Math 1 Two-hour Exam December 7, 2015

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Problem 1

```
> restart;with(LinearAlgebra):
```

Given

```
> A:=<<1,0,0,0>|<0,a-1,a,0>|<0,a,a-1,0>|<1,0,0,1>>;
```

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a-1 & a & 0 \\ 0 & a & a-1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

```
> b:=<1,1,-1,-1>;
```

$$b := \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

where $a \in \mathbb{R}$.

Question 1

a=2:$\mathbf{A}\mathbf{x}=\mathbf{b}$.

The augmented matrix is

```
> T:=<subs(a=2,A)|b>;
```

$$T := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

that has the reduced row echelon form

```
> trap('T'):= ReducedRowEchelonForm(T);
```

$$trap(T) := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

From this we read

$$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Question 2

By expansion along the first column and then along the third column we get

det(**A**)=det$\left(\begin{bmatrix} a-1 & a \\ a & a-1 \end{bmatrix}\right)= (a-1)^2 - a^2 = 1-2\,a$ for all $a \in \mathbb{R}$.

With Maple we get

```
> Determinant(A);
```

$$-2\,a+1$$

Since det(**A**)=0 <=> a=$\frac{1}{2}$, we get

$\rho(\mathbf{A})=4$ for a $\neq \frac{1}{2}$.

For $a=\frac{1}{2}$, $\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, that has the reduced row echelon form

```
> trap('A'):=ReducedRowEchelonForm(subs(a=1/2,A));
```

$$trap(A) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we read that $\rho(\mathrm{A}) = 3$ for $a=\frac{1}{2}$.

Question 3

The matrix equation **Ax**=**b** has the solution set

x=$\begin{bmatrix} 2 \\ -2 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2+t \\ t \\ -1 \end{bmatrix}$, $t \in \mathbb{R}$,

i.e. the number of free parameters =

$n - \rho(\mathbf{A}) = 4 - \rho(\mathbf{A}) = 1 \Leftrightarrow \rho(\mathbf{A}) = 3 \Leftrightarrow a = \frac{1}{2}$ according to Question 2.

For a = $\frac{1}{2}$ we get

```
> x=LinearSolve(subs(a=1/2,A),b,free=t);
```

$$x = \begin{bmatrix} 1 \\ -2 + t_3 \\ t_3 \\ -1 \end{bmatrix}$$

that is exactly the wanted solution set.

Thus the matrix equaiton has the solution x = $\begin{bmatrix} 2 \\ -2 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, t \in \mathbb{R} \Leftrightarrow a = \frac{1}{2}$.

Problem 2

$f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is linear and **e**F**e** = $\left[ef(\mathbf{e}_1)\ ef(\mathbf{e}_2)\ ef(\mathbf{e}_3) \right] = \mathbf{F} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$.

Question 1

$\mathbf{v} = (1,-3,-3) \in \mathbb{R}^3$.

Since $ef(\mathbf{v}) = e\mathbf{F}e\ e\mathbf{v} = \mathbf{F}\ e\mathbf{v} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ then $f(\mathbf{v}) = (-2, 4)$.

Question 2

$\mathbf{x} \in \ker f \Rightarrow f(\mathbf{x}) = \mathbf{0} \Leftrightarrow ef(\mathbf{x}) = 0 \Leftrightarrow$ eFe ex $= \mathbf{0} \Leftrightarrow$

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\mathbf{F} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \rightarrow \text{trap}(\mathbf{F}) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

I.e. $\mathbf{x} \in \ker f \Leftrightarrow x_1 - x_2 + 2x_3 = 0 \Leftrightarrow x = t_1(1, 1, 0) + t_2(-2, 0, 1),\ t_1, t_2 \in \mathbb{R}$.

((1,1,0),(-2,0,1)) is therefore a basis for ker$f$ and dim$f(\mathbb{R}^3) = \rho(\mathbf{F}) = 1$.

$P_2(\mathbb{R})$ with the monomial basis m = $(1, x, x^2)$ and $P_1(\mathbb{R})$ with the monomial basis $(1, x)$.

$g : P_2(\mathbb{R}) \rightarrow P_1(\mathbb{R})$ is linear and given by $g(1) = 1 - 2x,\ g(x) = -1 + 2x$ and $g(x^2) = 2 - 4x$.

Question 3

m**G**m = [mg(1) mg(x) mg(

$$x^2\Big)\Big]\Big] = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} = \mathbf{F}.$$ We then have

$g(P(\text{x})) = -2 + 4\text{x} \Rightarrow \text{m}g(P(\text{x})) = \text{m}\mathbf{G}\text{m m}P(\text{x}) = \mathbf{F}\ \text{m}P(\text{x}) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

If we put $P(x) = 1 - 3x - 3x^2$ then it follows from Question 1 that $g(P(x)) = -2 + 4x$.
If we put $Q(x) = -x^2$ then it immediately follows that $g(-x^2) = -g(x^2) = 2 - 4x$.

Problem 3

$\mathbf{A} \in \mathbb{R}^{2\,x2}$ has the eigenvalues -2 and 5 with $E_{-2} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ and $E_5 = \text{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$.

Question 1

$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \in E_5$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\in E_{-2}$ are linearly independent because the corresponding eigenvalues are different.

If we put $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ then $\mathbf{V}$ is regular and $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$.

$\mathbf{C} = \mathbf{A} + \mathbf{B}$ where $\mathbf{B} = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$, $b \in \mathbb{R}$.

