 Math 1 Two-hours test 17.5.2014

Problem 1.

$f(x) = 2\cos x - \sin 2x \,,\; x \in \mathbb{R}.$

1. $f'(x) = -2\sin x - 2\cos 2x$

   $f''(x) = -2\cos x + 4\sin 2x$

   $f'''(x) = 2\sin x + 8\cos 2x$

2. With the developmentpoint $x_0 = 0$ we get

   $\underline{P_2(x)} = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = \underline{2 - 2x - x^2}.$

3. According to Taylor's formula with $x_0 = 0$ there exists a ξ between $\frac{1}{10}$ and 0, such that

$f\left(\frac{1}{10}\right) = P_2\left(\frac{1}{10}\right) + \frac{1}{3!}f'''(\xi)\left(\frac{1}{10}\right)^3 = P_2\left(\frac{1}{10}\right) + \frac{1}{6}\left(2\sin\xi + 8\cos 2\xi\right)\frac{1}{10^3}$

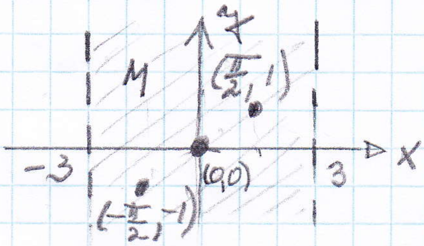
If we use $P_2\left(\frac{1}{10}\right) = \frac{179}{100} = 1{,}79$ instead of $f\left(\frac{1}{10}\right)$ then the error $\left|f\left(\frac{1}{10}\right) - P_2\left(\frac{1}{10}\right)\right| = \left|R_2\left(\frac{1}{10}\right)\right| = \frac{1}{6\cdot 10^3}\left|2\sin\xi + 8\cos 2\xi\right| \leq$

$\frac{10}{6\cdot 10^3} = \frac{1}{600}$, since $0 < 2\sin\xi + 8\cos 2\xi \leq 2 + 8 = 10$, $\xi \in \left[0; \frac{1}{10}\right]$.

Since $f\left(\frac{1}{10}\right) - P_2\left(\frac{1}{10}\right) = R_2\left(\frac{1}{10}\right) > 0$ and $\frac{1}{600} = 0{,}001666\ldots < 0{,}0017$

we get $P_2\left(\frac{1}{10}\right) < f\left(\frac{1}{10}\right) < P_2\left(\frac{1}{10}\right) + 0{,}017$. Therefore $1{,}7900 < f\left(\frac{1}{10}\right) < 1{,}7917$.

Problem 2.



$f(x,y) = \frac{1}{2}y^2 - y\sin x.$

$M = \{(x,y) \in \mathbb{R}^2 \mid -3 < x < 3\}$

1. $f'_x(x,y) = -y\cos x$, $f'_y(x,y) = y - \sin x$.

   $f''_{xx}(x,y) = y\sin x$, $f''_{xy}(x,y) = f''_{yx}(x,y) = -\cos x$, $f''_{yy}(x,y) = 1$

2. $\nabla f(x,y) = (f'_x(x,y), f'_y(x,y)) = (-y\cos x, y - \sin x) = (0,0)$ and $(x,y) \in M \Leftrightarrow (y = 0 \wedge \sin x = 0 \wedge x \in\,]-3;3[) \vee (\cos x = 0 \wedge y - \sin x = 0 \wedge x \in\,]-3;3[)$. This gives the points $\underline{(0,0)}$, $\underline{(-\frac{\pi}{2}, -1)}$ and $\underline{(\frac{\pi}{4}, 1)}$. <u>These three points are the only stationary points in M.</u>

3. The Hessian matrix for f in the point (x,y) is

Problem 2 continued

$$\underline{H}(x,y) = \begin{bmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{xy}(x,y) & f''_{yy}(x,y) \end{bmatrix} = \begin{bmatrix} y\sin x & -\cos x \\ -\cos x & 1 \end{bmatrix}.$$

If f has a local minimum or a local maximum in a point in M, then the point must be a stationary point in M, since f do not have any exception points in M.

$$\underline{H}(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}. \quad \begin{vmatrix} -\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0 \Leftrightarrow \lambda = \begin{cases} \frac{1+\sqrt{5}}{2} > 0 \\ \frac{1-\sqrt{5}}{2} < 0 \end{cases}$$

f has neither a local minimum nor a local maximum in the stationary point $(0,0)$.

$$\underline{H}(-\tfrac{\pi}{2},-1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{H}(\tfrac{\pi}{2},1). \quad \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \Leftrightarrow \lambda = \begin{cases} 1 > 0 \\ 1 > 0 \end{cases}.$$

f has a proper local minimum in the stationary points $(-\frac{\pi}{2},-1)$ and $(\frac{\pi}{2},1)$ with the values $f(-\frac{\pi}{2},-1) = -\frac{1}{2} = f(\frac{\pi}{2},1)$.

