Mat 1 2-hour exam 2013-12-7

PROBLEM 1

$$\underline{T} = [\underline{A}|\underline{b}] \to trap(\underline{T}) = \left[\begin{matrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{matrix}\left|\begin{matrix} -2 \\ 0 \\ 4 \end{matrix}\right.\right].$$

1. Number of equations = 3 and number of unknowns = 4.

$$\rho(\underline{A}) = 3 \quad \text{and} \quad \rho(\underline{T}) = 3$$

2. The completely reduced linear system of equations is

$$\begin{matrix} x_1 & & & +2x_4 & = -2 \\ & x_2 & & -x_4 & = 0 \\ & & x_3 & & = 4 \end{matrix} \iff \begin{matrix} x_1 = -2-2x_4 \\ x_2 = x_4 \\ x_3 = 4 \end{matrix}$$

$\underline{x}_1 = (1,0,4,1)$, $\underline{x}_2 = (1,-2,0,1)$ and $\underline{x}_3 = (0,-1,4,-1)$

By substitution we see that only $\underline{x}_3 = (0,-1,4,-1)$ is a solution to the system of equations.

3. All the solutions to the corresponding homogeneous linear system of equations are

$$(x_1, x_2, x_3, x_4) = t(-2,1,0,1), \quad t \in \mathbb{R}.$$

PROBLEM 2.

$\underline{v}_1 = (1,0,1)$, $\underline{v}_2 = (0,1,0)$ and $\underline{v}_3 = (1,0,-1)$.

1.
$$_e\underline{V} = \left[{}_e\underline{v}_1 \; {}_e\underline{v}_2 \; {}_e\underline{v}_3\right] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \quad \rho(_e\underline{V}) = 3.$$

Since $\rho(_e\underline{V}) = 3$ $\underline{v}_1, \underline{v}_2$ and $\underline{v}_3$ are three linearly independent vectors in $\mathbb{R}^3$ and therefore $(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ is a basis for $\mathbb{R}^3$.

2. From $\underline{A}\underline{v}_1 = \underline{v}_1$, $\underline{A}\underline{v}_2 = 2\underline{v}_2$ and $\underline{A}\underline{v}_3 = -\underline{v}_3$ we read that $\underline{v}_1$ is an eigenvector for $\underline{A}$ with corresponding eigenvalue 1, $\underline{v}_2$ is an eigenvector for $\underline{A}$ with corresponding eigenvalue 2 and $\underline{v}_3$ is an eigenvector for $\underline{A}$ with corresponding eigenvalue -1.

The numbers 1, 2 and -1 are thus three different eigenvalues for the 3×3 matrix $\underline{A}$. All eigenvalues for $\underline{A}$ are

PROBLEM 2 (cont.)

Therefor 1, 2 and with $gm(1) = am(1) = 1$, $gm(2) = am(2) = 1$ and $gm(-1) = am(-1) = 1$.
$(1 \leq gm(\lambda) \leq am(\lambda))$

$f: \mathbb{R}^3 \to \mathbb{R}^3$ linear and ${}_e\underline{F}_e = \underline{A}$. I.e. ${}_e f(x) = {}_e\underline{F}_{e\,e}\underline{x} = \underline{A}\,{}_e\underline{x}$.

3. Since $\underline{v} \in \mathbb{R}^3$ is an eigenvector for $f \Leftrightarrow {}_e\underline{v} = \underline{v}$ is an eigenvector for ${}_e\underline{F}_e = \underline{A}$, then it follows from 1. and 2., that $(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors for f. Hence

$$\underline{{}_vF_v} = [\,{}_vf(\underline{v}_1)\ {}_vf(\underline{v}_2)\ {}_vf(\underline{v}_3)] = \underline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}} = \underline{\Delta}$$

(We have $f(\underline{v}_1) = \underline{v}_1 = 1\underline{v}_1 + 0\cdot\underline{v}_2 + 0\cdot\underline{v}_3$, $f(\underline{v}_2) = 2\underline{v}_2 = 0\underline{v}_1 + 2\underline{v}_2 + 0\underline{v}_3$ and $f(\underline{v}_3) = -\underline{v}_3 = 0\underline{v}_1 + 0\underline{v}_2 - 1\underline{v}_3$)

4. $${}_e\underline{M}_v = {}_e\underline{V} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix},\quad {}_v\underline{M}_e = {}_e\underline{M}_v^{-1} = {}_e\underline{V}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

$$\underline{\underline{A}} = {}_e\underline{F}_e = {}_e\underline{M}_v\,{}_v\underline{F}_v\,{}_v\underline{M}_e = {}_e\underline{V}\cdot\underline{\Delta}\cdot\underline{V}^{-1} = \underline{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}}.$$