Question 2

$$\mathbf{C}\mathbf{v}_1 = (\mathbf{A} + \mathbf{B})\mathbf{v}_1 = \mathbf{A}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 5\begin{bmatrix} 2 \\ -1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \end{bmatrix} = (5 + b)\mathbf{v}_1.$$

I.e. $\mathbf{v}_1$ is a proper eigenvector for $\mathbf{C}$ with the corresponding eigenvalue 5 + b .

$$\mathbf{C}\mathbf{v}_2 = (\mathbf{A} + \mathbf{B})\mathbf{v}_2 = \mathbf{A}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-2 + b)\mathbf{v}_1.$$

I.e. $\mathbf{v}_2$ is a proper eigenvector for $\mathbf{C}$ with the corresponding eigenvalue 5 + b .
Therefore all eigenvalues for the 2x2 matrix $\mathbf{C}$ are 5 + b and -2 + b with corresponding

eigenvectorspace $E_5 = \text{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$ and $E_{-2} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

Problem 4

```
> restart;with(LinearAlgebra):
```

$x_1'(t) = x_2(t)$

$x_2'(t) = 3\,x_1(t) - 2\,x_2(t)$

$$\Leftrightarrow \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Question 1

```
> A :=<<0,3>|<1,-2>>;
```

$$A := \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$$

```
> Eigenvectors(A,output=list);
```

$$\left[\left[-3, 1, \left\{\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right\}\right], \left[1, 1, \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}\right]\right]$$

From this we read that all eigenvalues for the system matrix A are 1 and -3 with the corresponding eigenvectorspace $E_1 = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ and E_{-3} span$\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$.

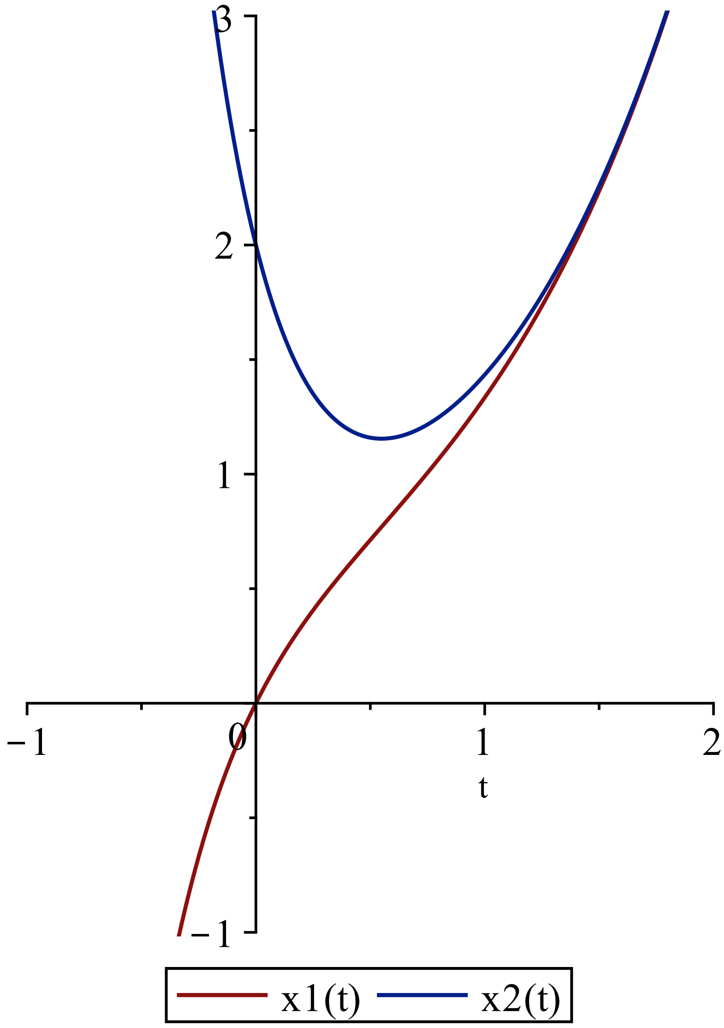
$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in E_1$ and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ are linearly independent,

because the corresponding eigenvalues are different.

Then all solutions to the system of diffential equations are

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \mathrm{e}^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-3t} \begin{bmatrix} -1 \\ 3 \end{bmatrix}, t \in \mathbb{R}, c_1, c_2 \in \mathbb{R}.$$

Question 2



From the figure se see that $(x_1(0), x_2(0)) = (0, 2)$.

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow$$

$c_1 - c_2 = 0$ and $c_1 + 3\,c_2 = 2 \Leftrightarrow (c_1, c_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

The wanted solution is then

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2}\mathrm{e}^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\mathrm{e}^{-3\,t} \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$ $t \in \mathbb{R}$. Or written separately

$$x_1(t) = \frac{1}{2}\mathrm{e}^{t} - \frac{1}{2}\mathrm{e}^{-3\,t}$$

$$x_2(t) = \frac{1}{2}\mathrm{e}^{t} + \frac{3}{2}\mathrm{e}^{-3\,t}$$

where $t \in \mathbb{R}$.

Question 3

Assume that there exists a $t \in \mathbb{R}$ such that

$x_1(t) = x_2(t) \Leftrightarrow \frac{1}{2}\mathrm{e}^{t} - \frac{1}{2}\mathrm{e}^{-3\,t} = \frac{1}{2}\mathrm{e}^{t} + \frac{3}{2}\mathrm{e}^{-3\,t} \Leftrightarrow -\frac{1}{2} = \frac{3}{2}$, which is a contradiction.

So $x_1(t) \neq x_2(t)$ for all $t \in \mathbb{R}$.
The two graphs in the figure do no intersect for any $t \in \mathbb{R}$.