DANMARKS TEKNISKE UNIVERSITET
Written two-hours exam: December 5, 2016.
Course name: Advanced Engineering Mathematics 1 **Course no.** 01006
Allowed helping aids: All helping aids allowed by DTU can be brought to the exam and used.
Weighting: The four problems will have equal weight.
All answers should be supported by valid arguments, and intermediate calculations should be included to an appropriate extent.

PROBLEM 1

Let a be an arbitrary real number. An inhomogeneous linear system of equations consisting of three equations in four unknowns x_1, x_2, x_3 and x_4 has the augmented matrix $\mathbf{T}$ given by

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & a & -2 & -4 & 2 \\ 0 & 0 & 1-a^2 & 0 & 1-a \end{bmatrix}.$$

1. State the complete solution to the system of equations in each of the cases where $a=-1$, $a=0$ and $a=1$.
2. Determine those values of a for which $(x_1,x_2,x_3,x_4) = (1,0,-1,0)$ is a solution.

PROBLEM 2

In $\mathbb{R}^3$ the vectors $\mathbf{v}_1 = (-1,1,0)$, $\mathbf{v}_2 = (1,-2,1)$, $\mathbf{v}_3 = (1,-4,0)$ and $\mathbf{u} = 2\mathbf{v}_1 - \mathbf{v}_2$ are given.

1. Explain that the set of vectors $v = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis for $\mathbb{R}^3$, and state the change of basis matrix ${}_e\mathbf{M}_v$ that changes from v-coordinates to standard e-coordinates.
2. State the coordinate vector ${}_v\mathbf{u}$ for $\mathbf{u}$ with respect to the basis v, and determine the coordinate vector ${}_e\mathbf{u}$ for $\mathbf{u}$ with respect to the standard basis e.

For a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ it applies that $f(\mathbf{v}_1) = 3\mathbf{v}_1$, $f(\mathbf{v}_2) = 3\mathbf{v}_2$ and $f(\mathbf{v}_3) = \mathbf{0}$.

3. Explain that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are eigenvectors for f, and determine the eigenvalues to which they correspond.
4. Determine the mapping matrix ${}_v\mathbf{F}_v$ for f with respect to basis v and the mapping matrix ${}_e\mathbf{F}_e$ for f with respect to the standard basis e.
5. Is $\mathbf{u}$ an eigenvector for f?

PROBLEM 3

Consider the vector space $C^\infty(\mathbb{R})$ of infinitely many times differentiable real functions defined on $\mathbb{R}$. Consider further in $C^\infty(\mathbb{R})$ the subspace $P_2(\mathbb{R})$ of polynomials of at most second degree equipped with the monomial basis $(1, t, t^2)$ and the subspace $P_3(\mathbb{R})$ of polynomials of at most third degree equipped with the monomial basis $(1, t, t^2, t^3)$.

A linear map $f : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is given by the rule

$$(\star) \qquad f(x(t)) = x'(t) - t \cdot x(t) \ .$$

1. Determine the complete solution to the homogeneous differential equation $f(x(t)) = 0$.

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Now we consider f given by the rule $(\star)$ as a map from $P_2(\mathbb{R})$ to $C^\infty(\mathbb{R})$.

2. Determine the images $f(1)$, $f(t)$ and $f(t^2)$ of the basis vectors in $P_2(\mathbb{R})$ and explain that the image space $f(P_2(\mathbb{R}))$ is a subspace in $P_3(\mathbb{R})$.

Finally we consider f given by the rule $(\star)$ as a map from $P_2(\mathbb{R})$ to $P_3(\mathbb{R})$.

3. State the mapping matrix ${}_m\mathbf{F}_m$ for f with respect to the monomial basis in $P_2(\mathbb{R})$ and the monomial basis in $P_3(\mathbb{R})$. Determine the kernel for f and the dimension of the image space $f(P_2(\mathbb{R}))$.

PROBLEM 4

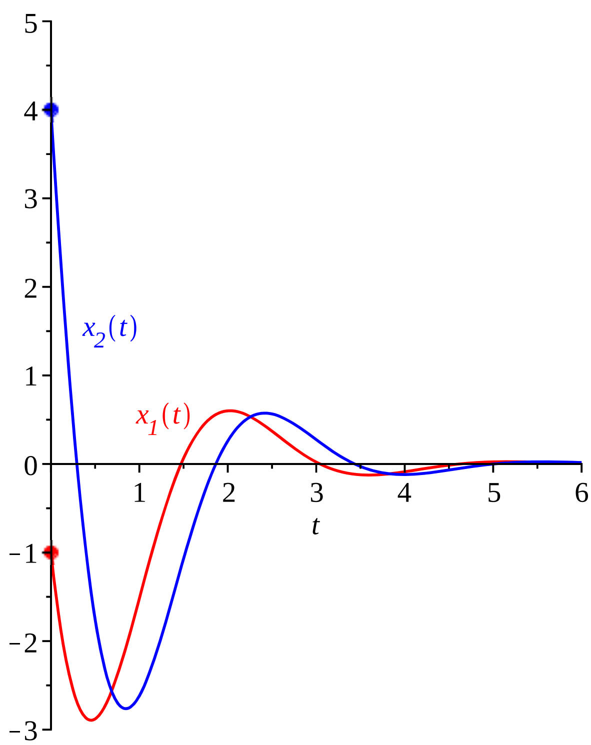
A homogeneous linear 1. order system of differential equations with constant coefficients is given by

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad t \in R$$

where $\mathbf{A}$ is a real 2×2-matrix. About $\mathbf{A}$ it is further given that it has the eigenvector $\mathbf{v} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = -1 + 2i$.

1. Determine the other eigenvalue of $\mathbf{A}$ and state a corresponding eigenvector. State the complete *complex* solution to the system of differential equations.

2. Determine the solution to the system of differential equations that is illustrated in the figure. The answer must be expressed in terms of real numbers and real functions.



End of exam.