TECHNICAL UNIVERSITY OF DENMARK
Written 2-hours exam in the fall curriculum December 8, 2019.
Course Name: Advanced Engineering Mathematics 1. **Course no.** 01006
Allowed helping aids: All helping aids allowed by DTU can be brought and used.
Weighting: The four exercises are equally weighted.
All answers must be argued and intermediate calculations included to an appropriate extend.

PROBLEM 1

Let a be an arbitrary real number. A homogeneous system of linear equations is given by

$$\begin{aligned} x_1+x_2+x_3+x_4&=0\\ 2x_1-x_2+8x_3-4x_4&=0\\ x_1-2x_2+7x_3+a\cdot x_4&=0 \end{aligned}$$

1. For $a=1$ state the complete solution to the system of equations in standard parametric form.
2. For a certain value of a the coefficient matrix for the system of equations has the rank 2. State for this value of a two linearly independent solutions to the system of equations.

PROBLEM 2

A 2-dimensional vector space V has the base $v=(\mathbf{v}_1,\mathbf{v}_2)$. A linear map

$$f:V\to\mathbb{R}^3$$

has with respect to the base v in V and the standard base e in $\mathbb{R}^3$ the mapping matrix

$${}_e\mathbf{F}_v=\begin{bmatrix}1&2\\0&-1\\2&0\end{bmatrix}.$$

1. State the coordinate vector for the vector

   $$\mathbf{v}_3=2\mathbf{v}_1-5\mathbf{v}_2$$

   with respect to the base v, and determine the image vector $f(\mathbf{v}_3)$.
2. Solve the equation

   $$f(\mathbf{v})=(1,2,10).$$

3. Determine the dimension of the image space $f(V)$ and the dimension of $\ker(f)$.
4. State (and argue) a vector in $\mathbb{R}^3$ that does not belong to $f(V)$.

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PROBLEM 3

In the vector space $\mathbb{R}^3$ equipped with the usual dot product three vectors are given

$$\mathbf{v}_1 = (1, -1, 1)\,,\ \mathbf{v}_2 = (1, 0, -1) \text{ and } \mathbf{v}_3 = (1, 1, 0)\,.$$

A matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$ has the eigenspaces $E_6 = \text{span}\{\mathbf{v}_1\}$ and $E_{-3} = \text{span}\{\mathbf{v}_2, \mathbf{v}_3\}$.

1. Consider the matrix $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$. State a diagonal matrix $\mathbf{\Lambda}$ that fulfill the identity
$$\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\,\mathbf{V}\,.$$
2. Explain that every vector in E_6 is orthogonal to every vector in E_{-3} .
3. Determine a positive orthogonal matrix $\mathbf{Q}$ that fulfill the identity
$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{T}}$$
where $\mathbf{\Lambda}$ is the diagonal matrix from question 1.

PROBLEM 4

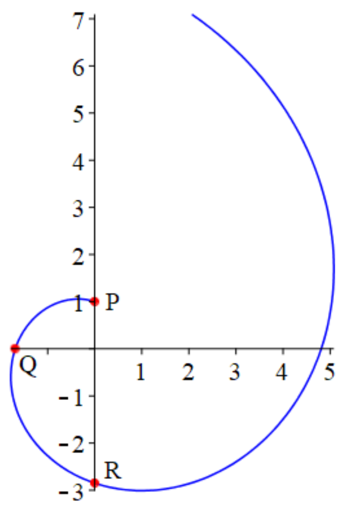
For a given matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ we consider a linear system of first order differential equations in the form

$$\begin{bmatrix} x_1{}'(t) \\ x_2{}'(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ t \in \mathbb{R}\,.$$

It is given that the complete complex solution to the system is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \cdot \mathrm{e}^{(2+6i)t} \cdot \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 \cdot \mathrm{e}^{(2-6i)t} \cdot \begin{bmatrix} -i \\ 1 \end{bmatrix} \text{ where } c_1, c_2 \in \mathbb{C} \text{ and } t \in \mathbb{R}\,.$$

1. Find the eigenvalues for $\mathbf{A}$ and the corresponding eigenspaces.
2. Show that the particular solution where $c_1 = c_2 = \frac{1}{2}$, satisfies the initial condition
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
3. Consider the particular solution in the previous question. The figure shows the trajectory, the point $(x_1(t), x_2(t))$ traverses in the time $t \in [0, 1]$. The point P is the initial point, while Q is the first point of intersection with the first axis, and R is the first point of intersection with the second axis, see the figure.



Determine the values of t where the point $(x_1(t), x_2(t))$ is passing Q and R, respectively.

THE END