Math 1 Two-hours test May 18, 2015

Problem 1.

$f(x,y) = x^2 y + \frac{1}{3} y^3 - y, \quad (x,y) \in \mathbb{R}^2.$

1. $\nabla f(x,y) = (f'_x(x,y), f'_y(x,y)) = (2xy, x^2+y^2-1) = (0,0) \Leftrightarrow$
$x=0$ and $y = \pm 1$ or $y = 0$ and $x = \pm 1$.

The stationary points for f is then $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$.

2. If f has a proper local maximum or a proper local minimum in a point, then the point must be a stationary point, since f has no exception points.

The Hessian matrix for f in the point (x,y) is

$$\underline{H}(x,y) = \begin{bmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{bmatrix} = \begin{bmatrix} 2y & 2x \\ 2x & 2y \end{bmatrix}.$$

$\underline{H}(x,y) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \ \lambda = \begin{cases} 2 > 0 \\ -2 < 0 \end{cases}. \quad \underline{H}(0,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \ \lambda = \begin{cases} 2 > 0 \\ 2 > 0 \end{cases}$

f has neither a proper local maximum nor a proper local minimum in the stationary point $(1,0)$.

f has a proper local minimum in the stationary point $(0,1)$ with the value $f(0,1) = -\frac{2}{3}$.

f has neither a proper local maximum nor a proper local minimum in $(0,0)$ since $(0,0)$ is not a stationary point.

3. $\nabla f(0,0) = (0,-1)$. $\underline{e}$ is unit vector.
$f'((0,0);\underline{e}) = \nabla f(0,0) \cdot \underline{e} = 0 \Leftrightarrow \underline{e} \perp \nabla f(0,0) \Leftrightarrow$
$\underline{e} = (1,0)$ or $\underline{e} = (-1,0)$.

The positive and the negative directions of the x-axis are exactly the wanted directions in which the directional derivative of f in $(0,0)$ assumes the value 0

Problem 2

$\underline{\underline{A}} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix} = \underline{\underline{A}}^T$. $\underline{v}_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is an eigenvector for $\underline{\underline{A}}$ corresponding to the eigenvalue 2 and $\underline{v}_2 = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is an eigenvalue for $\underline{\underline{A}}$ corresponding to the eigenvalue 0.

1. Since $\underline{v}_1$ and $\underline{v}_2$ are unit vectors and since $\underline{v}_1 \perp \underline{v}_2$ then $(\underline{v}_1, \underline{v}_2)$ is an orthonormal basis for $(\mathbb{R}^2, \bullet)$.

2. $\frac{1}{2}x^2 + \sqrt{3}xy + \frac{3}{2}y^2 + \frac{\sqrt{3}}{2}x - \frac{1}{2}y = -2 \iff$

$$[x\ y] \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -2$$

$(0; \underline{v}_1, \underline{v}_2)$ is a new ordinary orthogonal coordinate system in the plane given by a rotation of the old coordinate system $\pi/3$ about $(0,0)$.

$\underline{\underline{Q}} = [\underline{v}_1\ \underline{v}_2] = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ are positive orthogonal and

$$\underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} = \underline{\underline{Q}}^{-1} \underline{\underline{A}}\, \underline{\underline{Q}} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}. \quad \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{Q}} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \iff \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \underline{\underline{Q}}^T \begin{bmatrix} x \\ y \end{bmatrix}.$$

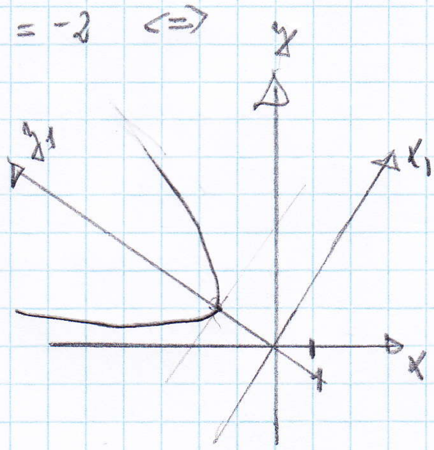
((x,y) are old coordinates and (x_1, y_1) are new coordinates for a point in the plane.)

The equation for the parabola in the new coordinate system is

$$[x_1\ y_1]\, \underline{\underline{Q}}^T \underline{\underline{A}}\, \underline{\underline{Q}} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = -2 \iff$$

$$2x_1^2 - y_1 = -2 \iff y_1 - 2 = 2x_1^2$$

I.e a parabola with vertex $T(x_1, y_1) = (-\sqrt{3}, 1)$ in the old coordinate system.



