Math 1. Two-hours Exam December 9, 2018.

JE/JKL 8.12.18

Problem 1

```
> restart:
```

A circle C in the complex number plane is given by the equation $|z-5|=5$.
I.e. C is the circle with center (5, 0) and radius 5.

Question 1

```
> z1:=1+3*I;
```

$$z1 := 1+3\,\mathrm{I} \tag{1.1.1}$$

$|z_1-5| = |-4+3\,i| = \sqrt{16+9} = \sqrt{25} = 5.$

This shows that z_1 lies on C.

Using Maple we get directly

```
> abs(z1-5);
```

$$5 \tag{1.1.2}$$

$$\frac{1}{z_1} = \frac{1}{1+3\,i} = \frac{1-3\,i}{(1+3\,i)(1-3\,i)} = \frac{1}{10} - \frac{3}{10}i.$$

This shows that $\mathrm{Re}\left(\frac{1}{z_1}\right) = \frac{1}{10}$.

Using Maple we get directly

```
> evalc(1/z1);
```

$$\frac{1}{10} - \frac{3\,\mathrm{I}}{10} \tag{1.1.3}$$

Or even more directly

```
> Re(1/z1);
```

$$\frac{1}{10} \tag{1.1.4}$$

Question 2

```
> z2:=5*sqrt(3)*exp(I*Pi/6);
```

$$z2 := 5\sqrt{3}\left(\frac{\sqrt{3}}{2} + \frac{\mathrm{I}}{2}\right) \tag{1.2.1}$$

```
> evalc(z2);
```

$$\frac{15}{2} + \frac{5\,\mathrm{I}\sqrt{3}}{2} \tag{1.2.2}$$

$$\frac{1}{z_2} = \frac{\overline{z_2}}{z_2\,\overline{z_2}} = \frac{1}{10} - \frac{\sqrt{3}\,i}{30}.$$

This shows that Re(

$\frac{1}{z_2}) = \frac{1}{10}$.

Using Maple we get directly

```
> evalc(1/z2);
```

$$\frac{1}{10} - \frac{I\sqrt{3}}{30} \qquad \textbf{(1.2.3)}$$

Or even more directly

```
> Re(1/z2);
```

$$\frac{1}{10} \qquad \textbf{(1.2.4)}$$

Question 3

```
> assume(x,real):assume(y,real);interface(showassumed=0):
> z:=x+I*y;
```

$$z := x + I\,y \qquad \textbf{(1.3.1)}$$

```
> Re(1/z);
```

$$\frac{x}{x^2 + y^2} \qquad \textbf{(1.3.2)}$$

$\operatorname{Re}(\frac{1}{z}) = \frac{x}{x^2 + y^2} = \frac{1}{10} \Leftrightarrow x^2 + y^2 = 10\,x \Leftrightarrow x^2 + y^2 - 10\,x = 0 \Leftrightarrow (x-5)^2 + y^2 = 25 \Leftrightarrow$

$|x - 5 + iy| = 5 \Leftrightarrow |x + iy - 5| = 5 \Leftrightarrow |z - 5| = 5$.

This shows that every number z that fulfill $\operatorname{Re}(\frac{1}{z}) = \frac{1}{10}$ lies on C.

In particular it follows from Question 2 that z_2 lies on C.

Problem 2

```
> restart;with(LinearAlgebra):
```

The augmented matrix **T** = [**A**| **b**] for an inhomogeneous linear system of equation has the totally reduced form

```
> trap(T):=<<1,0,0,0>|<0,1,0,0>|<-4,0,0,0>|<0,0,1,0>|<-2,1,3,0>|<1,
  0,-5,0>>;
```

$$trap(T) := \begin{bmatrix} 1 & 0 & -4 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \textbf{(2.1)}$$

Question 1

The completely reduced linear system of equations is then

$$\begin{matrix} x_1 - 4\,x_3 - 2\,x_5 = 1 \\ x_2 + x_5 = 0 \\ x_4 + 3\,x_5 = -5 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} x_1 = 1 + 4\,x_3 + 2\,x_5 \\ x_2 = -x_5 \\ x_4 = -5 - 3\,x_5 \end{matrix}$$

If we put $x_3 = t_1$ and $x_5 = t_2$ then the complete solution in standard parameter form is

$(x_1, x_2, x_3, x_4, x_5) = (1, 0, 0, -5, 0) + t_1(4, 0, 1, 0, 0) + t_2(2, -1, 0, -3, 1)$, $t_1, t_2 \in \mathbb{R}$.

