Forest Fire

Theme Exercise on Vector Analysis

01006 Advanced Engineering Mathematics 1 - SPRING 2020

Figure 1: Fire!!

1 Objective

This theme exercise is about a well-known kind of catastrophe: On a calm summer's day a bone dry forest suddenly catches fire and the firefront immediately spreads in circles centered at the point of ignition.

But if the wind is rising and thereby brings oxygen to the fire from a given direction the fire front is far from being circular.

The objective of this theme exercise is to find the time evolution of forest fires, both with respect to the size of the area destroyed by the fire at a given point in time, and the form of the fire front curve at time t after the ignition.

Of course there are numerous good reasons to worry about these questions. It is also evident from the following quotations from [Wik1] and [FA], respectively:

> "A wildfire, also known as a forest fire, vegetation fire, grass fire, brush fire, or bush fire (in Australia), is an uncontrolled fire often occurring in wild land areas, but which can also consume houses or agricultural resources. Common causes include lightning, human carelessness and arson.
>
> One main component of Carboniferous north hemisphere coal is charcoal left over by forest fires. The earliest known evidence of a wildfire dates back to Late Devonian period (about 365 million years ago).

The powerful updraft caused by a large wildfire will draw air from surrounding areas. These self-generated winds can lead to a phenomenon known as a firestorm.

... [M]odels predict an elliptical shape [of the fire's front line] when the ground is flat and the vegetation is homogeneous.

... All the large catastrophic fires in the United States have been wind driven events where the amount of fuel (trees, shrubs, etc.) has not been the most important factor in the fire spread."

"A working knowledge of the effects of wind and other weather elements on fire behaviour, supported by accurate fire intelligence, is vital for good suppression planning. Without good fire behaviour information firefighters are unable to:

- determine the number of firefighters and level of equipment necessary;
- identify the location of suitable areas for backburning; and
- ensure that the general public is informed about the precise fire situation."

Further see [Wik2] that contains a list of some of the largest known forest fires in recent times.

Figure 2: Fire fighting.

2 Modelling of Forest, Ignition, Wind, and Loss

To be specific we consider a rectangular horizontal forest field S. We assume that the forest is completely homogeneous with constant density: ρ trees per unit area. All the trees have the same height and the same combustion properties. We make the somewhat crude assumption that we can consider the evolution of the fire as a plane 2-dimensional problem - including that the fire spreads with a well-defined fire front *curve*.

1. Why is this a crude assumption?

Therefore we introduce a 2D-coordinate system in the plane of the forest such that the origin lies in the middle of the forest and such that the forest besides is given by:

$$S = \{(x,y) \in \mathbb{R}^2 \mid -L \leq x \leq L \quad \text{and} \quad -M \leq y \leq M\} \quad , \tag{1}$$

where L and M are given values for the (half) length and (half) width of the forest: $L \in]0, \infty]$, and $M \in]0, \infty]$, respectively. The area of the forest is thus (when L and M both are finite): $\mathrm{A}(S) = 4LM$.

The point of ignition is denoted by $p = (x_0, y_0) \in S$.

The wind is assumed constant in magnitude and direction and is therefore represented by a constant plane vector field $\boldsymbol{W}$ with the length W in the (x, y)-plane. Thus a constant angle θ and a constant wind speed W exist, such that

$$\boldsymbol{W}(x, y) = (W\cos(\theta), W\sin(\theta)) \quad \text{for all} \quad (x, y) \in \mathbb{R}^2 \quad . \tag{2}$$

We let $\Omega(t) \in S$ denote the part of S destroyed by the fire at time $t > 0$. The corresponding area destroyed is then

$$A(t) = \mathrm{Area}(\Omega(t)) \quad , \quad t > 0 \quad . \tag{3}$$

Part of the exercises below is about finding this area in different given situations. The area corresponds precisely to the number of trees lost to the fire: $\mathrm{Tab}(t) = \rho A(t)$.

Initially the fire will spread as an ellipse with growing semi-axes and with a constant translational velocity (in the wind direction). Therefore the fire will typically form a main front, a tail front and two flanks. The push is largest at the main front where most trees are being burned per unit time. The push is small - but not necessarily negligible - in the tail where the fire front is moving backwards against the wind.

If there is no external wind, that is, $W = 0$, the spreading of the fire will be perfectly symmetrical with respect to the point of ignition at any time $t > 0$ - if indeed the forest is sufficiently large. This short theme exercise only treats this special case: the more general methods are discussed in e.g. [GH] and [M]

3 Model without Wind

The circular spreading of the fire front curve (with $W = 0$) can be modelled in the following way, where ϕ denotes the direction-angle-parameter, $\phi \in [-\pi, \pi]$. Time is denoted by $t > 0$ and $a > 0$ is a constant that depends on the quality of the forest, density, tree height, dryness, etc.:

$$\begin{aligned} x(t, \phi) &= at\cos(\phi) \\ y(t, \phi) &= at\sin(\phi) \quad . \end{aligned} \tag{4}$$

In particular the model above yields that the radius of the circular destroyed field $\Omega(t)$ increases proportion to time t.