The axis of symmetry is the y_1-axis, which is the line with the equation $y = -\frac{\sqrt{3}}{2}x$ in the old coordinate system.

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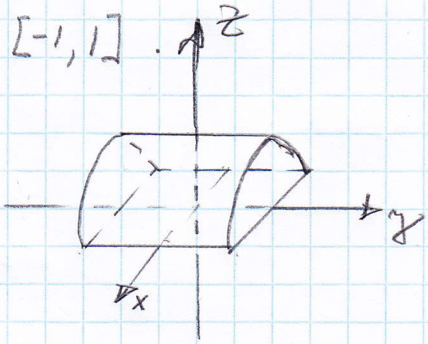
Problem 3

$h(x,y) = 1 - x^2$. The graph surface F is given by

$F = \{(x,y,z) \mid -1 \le x \le 1,\ -1 \le y \le 1,\ z = h(x,y)\}$.

1. The parametric representation for F:

$\underline{r}(u,v) = (u, v, 1-u^2)$, $u \in [-1,1]$, $v = [-1,1]$.



$\underline{r}'_u(u,v) = (1, 0, -2u)$

$\underline{r}'_v(u,v) = (0, 1, 0)$.

$\underline{N}(u,v) = \underline{r}'_u \times \underline{r}'_v = (2u, 0, 1)$ has a positive z-coordinate.

$\underline{V}(x,y,z) = (x^2, z - 2xy, 4z)$, $(x,y,z) \in \mathbb{R}^3$.

2. $\underline{V}(\underline{r}(u,v)) = (u^2, 1-u^2-2uv, 4-4u^2)$.

$\underline{V}(\underline{r}(u,v)) \cdot \underline{N}(u,v) = 2u^3 - 4u^2 + 4$.

$\underline{\text{Flux}(\underline{V}, F_{\underline{r}})} = \int_{F_{\underline{r}}} \underline{V} \cdot \underline{n}_{F_{\underline{r}}}\, d\mu = \int_{v=-1}^{1} \left(\int_{u=-1}^{1} \underline{V}(\underline{r}(u,v)) \cdot \underline{N}(u,v)\, du \right) dv$

$= \int_{v=-1}^{1} \left(\int_{u=-1}^{1} (2u^3 - 4u^2 + 4)\, du \right) dv = \int_{v=-1}^{1} \left[\frac{u^4}{2} - \frac{4u^3}{3} + 4u \right]_{u=-1}^{1} dv = \int_{v=-1}^{1} \frac{16}{3} dv = \underline{\frac{32}{3}}$

A solid region Ω in space is given by:

$\Omega = \{(x,y,z) \mid x \in [1,1],\ y \in [-1;1],\ z = [0, 1-x^2]\}$.

3. The parametric representation for Ω [$\underline{\Omega}(u,v,w)$ is the line segment from $(u,v,0)$ to $(u,v,1-u^2)$]: $\underline{\Omega}(u,v,w) = (u, v, w(1-u^2))$, $u \in [-1,1]$, $v \in [-1,1]$, $w \in [0,1]$.

$J_{\underline{\Omega}}(u,v,w) = \left| \underline{\Omega}'_u \ \ \underline{\Omega}'_v \ \ \underline{\Omega}'_w \right| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2uw & 0 & 1-u^2 \end{vmatrix} = 1 - u^2 = \text{Jacobi}_{\underline{\Omega}}(u,v,w)$ since $u \in [-1,1]$.

$\underline{\text{Vol}(\Omega)} = \int_{\Omega_{\underline{\Omega}}} 1\, d\mu = \int_{w=0}^{1} \left(\int_{v=-1}^{1} \left(\int_{u=-1}^{1} \text{Jacobi}_{\underline{\Omega}}(u,v,w)\, du \right) dv \right) dw$

$= \int_{w=0}^{1} \left(\int_{v=-1}^{1} \left(\int_{u=-1}^{1} (1-u^2)\, du \right) dv \right) dw = \int_{w=0}^{1} \left(\int_{v=-1}^{1} \left[u - \frac{u^3}{3} \right]_{u=-1}^{1} dv \right) dw = \int_{w=0}^{1} \left(\int_{v=-1}^{1} \frac{4}{3} dv \right) dw = \underline{\frac{8}{3}}$

4. $\text{div}(\underline{V}) = \nabla \cdot \underline{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 2x - 2y + 4 = 4$.