Using Maple we get

```
> LinearSolve(trap(T),free=t);
```

$$\begin{bmatrix} 1 + 4\,t_3 + 2\,t_5 \\ -t_5 \\ t_3 \\ -5 - 3\,t_5 \\ t_5 \end{bmatrix} \tag{2.1.1}$$

If we put $t_3 = t_1$ and $t_5 = t_2$ we find again

$(x_1, x_2, x_3, x_4, x_5) = (1, 0, 0, -5, 0) + t_1(4, 0, 1, 0, 0) + t_2(2, -1, 0, -3, 1)$, $t_1, t_2 \in \mathbb{R}$.

$f: \mathbb{R}^5 \to \mathbb{R}^4$ is linear and given by the fact that ${}_{e}\mathbf{F}_{e} = \mathbf{A}$.

Question 2

From question 1 we get.

$\mathbf{x} \in \ker(f) \Leftrightarrow f(\mathbf{x}) = \mathbf{0} \Leftrightarrow {}_{e}\mathbf{F}_{e}\;{}_{e}\mathbf{x} = \mathbf{0} \Leftrightarrow \mathbf{A}\mathbf{x} = \mathbf{0} \Leftrightarrow \mathbf{x} = t_1(4, 0, 1, 0, 0) + t_2(2, -1, 0, -3, 1)$,

$t_1, t_2 \in \mathbb{R}$.

I.e. $\ker(f) = \text{span}\{(4, 0, 1, 0, 0), (2, -1, 0, -3, 1)\}$, where the two vectors are linearly independent.

From this it follows that $\dim(\ker(f)) = 2$.

$\dim(f(\mathbb{R}^5)) = \dim(\mathbb{R}^5) - \dim(\ker(f)) = 5-2 = 3$.

Or more easily:

$\dim(f(\mathbb{R}^5)) = \rho({}_{e}\mathbf{F}_{e}) = \rho(\mathbf{A}) = 3$, which can be read from 5 first columns in trap($\mathbf{T}$).

$\dim(\ker(f)) = \dim(\mathbb{R}^5) - \dim(f(\mathbb{R}^5)) = 5-3 = 2$.

Problem 3

```
> restart;with(LinearAlgebra):
> prik:=(x,y)->VectorCalculus[DotProduct](x,y):
  kryds:=(x,y)->convert(VectorCalculus[CrossProduct](x,y),Vector):
```

$\mathbb{R}^3$ equipped with the usual dot product.

```
> v1:=<1,1,1>;
```

$$v1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{3.1}$$

```
> v2:=<1,0,-1>;
```

$$v2 := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{3.2}$$

```
> v3:=<-1,1,0>;
```

$$v3 := \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \textbf{(3.3)}$$

Question 1

The coordinate matrix for the vectors **v**1, **v**2 and **v**3 with respect to the standard basis *e* is

```
> eV:=<v1|v2|v3>;
```

$$eV := \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \textbf{(3.1.1)}$$

Since

```
> rho('eV')=Rank(eV);
```

$$\rho(eV) = 3 \quad \textbf{(3.1.2)}$$

v1, **v**2 and **v**3 are three linearly independent vectors in $\mathbb{R}^3$ and thus $v = (\mathbf{v}1, \mathbf{v}2, \mathbf{v}3)$ is a basis for $\mathbb{R}^3$.

$f: \mathbb{R}^3 \to \mathbb{R}^3$ is linear and given by the fact that for every vector $\mathbf{u} \in \mathrm{span}\{\mathbf{v}1\}$ it applies that
$f(\mathbf{u}) = 5\mathbf{u}$
and for every vector $\mathbf{u} \in \mathrm{span}\{\mathbf{v}2, \mathbf{v}3\}$ it applies that
$f(\mathbf{u}) = -4\mathbf{u}$.

Question 2

The given properties of *f* shows that 5 is an eigenvalue for *f* with corresponding eigenvector-space $E_5 = \mathrm{span}\{\mathbf{v}1\}$ and that −4 is an eigenvalue for *f* with corresponding eigenvector-space $E_{-4} = \mathrm{span}\{\mathbf{v}2, \mathbf{v}3\}$.
Thus $v = (\mathbf{v}1, \mathbf{v}2, \mathbf{v}3)$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors for *f*.
Since $f(\mathbf{v}1) = 5\mathbf{v}1 = 5\mathbf{v}1 + 0\mathbf{v}2 + 0\mathbf{v}3$, $f(\mathbf{v}2) = -4\mathbf{v}2 = 0\mathbf{v}1 - 4\mathbf{v}2 + 0\mathbf{v}3$ and
$f(\mathbf{v}3) = -4\mathbf{v}3 = 0\mathbf{v}1 + 0\mathbf{v}2 - 4\mathbf{v}3$, then

```
> vFv:=DiagonalMatrix(<5,-4,-4>);
```

$$vFv := \begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \quad \textbf{(3.2.1)}$$

The change of base matrix, that changes from *v*-coordinates to standard *e*-coordinates, is

```
> eMv:=eV;
```

$$eMv := \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \textbf{(3.2.2)}$$

and the change of base matrix that changes from standard *e*-coordinates to coordinates, is

```
> vMe:=eMv^(-1);
```

$$vMe := \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \quad (3.2.3)$$

We then have

```
> eFe:=eMv.vFv.vMe;
```

$$eFe := \begin{bmatrix} -1 & 3 & 3 \\ 3 & -1 & 3 \\ 3 & 3 & -1 \end{bmatrix} \quad (3.2.4)$$

which is seen to be symmetric.