2. Assume that the spreading model above (4) applies and that the forest is enormously large, $L = M = \infty$. What is the area $A(t)$ of $\Omega(t)$ as a function of time t when the fire is ignited in a given point (x_0, y_0) at time $t = 0$?

3. Assume again (4) and now more realistically assume that the forest has a finite extent given by finite values of L and M, but also assume (somewhat less realistically) that the model (4) applies no matter how many trees are left in the forest. How long time is elapsed from the ignition, corresponding to $t = 0$ at the point (x_0, y_0) until the forest is totally destroyed?

For every fixed direction ϕ_0 we get from the model (4) a t-parametrized curve along which the "fire front particle", corresponding to that direction, is moving:

$$\boldsymbol{r}_{\phi_0}(t) = (x(t,\phi_0), y(t,\phi_0)) = (at\cos(\phi_0), at\sin(\phi_0)) \quad . \tag{5}$$

4. Determine the velocity and speed of the "fire front particle"for any given value of ϕ_0 and at any given time t.

We now introduce the following modified time-dependent *explosion vector field* in the plane (cf. eNote 24, the examples 24.8 and 24.15):

$$\boldsymbol{V}_t(x,y) = \left(\frac{x}{t}, \frac{y}{t}\right) \quad , \quad \text{for all} \quad t > 0 \quad . \tag{6}$$

For every fixed $t = t_0$ $\boldsymbol{V}_{t_0}(x,y)$ is in fact an explosion vector field in the plane - apart from the constant factor t_0. It is Gauss' theorem for these kinds of plane vector fields that we will use in the following for finding an alternative determination of the area function $A(t)$.

5. Show that the curves $\boldsymbol{r}_{\phi_0}(t)$ are flow curves for the time-dependent vector field $\boldsymbol{V}_t(x,y)$ in the following sense (being four equivalent formulations of the property that the tangent vector field for the curves at every point is exactly the value of the vector field $\boldsymbol{V}$ at this point):

$$\begin{aligned} \frac{d}{dt}\boldsymbol{r}_{\phi_0}(t) &= \boldsymbol{V}_t(x(t,\phi_0), y(t,\phi_0)) \\ \boldsymbol{r}'_{\phi_0}(t) &= \left(\frac{x(t,\phi_0)}{t}, \frac{y(t,\phi_0)}{t}\right) \\ (x'(t,\phi_0), y'(t,\phi_0)) &= \left(\frac{x(t,\phi_0)}{t}, \frac{y(t,\phi_0)}{t}\right) \\ (t\,x'(t,\phi_0), t\,y'(t,\phi_0)) &= (x(t,\phi_0), y(t,\phi_0)) \end{aligned} \tag{7}$$

6. Determine the divergence of the plane vector field $\boldsymbol{V}_{t_0}(x,y)$ for every fixed time $t_0 > 0$ and use Gauss' divergence theorem for plane vector fields (see Theorem 1 in Section 5 below) in order to verify the following differential equation for the area function $A(t)$:

$$\frac{d}{dt}A(t)_{|t_0} = 2\frac{A(t_0)}{t_0} \quad . \tag{8}$$

7. Find all solutions to the differential equation

$$\frac{d}{dt}A(t) = 2\frac{A(t)}{t} \quad , \quad t > 0 \quad . \tag{9}$$

Then use the initial condition $A(1) = \pi a^2$ to find the area function $A(t)$ for the circular fire as it is represented by (4). Compare with Exercise 2.

4 Model with Wind Speed and Direction Constant

As already remarked in the introduction the wind plays a very large role for the spreading of forest fires. An obvious idea is to extend the circular model to an "elliptic"model with translation in the direction of the wind. This is exactly what is done in [R], from which we quote:

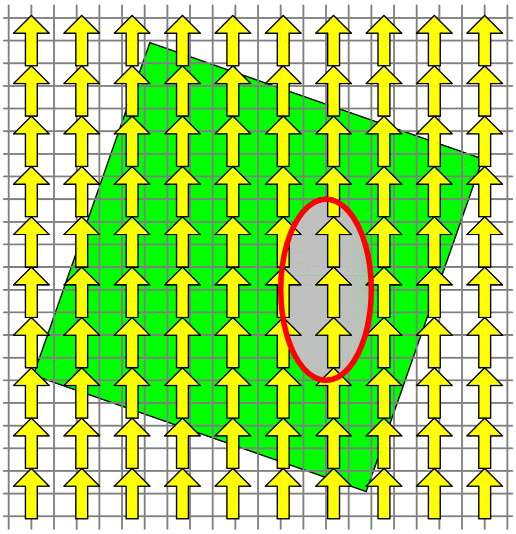

Figure 3: Snapshot of an elliptical fire zone.

