DANMARKS TEKNISKE UNIVERSITET

Written exam December 6, 2014.

Course name: Matematik 1. **Course no.** 01005

Allowed helping aids: All helping aids allowed by DTU can be brought and used.

Weighting: The four problems will have equal weight.

All answers should be supported by valid arguments, and intermediate calculations should be included to an appropriate exetend. It is not allowed to communicate with others during the exam, neither directly nor electronically.

PROBLEM 1

A real linear system of equations is given by:

$$\begin{aligned} x_1+3x_3 &= -3 \\ x_1-2x_2+x_3 &= 1 \\ -2x_1+4x_2-2x_3 &= -2 \\ x_1+x_2+4x_3 &= -5 \end{aligned}$$

1. Determine the reduced row echelon form of the augmented matrix corresponding to the system of equations, and state the complete solution to the system of equations in standard parametric form.

A linear map $f:\mathbb{R}^3 \mapsto \mathbb{R}^4$ has with respect to the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^4$ the mapping matrix

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & 1 & 4 \end{bmatrix}.$$

2. Determine the kernel for f, and state a basis for the range $f(\mathbb{R}^3)$.

PROBLEM 2

A linear second-order differential equation with constant coefficients has the form

$$x''(t) - 8x'(t) + 16x(t) = q(t).$$

where t is a real variable and $q(t)$ is a continuous function.

1. Assume $q(t) = 0$ for $t \in \mathbb{R}$. Find the solutions to the characteristic equation of the differential equation and state by use of this the complete real solution to the differential equation.

2. Assume $q(t) = e^{2it}$ for $t \in \mathbb{R}$. Determine a number $c \in \mathbb{C}$ such that the function ce^{2it} is a solution to the differential equation.

3. Assume $q(t) = 4\cos(2t)$ for $t \in \mathbb{R}$. Determine by use of the results in question 1 and 2 the complete real solution to the differential equation.

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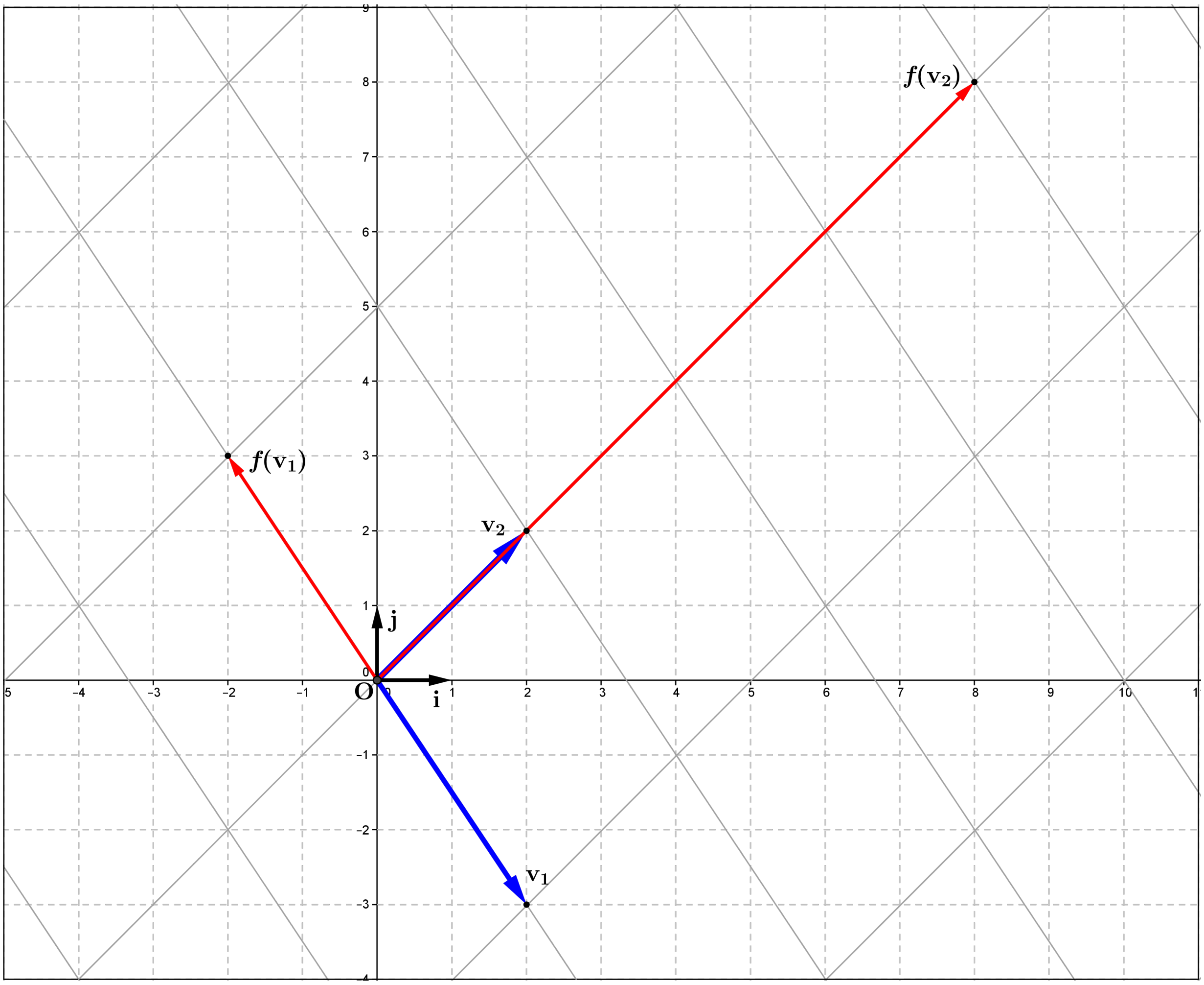
Problem 3

