THE TECHNICAL UNIVERSITY OF DENMARK
Written 2-hour test in the Autumn syllabus, December 6 2022.
Course title: Advanced Engineering Mathematics 1. **Course no.:** 01006
Permitted aids: You may bring and use all aids permitted by DTU.
Weight: The four problems weigh equally.
All answers must be well-reasoned, and relevant calculations must be shown to an appropriate extent.

PROBLEM 1

Let a, b and c be arbitrary real numbers. We consider the matrix

$$\mathbf{A} = \begin{bmatrix} a & 1 & 2 & 1 \\ 1 & a & 1 & 2 \\ 2 & 1 & a & 1 \end{bmatrix}$$

and the vectors $\mathbf{v}_1 = (7, 1, -3)$ and $\mathbf{v}_2 = (4, b, c)$.

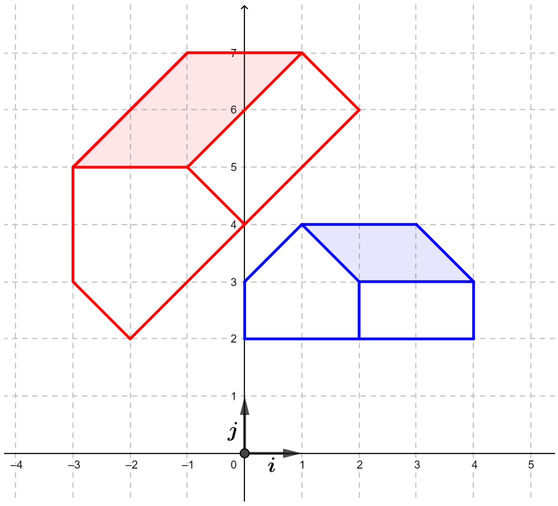
1. Let $a = 0$. Determine on standard parametric form the general solution to the matrix equation
$$\mathbf{A} \cdot \mathbf{x} = \mathbf{v}_1 .$$
2. Let $a = 2$. Determine the values of b and c for which the matrix equation
$$\mathbf{A} \cdot \mathbf{x} = \mathbf{v}_2$$
has solutions.

PROBLEM 2

Let $G2$ denote the set of vectors in a 2D plane that are positioned from the origin in a standard $(O, \mathbf{i}, \mathbf{j})$ coordinate system in which the standard basis $(\mathbf{i}, \mathbf{j})$ is denoted e.

A linear map $f : G2 \to G2$ is below illustrated as follows: A blue point P with position vector $\overrightarrow{OP}$ is mapped to a red point which is the end point of $f(\overrightarrow{OP})$. The red figure is in this way the image by f of the blue figure.

The problem sheet continues ⟶



We consider the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ given by ${}_e\mathbf{v}_1 = (3,4)$ and ${}_e\mathbf{v}_2 = (0,2)$.

1. Justify that the vector set $v = (\mathbf{v}_1, \mathbf{v}_2)$ is a new basis for $G2$. Determine the change-of-basis matrix ${}_e\mathbf{M}_v$.
2. State the image vectors $f(\mathbf{v}_1)$ and $f(\mathbf{v}_2)$, and determine the mapping matrices ${}_e\mathbf{F}_v$ and ${}_e\mathbf{F}_e$ of f.
3. A vector $\mathbf{w}$ is given by ${}_e\mathbf{w} = (1,4)$. Compute the angle between $\mathbf{w}$ and its image $f(\mathbf{w})$. Justify that the angle between an arbitrary other proper vector and its image vector is identical to the angle between $\mathbf{w}$ and $f(\mathbf{w})$.

PROBLEM 3

The characteristic polynomial of a real 3×3 matrix $\mathbf{A}$ is given on fully factorized form:

$$P(\lambda) = (\lambda + 2) \cdot \left(\lambda - 1 + \frac{i}{2}\right) \cdot \left(\lambda - 1 - \frac{i}{2}\right), \lambda \in \mathbb{C}.$$

1. State the eigenvalues of $\mathbf{A}$.
2. We are furthermore informed that the vector $\mathbf{u}_1 = (1,0,0)$ belongs to the eigenspace E_{-2}, and that the vector $\mathbf{u}_2 = (0, i, -1)$ belongs to the eigenspace $E_{1-\frac{i}{2}}$. Based on this, state $\mathbf{A} \cdot \mathbf{u}_1$ and $\mathbf{A} \cdot \mathbf{u}_2$.
3. State a 3×3 matrix that matches the above information about $\mathbf{A}$.

PROBLEM 4

We are given the matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 4 & -2 \end{bmatrix}.$$

1. Compute the eigenvalues of $\mathbf{A}$, and provide for each of them a corresponding proper eigenvector.

The problem sheet continues ⟶

An inhomogeneous system of linear differential equations is given by

$$x_1'(t) + 3x_1(t) = -1 + 6t$$
$$x_2'(t) - 4x_1(t) + 2x_2(t) = -8t\,.$$

2. Determine the general solution to the homogeneous system of differential equations that corresponds to the above given inhomogeneous system.

3. A particular solution $(x_1(t), x_2(t))$ to the given inhomogeneous system of differential equations exists so that $x_1(t) = at + b$ and $x_2(t) = ct + d$. Substitute this guess on a particular solution into the system of differential equations and determine in this way the numbers a, b, c and d.

4. Use the results from questions 2 and 3 to determine the general solution to the given inhomogeneous system of differential equations.

End of the problem sheet.