Problem 3.

Space curve $K_{\underline{r}}$ given by

$$\underline{r}(t) = (e^t - e^{-t}, e^t + e^{-t}, 1-2t), \quad t \in [0;1]$$

1. $\underline{r}'(t) = (e^t + e^{-t}, e^t - e^{-t}, -2)$.

$$\underline{|\underline{r}'(t)|} = \sqrt{(e^t+e^{-t})^2 + (e^t-e^{-t})^2 + 4} = \sqrt{2e^{2t} + 2e^{-2t} + 4}$$
$$= \sqrt{2}\sqrt{(e^t+e^{-t})^2} = \underline{\sqrt{2}(e^t + e^{-t})}.$$

$$\underline{L(K_{\underline{r}})} = \int_{K_{\underline{r}}} ds = \int_0^1 |\underline{r}'(t)|\,dt = \sqrt{2}\int_0^1 (e^t + e^{-t})\,dt = \underline{\sqrt{2}(e - e^{-1})}.$$

$\underline{V}(x,y,z) = (y, x, -2)$, $(x,y,z) \in \mathbb{R}^3$. (First order).

2. $\underline{V}(\underline{r}(t)) = (e^t + e^{-t}, e^t - e^{-t}, -2)$.

$$\underline{V}(\underline{r}(t)) \cdot \underline{r}'(t) = (e^t + e^{-t})^2 + (e^t - e^{-t})^2 + 4 = 2e^{2t} + 2e^{-2t} + 4.$$

$$\underline{\mathrm{Tan}(\underline{V}, K_{\underline{r}})} = \int_{K_{\underline{r}}} \underline{V} \cdot \underline{t}\,ds = \int_0^1 \underline{V}(\underline{r}(t)) \cdot \underline{r}'(t)\,dt = \int_0^1 (2e^{2t} + 2e^{-2t} + 4)\,dt$$

$$= \left[e^{2t} - e^{-2t} + 4t\right] = \underline{e^2 - e^{-2} + 4}.$$

Problem 3, cont.

3. Since $\underline{V}(\underline{r}(t)) = (e^t + e^{-t}, e^t - e^{-t}, -2) = \underline{r}'(t)$ for all $t \in [0;1]$ and since $\underline{r}(0) = (0,2,1)$, then $K_{\underline{r}}$ is the flow curve for $\underline{V}$, that starts in the point $(0,2,1)$ for $t=0$. (Cf. the existence and uniqueness theorem)

Problem 4

A solid field Ω in space is given by

$\underline{r}(u,v,w) = (u\cos w, u\sin w, v(2-u))$, where

$$u \in [1;2],\ v \in [0;1] \text{ and } w \in [0;2\pi].$$

1. $f(x,y,z) = 1$ and $g(x,y,z) = \frac{y}{2}$.

$\underline{r}'_u(u,v,w) = (\cos w, \sin w, -v)$.

$\underline{r}'_v(u,v,w) = (0, 0, 2-u)$

$\underline{r}'_w(u,v,w) = (-u\sin w, u\cos w, 0)$.

$$J_{\underline{r}}(u,v,w) = \left|\underline{r}'_u\ \underline{r}'_v\ \underline{r}'_w\right| = \begin{vmatrix} \cos w & 0 & -u\sin w \\ \sin w & 0 & u\cos w \\ -v & 2-u & 0 \end{vmatrix}$$

$$= -(2-u)u < 0, \text{ since } u \in [1;2].$$

$\text{Jacobi}_{\underline{r}}(u,v,w) = |J_{\underline{r}}(u,v,w)| = u(2-u) = 2u - u^2$.

$$\underline{\int_\Omega f\,d\mu} = \int_\Omega d\mu = \int_{w=0}^{\pi}\left(\int_{v=0}^{1}\left(\int_{u=1}^{2}(2u-u^2)\,du\right)dv\right)dw$$

$$= \pi\left[u^2 - \frac{1}{3}u^3\right]_1^2 = \underline{\frac{2\pi}{3}} = \mathrm{Vol}(\Omega)$$

$$\underline{\int_\Omega g\,d\mu} = \int_\Omega \frac{1}{2}y\,d\mu = \int_{w=0}^{\pi}\left(\int_{v=0}^{1}\left(\int_{u=1}^{2}\frac{1}{2}u\sin w\cdot \text{Jacobi}_{\underline{r}}(u,v,w)\,du\right)dv\right)dw$$

$$= \int_{w=0}^{\pi}\left(\int_{v=0}^{1}\left(\int_{u=1}^{2}\sin w\left(u^2 - \frac{1}{2}u^3\right)du\right)dv\right)dw = \left[-\cos w\right]_{w=0}^{\pi}\left[\frac{1}{3}u^3 - \frac{1}{8}u^4\right]_{u=1}^{2}$$

$$= \underline{\frac{11}{12}}.$$

$\underline{V}(x,y,z) = \left(\frac{1}{2}z^2, \frac{1}{4}y^2, -2y\right)$, $(x,y,z) \in \mathbb{R}^3$.