PROBLEM 3

1. $$\frac{dx(t)}{dt} + t\,x(t) = 0, \quad t \in \mathbb{R}.$$

$$P(t) = \int t\,dt = \tfrac{1}{2}t^2$$

$$\underline{x_{hom}(t) = c\,e^{-P(t)} = c\,e^{-\frac{1}{2}t^2},\quad t \in \mathbb{R},\ c \in \mathbb{R}.}$$

$f: C^1(\mathbb{R}) \to C^0(\mathbb{R})$ is linear and given that.

$$f(x(t)) = \frac{dx}{dt} + t\cdot x(t).$$

2. $x(t) \in \ker(f) \Leftrightarrow f(x(t)) = \dfrac{d\,x(t)}{dt} + t\,x(t) = 0$ for all $t \in \mathbb{R}$

$\Leftrightarrow x(t) = c\,e^{-\frac{1}{2}t^2}$, $c \in \mathbb{R}$. That is

$$\ker(f) = \operatorname{span}\{e^{-\frac{1}{2}t^2}\} = \{x(t) \in C^1(\mathbb{R}) \mid x(t) = c\,e^{-\frac{1}{2}t^2},\ x \in \mathbb{R}\}$$

PROBLEM 3 (CONT.)

3. $g(t) = f(t-2) = 1 + t(t-2) = t^2 - 2t + 1$.

$f(x(t)) = g(t) \iff \frac{dx(t)}{dt} + t\,x(t) = t^2 - 2t + 1, \quad t \in \mathbb{R}$.

Since $f(t-2) = g(t)$ ($t-2$ has generated the right-hand-side), then $x_0(t) = t-2$ is a particular solution to the equation. From 1. and the structural theorem we get all solutions to the equation:

$$\underline{x(t) = x_0(t) + x_{hom}(t) = t - 2 + Ce^{-\frac{1}{2}t^2}, \quad t \in \mathbb{R}, \quad C \in \mathbb{R}}.$$

PROBLEM 4.

The characteristic equation $\lambda^2 + a\lambda + 25 = 0$, $a \in \mathbb{R}$.

1. $\frac{d^2x(t)}{dt^2} + a\frac{dx(t)}{dt} + 25x(t) = 0$ for all $t \in \mathbb{R}$.

2. $\underline{a = 0}$: $\iff \lambda = \pm 5i$

All solutions to the differential equation is then

$$\underline{x(t) = c_1 \cos(5t) + c_2 \sin(5t), \quad t \in \mathbb{R}, \quad c_1, c_2 \in \mathbb{R}}$$

2. If $x(t) = e^{-4t}\cos 3t$ is a solution to the differential equation, then $-4 \pm 3i$ must be the roots of the characteristic polynomial. We then have

$$\lambda^2 + a\lambda + 25 = ((\lambda+4) - 3i)((\lambda+4) + 3i) = (\lambda+4)^2 + 9$$
$$= \lambda^2 + 8\lambda + 25. \quad \text{I.e.} \quad \underline{a = 8.}$$

3. $\underline{a = 8}$: $\frac{d^2x(t)}{dt^2} + 8\frac{dx(t)}{dt} + 25x(t) = 0, \quad t \in \mathbb{R}$.

$\lambda^2 + 8\lambda + 25 = 0 \iff \lambda = -4 \pm 3i$.

All solutions to the differential equation is then

$$x(t) = c_1 e^{-4t}\cos 3t + c_2 e^{-4t}\sin 3t, \quad t \in \mathbb{R}, \quad c_1, c_2 \in \mathbb{R}.$$

$$x'(t) = -3c_1 e^{-4t}\sin 3t - 4c_1 e^{-4t}\cos 3t + 3c_2 e^{-4t}\cos 3t - 4c_2 e^{-4t}\sin 3t,$$

$x(t)$ is a solution through $(0,3)$ with horizontal tangent $\iff$

$$\begin{cases} x(0) = c_1 = 3. \\ x'(0) = -4c_1 + 3c_2 = 0 \end{cases} \iff (c_1, c_2) = (3, 4).$$

PROBLEM 4 (CONT.)

The wanted particular solution is then

$\underline{x(t) = 3e^{-4t}\cos 3t + 4e^{-4t}\sin 3t}, \quad \underline{t \in \mathbb{R}}.$