Danmarks Tekniske Universitet
Written exam December 7, 2015.
Course name: Matematik 1. **Course no.** 01005
Allowed helping aids: All helping aids allowed by DTU can be brought with to exam and used.
Weighting: The four problems will have equal weight.
All answers should be supported by valid arguments, and intermediate calculations should be included to an appropriate extend. It is not allowed to communicate with others during the exam, neither directly nor electronically.

PROBLEM 1

Let a be an arbitrary real number. A matrix $\mathbf{A}$ and a vector $\mathbf{b}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & a-1 & a & 0 \\ 0 & a & a-1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

1. For $a = 2$, solve the matrix equation $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$.
2. For every value of a, determine the determinant of $\mathbf{A}$ and the rank of $\mathbf{A}$.
3. Find the value of a for which the matrix equation $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has the solutions

$$\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ -1 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

PROBLEM 2

A linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ is with respect to the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^2$ given by the matrix map

$$\mathbf{F} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}.$$

1. Let $\mathbf{v} = (1, -3, -3) \in \mathbb{R}^3$ be given. Determine $f(\mathbf{v})$.
2. Determine a basis for the kernel of f and determine the dimension of the image $f(\mathbb{R}^3)$.

Let $P_1(\mathbb{R})$ denote the set of real polynomials with degree at most 1, and let $P_2(\mathbb{R})$ denote the set of real polynomials with degree at most 2. A linear map $g\colon P_2(\mathbb{R}) \to P_1(\mathbb{R})$ is defined by

$$g(1) = 1 - 2 \cdot x, \quad g(x) = -1 + 2 \cdot x, \quad \text{and} \quad g(x^2) = 2 - 4 \cdot x.$$

3. Find two different polynomials $P(x), Q(x) \in P_2(\mathbb{R})$ that satisfy

$$g(P(x)) = g(Q(x)) = -2 + 4 \cdot x.$$

Exam continues on next page $\longrightarrow$

PROBLEM 3

A matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ has the eigenvalues -2 and 5 with corresponding eigenspaces

$$E_{-2} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad E_5 = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}.$$

1. Specify a regular matrix $\mathbf{V} \in \mathbb{R}^{2\times 2}$ satisfying $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$.

Let $\mathbf{C} \in \mathbb{R}^{2\times 2}$ be given as $\mathbf{C} = \mathbf{A} + \mathbf{B}$, where $\mathbf{B} = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$, $b \in \mathbb{R}$.

2. Show that $\mathbf{C}$ has the eigenvalues $-2 + b$ and $5 + b$ with corresponding eigenspaces

$$E_{-2+b} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad E_{5+b} = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}.$$

PROBLEM 4

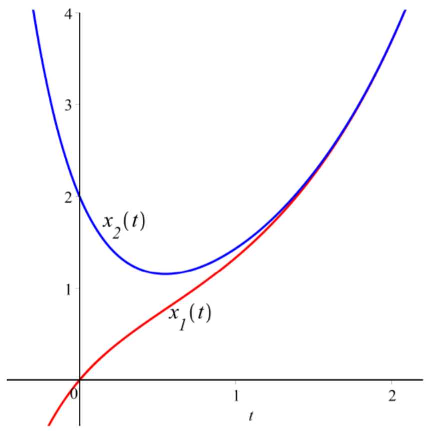
A system of differential equations is given by

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= 3 \cdot x_1(t) - 2 \cdot x_2(t) \end{aligned}$$

where $t \in \mathbb{R}$. Let $\mathbf{A}$ denote the corresponding coefficient matrix.

1. Determine all eigenvalues and eigenvectors for $\mathbf{A}$, and use this to write down the set of solutions to the system of differential equations.

2. A particular solution $(x_1(t), x_2(t))$ to the system of differential equations is illustrated below where the graphs of $x_1(t)$ and $x_2(t)$ are both plotted in the same plot. Find the formulae for $x_1(t)$ and $x_2(t)$.



3. Explain why the two graphs shown above do not intersect for any $t \in \mathbb{R}$.

End of exam.