Problem 4.

2. $\operatorname{div} \underline{V} = \nabla \cdot \underline{V} = \frac{1}{2} y = g(x,y,z)$.

$\partial\Omega$ is the closed surface of Ω with an outward pointing unit normal vector as orientation.

From Gauss' theorem we then get

$$\underline{\text{Flux}(\underline{V}, \partial\Omega)} = \int_\Omega \operatorname{div} \underline{V} \, d\mu = \int_\Omega g \, d\mu = \underline{\frac{11}{12}}. \quad (\text{From } 1.)$$

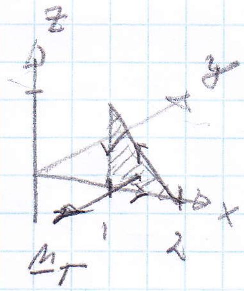
3. $\underline{U}(x,y,z)$, such that $\operatorname{div}(\underline{U}) = k$ and $\text{Flux}(\underline{U}, \partial\Omega) = \int_{\partial\Omega} \underline{u} \cdot \underline{n}_{\partial\Omega} \, d\mu = 2\pi$.

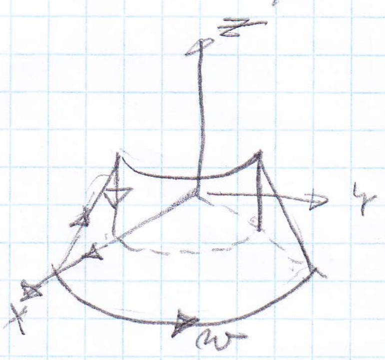
From Gauss' theorem we have

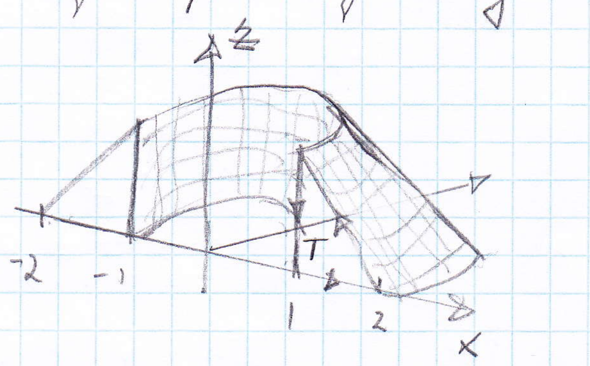
$$\text{Flux}(\underline{U}, \partial\Omega) = \int_{\partial\Omega} \operatorname{div}(\underline{U}) \, d\mu = k \int_\Omega d\mu \underset{\substack{\uparrow \\ (\text{From } 1.)}}{=} \frac{2k\pi}{3} = 2\pi \iff$$

$k = \operatorname{div}(\underline{U}) = 3$. $\quad$ $\underline{\text{Eg. } \underline{U}(x,y,z) = (x,y,z)}$.

4. If we put $w = 0$ we get the triangular field T given by







$\underline{r}'_u(u,v) = (1, 0, -w)$; $\underline{r}'_v(u,v) = (0, 0, 2-u)$.

$\underline{N}(u,v) = \underline{r}'_u \times \underline{r}'_v = (0, u-2, 0) = (2-u)(0,-1,0)$; $u \in [1,2]$.

$\operatorname{curl} \underline{V} = \nabla \times \underline{V} = (-2, z, 0)$. $\quad$ $\operatorname{curl} \underline{V}(\underline{r}(u,v)) = (-2, v(2-u), 0)$.

With the chosen orientation of the boundary curve ∂T for T we get $\underline{n}_T = (0,-1,0)$ and $\underline{n}_T = \dfrac{\underline{N}(u,v)}{|\underline{N}(u,v)|}$ for $u \in [1,2]$.

From Stokes' theorem we then have

$$\underline{\text{Circ}(\underline{V}, \partial T)} = \text{Flux}(\operatorname{curl} \underline{V}, T) = \int_T \underline{n}_T \cdot \operatorname{curl} \underline{V} \, d\mu$$

$$= \int_{v=0}^{1} \left(\int_{u=1}^{2} \underline{N}(u,v) \cdot \operatorname{curl} \underline{V}(\underline{r}(u,v)) \, du \right) dv = \int_{v=0}^{1} -v \left(\int_{u=1}^{2} (4 + u^2 - 4u) \, du \right) dv = \underline{-\frac{1}{6}}.$$