Mat 1. 2-hour exam 2014-12-8

PROBLEM 1.

$$\begin{aligned} x_1 \qquad\quad + 3x_3 &= -3 \\ x_1 - 2x_2 + x_3 &= 1 \\ -2x_1 + 4x_2 - 2x_3 &= -2 \\ x_1 + x_2 + 4x_3 &= -5 \end{aligned}$$

1.

$$\underline{T} = [\underline{A}\,|\,\underline{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & -3 \\ 1 & -2 & 1 & 1 \\ -2 & 4 & -2 & -2 \\ 1 & 1 & 4 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & -2 & -2 & 4 \\ 0 & 4 & 4 & -8 \\ 0 & 1 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \mathrm{Trap}(T).$$

The completely reduced linear system of equations:

$$\begin{cases} x_1 \quad + 3x_3 = -3 \\ \quad x_2 + x_3 = -2. \end{cases}$$ If we put $x_3 = t$ we get:

$$(x_1, x_2, x_3) = (-3, -2, 0) + t(-3, -1, 1), \quad t \in \mathbb{R}.$$

$f: \mathbb{R}^3 \to \mathbb{R}^4$ is linear and

$${}_e\underline{F}_e = \left[{}_e\underline{f}(\underline{e}_1)\ {}_e\underline{f}(\underline{e}_2)\ {}_e\underline{f}(\underline{e}_3)\right] = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & 1 & 4 \end{bmatrix} = \underline{A}.$$

2. $\underline{x} \in \ker f \iff f(\underline{x}) = \underline{0} \iff {}_e\underline{F}_e\,{}_e\underline{x} = \underline{0} \iff \underline{A}\underline{x} = \underline{0} \iff$

$\underline{x} = t(-3, -1, 1),\ t \in \mathbb{R}$. I.e. $\ker f = \mathrm{span}\{(-3, -1, 1)\}$.

Since $\dim f(\mathbb{R}^3) = \rho({}_e\underline{F}_e) = \rho(\underline{A}) = 2$ and since the two image vectors $f(\underline{e}_1) = (1, 1, -2, 1)$ and $f(\underline{e}_2) = (0, -2, 4, 1)$ are linearly independent then

$(f(\underline{e}_1), f(\underline{e}_2)) = ((1, 1, -2, 1), (0, -2, 4, 1))$ is a basis for $f(\mathbb{R}^3)$.

PROBLEM 2

$$x''(t) - 8x'(t) + 16x(t) = q(t), \quad t \in \mathbb{R}.$$

1. $q(t) = 0$: $x''(t) - 8x'(t) + 16x(t) = 0, \quad t \in \mathbb{R}$.

Characteristic equation: $\lambda^2 - 8\lambda + 16 = 0 \iff (\lambda - 4)^2 = 0 \iff$
$\lambda = 4$ (double).

PROBLEM 2 (cont.)

All real solutions:

$$\underline{x_{hom}(t) = c_1 e^{4t} + c_2 t \cdot e^{4t}, \quad t \in \mathbb{R}, \; c_1, c_2 \in \mathbb{R}.}$$

2. $q(t) = e^{2it}$: $\quad x''(t) - 8x'(t) + 16x(t) = e^{2it}, \quad t \in \mathbb{R}.$

$x(t) = c \cdot e^{2it}$ is a solution $\Leftrightarrow$

$x''(t) - 8x'(t) + 16x(t) = -4ce^{2it} - 16i \cdot ce^{2it} + 16ce^{2it} = e^{2it}$ for all $t \in \mathbb{R}$.

$\Leftrightarrow c(12 - 16i) = 1 \Leftrightarrow \underline{c} = \dfrac{1}{12-16i} = \dfrac{12+16i}{400} = \underline{\dfrac{3}{100} + \dfrac{4}{100} i}.$

3. $q(t) = 4\cos 2t$: $\quad x''(t) - 8x'(t) + 16x(t) = 4\cos 2t, \quad t \in \mathbb{R}.$

$e^{2it} = \cos 2t + i \sin 2t$. Since $4\cos 2t = 4\,\mathrm{Re}(e^{2it})$ we have

$x_0(t) = 4\,\mathrm{Re}\left(\left(\frac{3}{100} + \frac{4}{100} i\right)(\cos 2t + i \sin 2t)\right)$

$\quad = 4\left(\frac{3}{100}\cos 2t - \frac{4}{100}\sin 2t\right) = \frac{3}{25}\cos 2t - \frac{4}{25}\sin 2t$

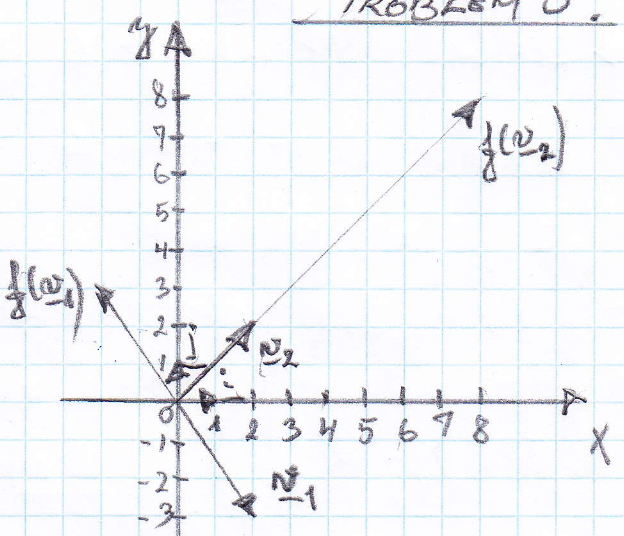
is a particular solution to the differential equation.

