Suggested solution to the2-hours test 7/12, E20, ver2

rev. 09.12.20

NB: The purpose of this solution proposal is to show how the tasks can be solved using simple Maple commands known from MapleDemos basic and simple explanations. For the sake of readability, there is probably a little more text in some places than you would spend time on in an exam situation, e.g. repetition of the questions asked. Note in particular that Problem 1 and 2 are variants of the Problems that were asked in Maple TA. In addition note that Problem 3 is not in the Essay Style format.

In ver2 of the solution proposal, the variant in Task 1 is the same as in the PDF version of the exam set.

Problem 1, asked and answered in Maple TA

```
> restart:
> with(LinearAlgebra):
```

Let a and b be real constants. An inhomogeneous system of linear equations has the augemented matrix **T** given by

$$\mathbf{T} = \begin{bmatrix} 1 & -3 & 5 & 5 \\ 0 & a-1 & 2 & 5 \\ 0 & 0 & a+1 & b \\ 0 & 0 & 0 & a^2-1 \end{bmatrix}.$$

1)

Determine for a = -1 and b =0 a solution to the system of equations.

```
> T:=<1,-3,5,5;0,-2,2,5;0,0,0,0;0,0,0,0>;
```

$$T := \begin{bmatrix} 1 & -3 & 5 & 5 \\ 0 & -2 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.1)}$$

```
> ReducedRowEchelonForm(T)
```

$$\begin{bmatrix} 1 & 0 & 2 & -\frac{5}{2} \\ 0 & 1 & -1 & -\frac{5}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2)}$$

The solution set:

```
> L:=<-5/2,-5/2,0>+t*<-2,1,1>;
```

$$L := \begin{bmatrix} -\frac{5}{2} - 2t \\ -\frac{5}{2} + t \\ t \end{bmatrix} \quad \textbf{(1.3)}$$

A possible answer is found for t = 0:

(-5/2,-5/2,0)

2)

From the solution set to the inhomogeneous system we read the solution set to the corresponding homogeneous system:

```
> Lhom:=t*<-2,1,1>;
```

$$Lhom := \begin{bmatrix} -2\,t \\ t \\ t \end{bmatrix} \tag{1.4}$$

The vector **w** = (-2,1,1) spans Lhom.

3)

Suppose that a = 1. For which value of b does the system of equations have solutions? State one solution for this value of b.

We suppose a = 1

```
> T:=<1,-3,5,5;0,0,2,5;0,0,2,b;0,0,0,0>;
```

$$T := \begin{bmatrix} 1 & -3 & 5 & 5 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 2 & b \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.5}$$

```
> RowOperation(T,[3,2],-1)
```

$$\begin{bmatrix} 1 & -3 & 5 & 5 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & b-5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.6}$$

For a=1 the system contains an inconsistent equation (cf. row 3) if not b = 5. Answer b=5.

4)

We assume as in the previous question that a = 1 and b = 5, and we shall find $\alpha, \beta, \gamma, \delta$ so that $(\alpha, 0, \beta)$ and $(\gamma, 1, \delta)$ are solutions.

```
> T:=<1,-3,5,5;0,0,2,5;0,0,2,5;0,0,0,0>;
```

$$T := \begin{bmatrix} 1 & -3 & 5 & 5 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.7}$$

```
> ReducedRowEchelonForm(T)
```

$$\begin{bmatrix} 1 & -3 & 0 & -\frac{15}{2} \\ 0 & 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.8}$$

The solution set:

```
> L:=<-15/2,0,5/2>+t*<3,1,0>;
```

$$L := \begin{bmatrix} -\frac{15}{2} + 3\,t \\ t \\ \frac{5}{2} \end{bmatrix} \qquad \textbf{(1.9)}$$

For $t = 0$ we find $\alpha = -\frac{15}{2}$ and $\beta = \frac{5}{2}$

For $t = 1$ we find $\gamma = -\frac{9}{2}$ and $\delta = \frac{5}{2}$.

5)

Examine whether values of a and b exist for which the system of equations has exactly one solution.

If not a is either -1 or 1, the system of equations contains an inconsistent equation, cf. row 4 in **T**.

If a = -1, then b = 0, or else the system of equations contains an inconsistent equation, cf. row 3 in **T**. The case b = 0 is treated in Question 1, where we found one free parameter.

If a = 1, then b = 5, or else the system of equations contains one inconsistent equation, cf. row 3 in **T**. The case b = 5 is treated in Question 3, where we found one free parameter.

Therefore values of a and b do not exist for which the system has exactly one solution.

Problem 2, asked and answered in Maple TA

A first-order inhomogeneous linear differential equation linear differential equation is given by

$$x'(t) + 3 \cdot x(t) = \mathrm{e}^{3 \cdot t} \cdot \cos(t)$$

1)

It is stated that (*) has a particular solution of the form

$$x_0(t) = a\,\mathrm{e}^{3 \cdot t} \cos(t) + b\,\mathrm{e}^{3 \cdot t} \sin(t) \text{ where } a, b \in \mathbb{R}.$$

Determine a and b.

```
> x0:=t->a*exp(3*t)*cos(t)+b*exp(3*t)*sin(t):
  x0(t)
```

$$a\,\mathrm{e}^{3\,t} \cos(t) + b\,\mathrm{e}^{3\,t} \sin(t) \qquad \textbf{(2.1)}$$

```
> diff(x0(t),t)+3*x0(t)=e^(3*t)*cos(t)
```

$$6\,a\,\mathrm{e}^{3\,t} \cos(t) - a\,\mathrm{e}^{3\,t} \sin(t) + 6\,b\,\mathrm{e}^{3\,t} \sin(t) + b\,\mathrm{e}^{3\,t} \cos(t) = e^{3\,t} \cos(t) \qquad \textbf{(2.2)}$$

$$(6\,a + b)\cos(t) + (6\,b - a)\sin(t) = \cos(t)$$

```
> lign1:=6*a+b=1;
```

$$lign1 := 6\,a + b = 1 \qquad \textbf{(2.3)}$$

```
> lign2:=6*b-a=0;
```

$$lign2 := 6\,b - a = 0 \qquad \textbf{(2.4)}$$

```
> solve({lign1,lign2},{a,b})
```

$$\left\{a = \frac{6}{37},\, b = \frac{1}{37}\right\} \qquad \textbf{(2.5)}$$

2)

Determine a solution to the corresponding homogeneous differential equation.