In the plane an ordinary orthogonal $(O, \mathbf{i}, \mathbf{j})$-coordinate system is given, in which all vectors are drawn from the origin. The corresponding standard basis $(\mathbf{i}, \mathbf{j})$ is denoted by an e. A new basis $v = (\mathbf{v}_1, \mathbf{v}_2)$ is drawn in the coordinate system, see the Figure below.



1. State the change of basis matrix ${}_e\mathbf{M}_v$ that changes from v-koordinater to e-coordinates.

By a linear map f of the set of vectors in the i plane into the set of vectors in the plane the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are mapped as shown in the Figure.

2. Explain why $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for f, and state their corresponding eigenvalues.

3. Determine the mapping matrix ${}_v\mathbf{F}_v$, corresponding to f and show that the mapping matrix ${}_e\mathbf{F}_v$ corresponding to f are given by

$$ {}_e\mathbf{F}_v = \begin{bmatrix} -2 & 8 \\ 3 & 8 \end{bmatrix}. $$

4. A vector $\mathbf{x}$ is given by its coordinate vector with respect to v like this: ${}_v\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Determine the coordinate vector for $f(\mathbf{x})$ with respect to the standard basis.

5. Detmine two numbers a and b that fulfills

$$ f(a\mathbf{i} + b\mathbf{j}) = 10\mathbf{i} + 5\mathbf{j}. $$

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PROBLEM 4

In a given system of two linear first-order differential equations with constant coefficients the two unknown functions are denoted $x_1(t)$ and $x_2(t)$. All the solutions to the system of differential equation are given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \mathrm{e}^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \mathrm{e}^{5t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ for } t \in \mathbb{R} \text{ og } c_1, c_2 \in \mathbb{R}.$$

1. Determine a solution to the system of differential equations that fulfills

$$x_1(0) = -1 \text{ and } x_2(0) = 13.$$

Let $\mathbf{A}$ denote the system matrix of the system of differential equations.

2. State the two eigenvalues for $\mathbf{A}$, and write for each eigenvalue the corresponding eigenspace.
3. State the system of differential equations.

`End of exam.`