From Gauss' theorem (since $\partial\Omega$ is a closed surface with outward pointing unit normal vector field): $\underline{\text{Flux}(\underline{V}, \partial\Omega)} = \int_{\Omega_{\underline{\Omega}}} \text{div}(\underline{V})\, d\mu = \int_{\Omega_{\underline{\Omega}}} 4\, d\mu = 4\, \text{Vol}(\Omega) = \underline{\frac{32}{3}}$.

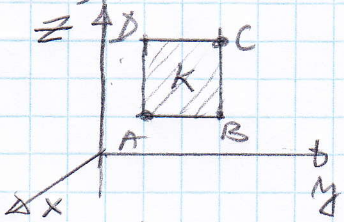
Problem 4.

$A(0,1,1)$, $B(0,3,1)$, $C(0,3,3)$, $D(0,1,3)$.

$\underline{U}(x,y,z) = (2xy, -z^2, x^2)$, $(x,y,z) \in \mathbb{R}^3$.

$f(x,y,z) = x^2y - \frac{1}{3}z^3$, $\underline{V}(x,y,z) = \underline{\nabla} f(x,y,z)$, $(x,y,z) \in \mathbb{R}^3$.

1.
$$\mathrm{curl}(\underline{U}) = \underline{\nabla} \times \underline{U} = \begin{vmatrix} \cdot & \cdot & \cdot \\ \partial_x & \partial_y & \partial_z \\ 2xy & -z^2 & x^2 \end{vmatrix} = (2z, -2x, -2x).$$



The parametric representation for K.

$\underline{r}(u,v) = (0,u,v)$, $u \in [1,3]$, $v \in [1,3]$.

$\underline{r}'_u(u,v) = (0,1,0)$; $\underline{r}'_v(u,v) = (0,0,1)$; $\underline{N}(u,v) = (1,0,0)$.

$\underline{\mathrm{curl}}(\underline{U})(\underline{r}(u,v)) = (2v,0,0)$

$\underline{\mathrm{curl}}(\underline{U})(\underline{r}(u,v)) \times \underline{N}(u,v) = 2v$.

$$\underline{\mathrm{Flux}(\mathrm{curl}(\underline{U}), K_{\underline{r}})} = \int_{K_{\underline{r}}} \mathrm{curl}(\underline{U}) \cdot \underline{n}_{K_r} = \int_{u=1}^{3} \left(\int_{v=1}^{3} 2v\,dv \right) du = \int_{u=1}^{3} \left[v^2 \right]_{v=1}^{3} du = \underline{16}.$$

2. l_{AD} : $\underline{s}(t) = (0,1,1) + t(0,0,2) = (0,1,1+2t)$, $t \in [0,1]$.

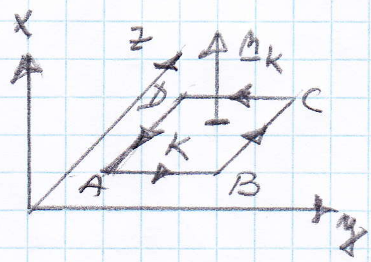
$\underline{s}'(t) = (0,0,2)$, $\underline{U}(\underline{s}(t)) = (0, -(1+2t)^2, 0)$

$$\underline{\mathrm{Tan}(\underline{U}, l_s)} = \int_{l_s} \underline{U} \cdot \underline{t}_s \, ds = \int_0^1 \underline{U}(\underline{s}(t)) \cdot \underline{s}'(t)\, dt = \int_0^1 0\,dt = \underline{0}.$$

Since $\underline{V}(x,y,z) = \underline{\nabla} f(x,y,z)$ then

$$\underline{\mathrm{Tan}(\underline{V}, l_s)} = f(D) - f(A) = -9 + \frac{1}{3} = \underline{-\frac{26}{3}}.$$

3. $\underline{\mathrm{Circ}}(\underline{V}, \partial K) = 0$, since $\underline{V}$ is a gradient fied.



With the chosen directions of the boundary curve ∂K for K we have

$\underline{n}_K = (1,0,0) = \underline{N}(u,v)$.

From Stokes' theorem we get

$$\underline{\mathrm{Circ}(\underline{U}, \partial K)} = \mathrm{Flux}(\mathrm{curl}(\underline{U}), K_{\underline{r}}) = \underline{16}$$