All real solutions is then according to the structural theorem

$\underline{x(t)} = x_0(t) + x_{hom}(t)$

$$= \underline{\frac{3}{25}\cos 2t - \frac{4}{25}\sin 2t + c_1 e^{4t} + c_2 t \cdot e^{4t}, \quad t \in \mathbb{R}, \; c_1, c_2 \in \mathbb{R}.}$$

PROBLEM 3.



$e = (\underline{i}, \underline{j})$ and $v = (\underline{v}_1, \underline{v}_2)$.

The vectorspace of geometrical vectors in the plane is denoted V_g^2.

$f: V_g^2 \to V_g^2$ is linear.

1. From the given Figure we read that

$${}_e\underline{M}_v = \left[{}_e\underline{v}_1 \; {}_e\underline{v}_2\right] = \begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}.$$

PROBLEM 3 (cont.)

2. Since it from the given figure is seen, that $f(\underline{v}_1) = -\underline{v}_1$ and $f(\underline{v}_2) = 4\underline{v}_2$, $\underline{v}_1$ is an eigenvector for f with corresponding eigenvalue -1 and $\underline{v}_2$ is an eigenvector for f with corresponding eigenvalue 4.

3. Since $f(\underline{v}_1) = -1\underline{v}_1 + 0\underline{v}_2$ and $f(\underline{v}_2) = 0\underline{v}_1 + 4\underline{v}_2$ we have

$$\underline{\underline{{}_vF_v}} = \left[{}_v f(\underline{v}_1) \; {}_v f(\underline{v}_2)\right] = \underline{\underline{\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}}}.$$

$$\underline{{}_e\underline{F}_v} = {}_e\underline{M}_v \, {}_v\underline{F}_v = \begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} = \underline{\begin{bmatrix} -2 & 8 \\ 3 & 8 \end{bmatrix}}.$$

4.

$${}_v\underline{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Leftrightarrow \underline{x} = -\underline{v}_1 + \underline{v}_2 = 5\underline{j}$$

$$\underline{{}_e f(\underline{x})} = {}_e\underline{F}_v \, {}_v\underline{x} = \begin{bmatrix} -2 & 8 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underline{\begin{bmatrix} 10 \\ 5 \end{bmatrix}}.$$

I.e. $f(\underline{x}) = 10\underline{i} + 5\underline{j}$.

5. $f(a\underline{i} + b\underline{j}) = 10\underline{i} + 5\underline{j}$.

Of 4. it follows that $f(5\underline{j}) = 10\underline{i} + 5\underline{j}$ and since $\underline{x} = 5\underline{j}$ is the only vector, that fulfills the equation $f(\underline{x}) = 10\underline{i} + 5\underline{j}$ ($\dim \ker f = 0$), then $\underline{a = 0 \text{ and } b = 5}$.

PROBLEM 4.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \underline{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad t \in \mathbb{R}.$$

All real solutions are

$$(*) \quad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}, \; c_1, c_2 \in \mathbb{R}.$$

PROBLEM 4 (cont.)

1. $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 13 \end{bmatrix} \Leftrightarrow \begin{cases} c_1 + 2c_2 = -1 \\ 2c_1 - c_2 = 13 \end{cases}$

From this we get $c_1 = 5$ and $c_2 = -3$.

The solution we are looking for is then

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 5e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3e^{5t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Leftrightarrow \underline{\begin{cases} x_1(t) = 5e^{3t} - 6e^{5t} \\ x_2(t) = 10e^{3t} + 3e^{5t} \end{cases}}, \ t \in \mathbb{R}$$

2. From (*) we read that

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a proper eigenvector for $\underline{A}$ with corresponding eigenvalue $\lambda_1 = 3$ and that $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a proper eigenvector for $\underline{A}$ with corresponding eigenvalue $\lambda_2 = 5$.

I.e.

$$\lambda_A = \underline{\begin{cases} 3 \text{ (single)}, & E_3 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}. \\ 5 \text{ (double)}, & E_5 = \operatorname{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}. \end{cases}}$$

3. If we put $\underline{V} = [\underline{v}_1 \ \underline{v}_2] = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and $\underline{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2)$

$= \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$, then $\underline{V}$ is regular and $\underline{A} = \underline{V}\,\underline{\Lambda}\,\underline{V}^{-1}$

$$= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{23}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{17}{5} \end{bmatrix}.$$

The system of differential equations is then:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} \frac{23}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{17}{5} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \Leftrightarrow \underline{\begin{cases} x_1'(t) = \frac{23}{5}x_1(t) - \frac{4}{5}x_2(t) \\ x_2'(t) = -\frac{4}{5}x_1(t) + \frac{17}{5}x_2(t) \end{cases}}, \ t \in \mathbb{R}.$$