Question 3

Since ${}_e\mathbf{F}_e$ is symmetric, then the two eigenvector-spaces E_5 and E_{-4} are orthogonal, which is also seen directly.

```
> q1:=v1/sqrt(prik(v1,v1));
```

$$q1 := \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix} \quad (3.3.1)$$

is an orthonormal basis for E_5 .

```
> q2:=v3/sqrt(prik(v3,v3));
```

$$q2 := \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \quad (3.3.2)$$

and

```
> q3:=kryds(q1,q2);
```

(3.3.3)

$$q3 := \begin{bmatrix} -\frac{\sqrt{3}\sqrt{2}}{6} \\ -\frac{\sqrt{3}\sqrt{2}}{6} \\ \frac{\sqrt{3}\sqrt{2}}{3} \end{bmatrix} \quad \textbf{(3.3.3)}$$

constitute an orthonormal basis for E_{-4} .

(**q**1, **q**2, **q**3) is then an orthonormal basis for $\mathbb{R}^3$ equipped with the ususal dot product consisting of eigenvectors for f , where the basis vector **q**2 is aligned with **v**3.

Problem 4

```
> restart;with(LinearAlgebra):
```

Given

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a & 3-a \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \ t \in \mathbb{R},$$

where a is an arbitrary real number.

The system matrix is

```
> A:=<<a,1>|<3-a,3>>;
```

$$A := \begin{bmatrix} a & 3-a \\ 1 & 3 \end{bmatrix} \quad \textbf{(4.1)}$$

Question 1

```
> a:=3:
```

The system matrix is in this case

```
> A;
```

$$\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix} \quad \textbf{(4.1.1)}$$

```
> g:=(x,y)->evalb(x[1]<y[1]):
> sort(Eigenvectors(A,output=list),g);
```

$$\left[\left[3, 2, \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}\right]\right] \quad \textbf{(4.1.2)}$$

From this it is read that the only eigenvalue for the system matrix **A** is 3 with am(3) = 2 and gm(3) = 1.

Since gm(3) < am(3) the system matrix **A** cannot be diagonalized (**A** does not have two linearly independent eigenvectors) and thus the system of equations cannot be solved by the diagonalization method.

Question 2

```
> a:='a':
```

```
> A;
```

$$\begin{bmatrix} a & 3-a \\ 1 & 3 \end{bmatrix} \quad \textbf{(4.2.1)}$$

Since

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{(4+i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ is a particular solution to the system of differential equations,

we have

$$(4+i)e^{(4+i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \mathbf{A}\, e^{(4+i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \Leftrightarrow (4+i) \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \mathbf{A} \begin{bmatrix} 1+i \\ 1 \end{bmatrix}.$$

From this it is read that $\lambda 1 = 4 + i$ is an eigenvalue for **A** and that $\mathbf{v}1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ is a corresponding eigenvector for **A**. Since **A** is a real 2×2- matrix, then $\lambda 2 = \overline{\lambda 1} = (4 - i)$ the other eigenvalue for **A** and $\mathbf{v}2 = \overline{\mathbf{v}1} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ is a corresponding eigenvector for **A**.

Since **v**1 and **v**2 are two linearly independent eigenvectors for **A (** the corresponding eigenvalues are different), then the complete complex solution to the system of differential equations

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c1\, e^{(4+i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} + c2\, e^{(4-i)t} \begin{bmatrix} 1-i \\ 1 \end{bmatrix}, t \in \mathbb{R}, c1, c2 \in \mathbb{C}.$$

Question 3

Since $4 + i$ is an eigenvalue for the system matrix **A**, then $4 + i$ is root in the characteristic polynomial for **A**, which is

```
> P:=lambda->Determinant(A-lambda*IdentityMatrix(2)):
> simplify(P(lambda));
```

$$\lambda^2 + (-a - 3)\lambda + 4a - 3 \quad \textbf{(4.3.1)}$$

```
> P(4+I);
```

$$5\,\mathrm{I} - \mathrm{I}\, a \quad \textbf{(4.3.2)}$$

From this it is seen that $P(4 + i) = 0 \Leftrightarrow a = 5$.

Thus. The value of a , that has been used in Question 2, is $a = 5$.