> "Under constant conditions for homogeneous, non-spotting fuels it is generally accepted that a fire ignited at a point will expand, at a constant rate, as an ellipse of the form:
>
> $$\begin{aligned} x(t,\phi) &= at\cos(\phi) \\ y(t,\phi) &= bt\sin(\phi)+ct \quad , \end{aligned} \tag{10}$$
>
> where t is time, the origin being the point of ignition and the y-axis being the wind direction. The forward rate v, the lateral rate u and the back rate w are defined as:
>
> $$\begin{aligned} v &= b+c \\ u &= a \\ w &= b-c \quad . \end{aligned} \tag{11}$$
>
> The Canadian Forest Fire Behaviour Prediction System (CFFBPS) assumes elliptical growth and has documented values of u, v, and w for a very large set of constant parameters affecting a fire. It has also been observed that, within certain limits, the ratio a/b is a function of wind speed only; this is also an assumption of the CFFBPS."

Note that in this model it is clearly assumed that the point of ignition $(x_0, y_0) = (0,0)$ and the wind is in the direction of the positive y-axis such that $\boldsymbol{W} = (0,W)$.

The new constants $b \geq a$ and $c \geq 0$ are, as is a, expressions for the fire technical properties of the forest material, now with the fire (and the corresponding supply of oxygen) as a new decisive parameter.

We assume that $b > c \geq 0$, such that the fire front-ellipses have a proper backwards spreading, that is, against the wind. Note that for fixed direction ϕ_0 the "fire-front-particles" also here move along the straight parametrized lines : $\boldsymbol{r}_{\phi_0}(t)$.

8. **8.** Determine the velocity of the "fire-front-particles"$\frac{d}{dt}\boldsymbol{r}_{\phi_0}(t) = \boldsymbol{r}'_{\phi_0}(t)$ and speed $\|\boldsymbol{r}'_{\phi_0}(t)\|$ for every value of ϕ_0 and at every given time t.
9. **9.** Where on the fire front curve are the most trees burned down pr. unit time?
10. **10.** As in Exercise 3: Find (for finite values of L and M and by use of the elliptic model in (10)) an expression for how long time elapses from the ignition in a given point (x_0, y_0) until the forest is totally destroyed by the fire. It is here assumed that the elliptic spreading model applies regardless of how many trees are left in the forest.

5 The 2-Dimensional, Plane Version of Gauss' Theorem

With arguments and exposition as in [M] Section 9.3 we get and motivate the following plane version of Gauss' divergence theorem:

Theorem 1 (Gauss' Theorem) *Let Ω denote a plane field with the boundary curve $\partial\Omega$ and outward-directed unit normal vector field $\mathbf{n}_{\partial\Omega}$ along the boundary curve. Then for every vector field $\mathbf{V}$ in the plane the following applies:*

$$\frac{d}{du}A(u)_{|_{u=0}} = \int_{\Omega} \mathrm{div}(\mathbf{V})\, d\mu = \int_{\partial\Omega} \mathbf{V}\cdot\mathbf{n}_{\partial\Omega}\, d\mu = \mathrm{Flux}(\mathbf{V}, \partial\Omega) \quad , \tag{12}$$

where the flux shall be computed with respect to the outward-directed unit normal vector field along the boundary curve of the given plane field.

Note that the last equality sign defines the flux of the plane vector field out through the boundary curve. The left-hand side, $\frac{d}{du}A(u)_{|_{u=0}}$, is the derivative (taken in $u = 0$) of the area as a function of the flow-parameter u for the flow curves of the vector field.

Litteratur

[A] M. E. Alexander, *Estimating the length-to-breadth ratio of elliptical forest fire patterns*, Proc. 8th Natl. Conf. on Fire and Forest Meteorology, Society of American Foresters, Washington, D.C., 1985.

[FA] L. G. Fogarty and M. E. Alexander, *A Field Guide for predicting Grassland Fire potential: Derivation and Use*, Forest and Rural Fire Research, Fire Technology Transfer Note, Number -20, July 1999.

[GH] Jan Glasa and Ladislav Halada, *On elliptical model for forest fire spread modeling and simulation*, Math. Comput. Simulation., 78(1):76-88, 2008.

[M] Steen Markvorsen, *From PA(X) to RPAM(X)*. In Erhard Behrends, Nuno Crato and JosÃl' Francisco Rodrigues, editors, *Raising Public Awareness of Mathematics* Springer. 2012.

[R] G. D. Richards, *An Elliptical Growth Model of Forest Fire Fronts and its Numerical Solution*, International Journal for Numerical Methods in Engineering, Vol. 30, 1163–1179 (1990).

[Wik1] Wikipedia: http://en.wikipedia.org/wiki/Wildfire

[Wik2] Wikipedia: http://en.wikipedia.org/wiki/List_of_wildfires