TECHNICAL UNIVERSITY OF DENMARK

Written 2-hours exam in the fall curriculum December 7, 2020: Essay Assignment.

Course Name: Advanced Engineering Mathematics 1. **Course no.** 01006

Allowed helping aids: All helping aids allowed by DTU can be used. For other rules, please refer to the document 'Syllabus & Rules' on the course homepage

Problem 1, variant. Asked and answered in Maple TA

Let a and b be real constants. An inhomogeneous system of linear equations has the augmented matrix $\mathbf{T}$ given by

$$\mathbf{T}=\begin{bmatrix} 1 & -3 & 5 & 5 \\ 0 & a-1 & 2 & 5 \\ 0 & 0 & a+1 & b \\ 0 & 0 & 0 & a^2-1 \end{bmatrix}.$$

1. Determine for $a=-1$ and $b=0$ one of the solutions $\mathbf{x}_0$ to the system of equations.
2. The values of the constants are unchanged $a=-1$ and $b=0$. State a vector $\mathbf{w}$ that spans the solution space for the corresponding homogeneous system of equations.
3. Suppose $a=1$. For which value of b does the system of equations have solutions?
4. We consider now the system of equations with the values for a and b as in the previous question. The vectors $(\alpha,0,\beta)$ and $(\gamma,1,\delta)$ are solutions to the inhomogeneous system. State the values of α,β,γ and δ.
5. Examine whether there are values of a and b for which the system of equations has exactly one solution.

Problem 2, variant. Asked and answered in Maple TA

A first-order inhomogeneous linear differential equation is given by:

$$x'(t)+3x(t)=\mathrm{e}^{3t}\cos(t).$$

1. It is stated that the differential equation has a particular solution of the form

   $$x_p(t)=a\cdot\mathrm{e}^{3t}\cos(t)+b\cdot\mathrm{e}^{3t}\sin(t) \text{ where } a,b\in\mathbb{R}.$$

   Determine a and b.
2. Determine a solution $x_h(t)$ to the corresponding homogeneous differential equation. We further stipulate that $x_h(t)$ must not be the zero solution.
3. From the previous questions we have $x_h(t)$ and $x_p(t)$. Which of the following three functions is a solution to the inhomogeneous differential equation

   $$4\cdot x_h(t)+x_p(t),\ x_h(t),\ x_h(t)+3\cdot x_p(t).$$
4. Justify your answers to the previous question using the structural theorem.

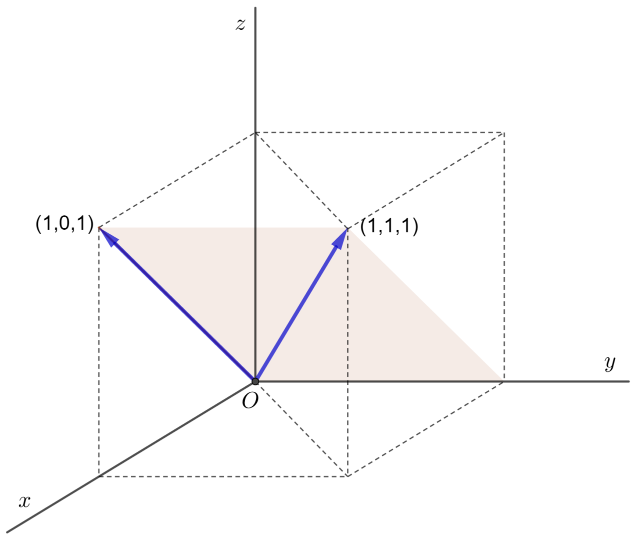
Regarding a homogeneous second-order linear differential equation, it is stated that its complete solution is given by:

$$x(t) = k_1 \cdot \mathrm{e}^{-5t}\cos(3t) + k_2 \cdot \mathrm{e}^{-5t}\sin(3t) \quad (*)\,.$$

5. Determine one of the roots λ in the characteristic equations corresponding to the differential equation.

6. Determine k_1 and k_2 so that the solution to $(*)$ fulfills $x(0) = 4$ and $x'(0) = 5$.

Problem 3. Essay Assignment. Answer is uploaded in pdf.



A subspace U i $\mathbb{R}^3$ is spanned by the vectors $\mathbf{u}_1 = (1,0,1)$ and $\mathbf{u}_2 = (1,1,1)$. It is also stated that U is the kernel for a linear map $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$.

1. Explain that $\dim(U) = 2$, and determine an orthonormal basis $(\mathbf{v}_1, \mathbf{v}_2)$ for U.

2. Explain that the number 0 is an eigenvalue for f, and determine $f(\mathbf{u}_1 + \mathbf{u}_2)$ and $f(2\mathbf{v}_1 - \mathbf{v}_2)$.

3. It is now further stated that f has the eigenvalue 4 and that E_4 is the orthogonal complement to U. Determine a diagonal matrix Λ and an orthogonal matrix $\mathbf{Q}$ so that

   $${}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}} = \mathbf{Q}\Lambda\mathbf{Q}^{\mathrm{T}}$$

   where ${}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}}$ denote the mapping matrix for f with respect to the standard base for $\mathbb{R}^3$.

4. Determine the mapping matrix ${}_{\mathrm{e}}\mathbf{G}_{\mathrm{e}}$ for another linear map $g : \mathbb{R}^3 \mapsto \mathbb{R}^3$ which also has U as the kernel, and which fulfills that $(1,2,3) \in g(\mathbb{R}^3)$.

THE END