TECHNICAL UNIVERSITY OF DENMARK
Written two-hours exam: December 10, 2017.
Course name: Advanced Engineering Mathematics 1 **Course no.** 01006
Allowed helping aids: All helping aids allowed by DTU can be brought to the exam and used.
Weighting: The four problems will have equal weight.
All answers should be supported by valid arguments, and intermediate calculations should be included to an appropriate extent.

PROBLEM 1

Let a be an arbitrary real number. An inhomogeneous linear system of equations is given by

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= a^2 + 10a - 3 \\ x_1 + 2x_2 - 5x_3 &= a^2 + 3 \end{aligned}$$

1. State for $a = 1$ the complete solution to the system of equations in standard parametric form.
2. For which value of a is the tuple of numbers

$$(x_1, x_2, x_3) = (-7, 7, -1)$$

a solution to the system of equations?

PROBLEM 2

A symmetric matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

1. State the characteristic polynomial for $\mathbf{A}$ in complete factorized form, and state the roots together with their algebraic multiplicity.
2. Determine an orthogonal matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{\Lambda}$ that fulfill

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{T}}.$$

3. Consider the linear map $f : \mathbb{R}^4 \to \mathbb{R}^4$ that has the mapping matirix $\mathbf{A}$ with respect to the standard basis in $\mathbb{R}^4$. Furthermore, consider the 2-dimensional subspace U in $\mathbb{R}^4$ that is given by

$$U = \left\{ \mathbf{u} \in \mathbb{R}^4 \mid f(\mathbf{u}) = -\mathbf{u} \right\}.$$

Show that the unit vector

$$\mathbf{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

belongs to U, and determine a vector $\mathbf{v}_2$ such that the set $(\mathbf{v}_1, \mathbf{v}_2)$ is an orthonormal basis for U.

The exam continues on the next page ⟶

PROBLEM 3

In the vector space $C^\infty(\mathbb{R}, \mathbb{C})$ of infinitely many times differentiable complex functions of a real variable a 4-dimensional subspace U is given by its basis:

$$a = \left(\cos(t), \sin(t), e^t\cos(t), e^{(1+i)\cdot t}\right).$$

A linear map $f : U \to U$ is given by the expression

$$f(x(t)) = x''(t) - 2x'(t) + 2x(t) \ .$$

1. Show that the two basis vectors $e^t\cos(t)$ and $e^{(1+i)\cdot t}$ belong to the kernel for f.
2. Determine the mapping matrix ${}_a\mathbf{F}_a$ for f.
3. Determine the coordinate vector for the function $5\cos(t)$ with respect to the basis a, and use ${}_a\mathbf{F}_a$ in the solution of the equation

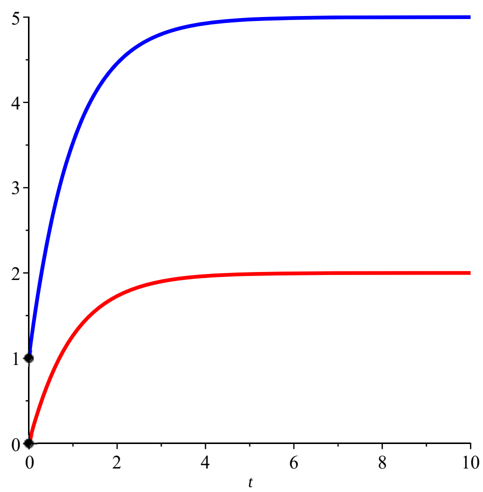
$$f(x(t)) = 5\cos(t).$$

PROBLEM 4

A linear system of first-order differential equations with constant coefficients is given by

$$\begin{bmatrix} x_1{}'(t) \\ x_2{}'(t) \end{bmatrix} = \begin{bmatrix} 4 & -10 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad t \in \mathbb{R}.$$

1. Determine by using the eigenvalues and the eigenvectors for the system matrix the complete solution to the system of differential equations.
2. Determine the solution to the system of differential equations that are illustrated in the figure, where the graph for x_1 is blue and the graph for x_2 red. Explain using the expressions found for $x_1(t)$ and $x_2(t)$ why $x_1(t)$ goes towards 5 and $x_2(t)$ towards 2, when t goes towards infinity.



3. State another non-constant solution to the system of differential equations that also fulfills that $x_1(t)$ goes towards 5 and $x_2(t)$ towards 2, when t goes toward infinity.

End of exam.