on the two visualizations.

|||| Problem 2 Distances in (x, y, z) Space

A tetrahedron T_1 in (x, y, z) space has the corners: origin $= (0, 0, 0)$, $A = (1, 1, 0)$, $B = (0, 0, 1)$ and $C = (1, -1, 1)$, see the figure.



a) Compute the volume of T_1.

We now consider a matrix $\mathbf{F}$ given by

$$\mathbf{F} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}.$$

b) Compute the eigenvalues of $\mathbf{F}$ and the corresponding eigenspaces.

Let G_3 denote the set of geometric vectors in (x, y, z) space drawn from the origin. A linear map $f : G_3 \to G_3$ has with respect to the standard basis in G_3 the mapping matrix $\mathbf{F}$ which is given above.

c) We consider the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$. Justify that the image vectors by f of the three vectors appear from a scaling (extension) of the three vectors in a direction away from the origin, and state for each of the vectors the scaling factor. The end points of the three image vectors constitute, along with the origin, a new tetrahedron T_2. Find a relation between the volumes of T_1 and T_2 and the mentioned scaling factors.

d) The corners in T_1 are themselves image points by f of the corners in a tetrahedron T_0. Determine the volume of T_0.

|||| Problem 3 Orthogonal Eigenspaces in $\mathbb{R}^3$

In $\mathbb{R}^3$ we consider the vector $\mathbf{u} = (1, -1, -1)$ and the subspace $U_1 = \operatorname{span}\{\mathbf{u}\}$. Let U_2 denote the orthogonal compliment in $\mathbb{R}^3$ to U_1 .

a) Determine a basis for $\mathbb{R}^3$ that is a composition of an orthogonal basis for U_1 and an orthogonal basis for U_2 .

Let a and b be arbitrary real numbers. A linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ has the eigenspaces $E_a = U_1$ and $E_b = U_2$.

b) Determine a mapping matrix ${}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}}$ of f with respect to the standard basis in $\mathbb{R}^3$.

We now use the values $a = 2$ and $b = -2$.

c) The vectors $\mathbf{w}_1 = (1, 1, 0)$ and $\mathbf{w}_2 = (1, 0, 1)$ are given. Show that the angle between $\mathbf{w}_1$ and $\mathbf{w}_2$ is the same as the angle between the images by f of $\mathbf{w}_1$ and $\mathbf{w}_2$. Does it apply generally that the angle between two vectors is equal to the angle between their image vectors?

End of problem sheet