$x'(t) = -3 \cdot x(t)$. Lhom is given by $c \cdot e^{-3 \cdot t}$ where c is a real number. We can choose:

$$x_h(t) = e^{-3 \cdot t}.$$

3-4)

Linhom is – according to the Structural Theorem and results in **1)** and **2)** – given by

$$x(t) = \frac{6}{37}\,e^{3\,t}\cos(t) + \frac{1}{37}\,e^{3\,t}\sin(t) + c\cdot e^{-3\cdot t},\quad c \in \mathbb{R}.$$

Only the first of the suggested three solutions has the required form.

About a second-order linear differential equation it is stated that the complete solution is given by

$$x(t) = k_1\,e^{-5\cdot t}\cos(3\,t) + k_2\,e^{-5\cdot t}\sin(3\,t) \qquad (*)$$

5)

Determine one of the roots in the characteristic equation.

We read: $\alpha = -5$ and $\beta = 3$, $\lambda = -5 + 3\,i$.

6)

Determine the two numbers k_1 and k_2 so that (*) is a solution that fulfills that $x(0) = 1$ and $x'(0) = 1$.

```
> restart:
> x:=t->k1*exp(-5*t)*cos(3*t)+k2*exp(-5*t)*sin(3*t)
```

$$x := t \mapsto k1\,e^{-5\,t}\cos(3\,t) + k2\,e^{-5\,t}\sin(3\,t) \qquad \textbf{(2.6)}$$

```
> x(0)=4
```

$$k1 = 4 \qquad \textbf{(2.7)}$$

```
> k1:=4;
```

$$k1 := 4 \qquad \textbf{(2.8)}$$

```
> D(x)(0)=5
```

$$-20 + 3\,k2 = 5 \qquad \textbf{(2.9)}$$

```
> k2:=solve(%,k2)
```

$$k2 := \frac{25}{3} \qquad \textbf{(2.10)}$$

Problem 3, Essay Assignment

```
> restart:
> with(LinearAlgebra):
  with(plots):
  prik:=(x,y)->VectorCalculus[DotProduct](x,y):
  kryds:=(x,y)->convert(VectorCalculus[CrossProduct](x,y),Vector):
```

Given the vectors

```
> u1:=<1,0,1>:
  u2:=<1,1,1>:
```

1)

The two vectors are not parallel, i.e. thus they are linearly independent. Therefore they constitute a basis for U which thus has dim(U) = 2.

Using the Gram-Schmidt algorithm:

```
> v1:=u1/sqrt(prik(u1,u1))
```

$$v1 := \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \qquad \textbf{(3.1)}$$

```
> w2:=u2-prik(u2,v1)*v1
```

$$w2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \textbf{(3.2)}$$

```
> v2:=w2/sqrt(prik(w2,w2))
```

$$v2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \textbf{(3.3)}$$

v1 and **v**2 have the length 1 and are orthogonal. Thus (**v**1,**v**2) is an orthonormal basis for U.

2)

Since U is the kernel for f, it applies for every vector **u** in V that f(**u**)=0**u**. Therefore U is identical to the eigenspace E[0].

Since **u**1+**u**2 and 2**v**1-v**2** lie in the kernel for *f*, it applies that f(**u**1+**u**2) = f(2**v**1-**v**2) = (0,0,0).

3)

The orthogonal complement is 1-dimensional and as a basis vector we can choose

```
> v3:=kryds(v1,v2)
```

$$v3 := \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \quad \textbf{(3.4)}$$

The set (**v**1,**v**2,**v**3) is thus an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors for f. If we put

```
> Q:=<v1|v2|v3>
```

$$Q := \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \quad \textbf{(3.5)}$$

Q is orthogonal.

We form a diagonal matrix of the eigenvalues that corresponds to (**v**1,**v**2,**v**3):

```
> Lambda:=DiagonalMatrix([0,0,4])
```

$$\Lambda := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \textbf{(3.6)}$$

Then it applies that $e\mathbf{F}e = \mathbf{Q}\cdot\mathbf{\Lambda}\cdot\mathbf{Q}^T$.

4)

NB: The Question can be solved in more ways and with more possible answers.

Here we put

```
> u3:=<1,0,0>
```

(3.7)

$$u3 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{3.7}$$

The set (**v**1,**v**2,**u**3) is then linearly independent and constitutes a new basis for $\mathbf{R}^3$ with the change of base matrix

```
> eMu:=<v1|v2|u3>
```

$$eMu := \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 1 \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \tag{3.8}$$

A linear map is determined by the image of the basis vectors. We can then determine a possible *g* by *g*(**u**3)=(1,2,3). We then have

```
> eGu:=<0,0,1;0,0,2;0,0,3>
```

$$eGu := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} \tag{3.9}$$

```
> eGe:=eGu.eMu^(-1)
```

$$eGe := \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 3 & 0 & -3 \end{bmatrix} \tag{3.10}$$