01006 MATH 1, FALL 21, ESSAY QUESTION – SOLUTION

Note: See some alternative solutions on the next page.

1. From the figure, $f(\mathbf{v}_1)$ is proportional to $\mathbf{v}_1$, therefore $\mathbf{v}_1$ is an eigenvector. $f(\mathbf{v}_2)$ is not proportional to $\mathbf{v}_2$, so this is not an eigenvector.

2. We have $f(\mathbf{v}_1) = 3\mathbf{v}_1 + 0 \cdot \mathbf{v}_2$, so $a = 3$, $b = 0$. To find c and d we solve:

$$f(\mathbf{v}_2) = (1,7) = c(3,1) + d(-1,3),$$

for c and d. The solution is $c = 1$, $d = 2$. The mapping matrix in the basis v is:

$${}_vF_v = [{}_vf(\mathbf{v}_1), {}_vf(\mathbf{v}_2)] = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}.$$

Finally, in the basis v, the coordinates for $\mathbf{v}_1$ and $\mathbf{v}_2$ are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively, so

$${}_vf(\mathbf{v}_1 + \mathbf{v}_2) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

3. (a) It's a square of sidelength $|\mathbf{v}_1| = \sqrt{10}$, so has area $A = 10$.
   (b) A linear map stretches areas by the absolute value of the determinant so

$$A = |10\det(f)| = 10|\det {}_vF_v| = 10 \cdot 6 = 60.$$

4. The change of basis matrix and its inverse are:

$${}_eM_v = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}, \qquad {}_vM_e = {}_eM_v^{-1} = \frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}.$$

So the mapping matrix in the standard basis is:

$${}_eF_e = {}_eM_v \, {}_vF_v \, {}_vM_e = \frac{1}{5} \begin{bmatrix} 13 & 6 \\ 1 & 12 \end{bmatrix}.$$

5. Working in the e-basis, we find the eigenvalues and vectors for ${}_eF_e$:

$$\lambda_1 = 3, \quad \mathbf{w}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 2, \quad \mathbf{w}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

This means, the mapping matrix is diagonal, $\operatorname{diag}(3,2)$ with respect to the basis $w = (\mathbf{w}_1, \mathbf{w}_2)$.

6. The map is diagonalizable if and only if all eigenvalues have the same geometric multiplicity as algebraic multiplicity. Working in the v basis, we have: ${}_vF_v = \begin{bmatrix} k & 1 \\ 0 & 7 \end{bmatrix}$. If we choose $k = 7$, we find there is one eigenvalue:

$$\lambda = 7, \quad am(\lambda) = 2, \quad gm(\lambda) = 1.$$

Therefore, f is not diagonalizable for $k = 7$. (For other values of k it is diagonalizable).

Some alternative solutions

1 $f(\mathbf{v}_1) = (9,3) = 3(3,1) = 3\mathbf{v}_1$, therefore $\mathbf{v}_1$ is an eigenvector.
On the other hand, if we try to solve $f(\mathbf{v}_2) = \lambda \mathbf{v}_2$ we would have:

$$f(\mathbf{v}_2) = (1,7) = \lambda(-1,3).$$

This gives the incompatible pair of equations:

$$\begin{cases} 1 = -\lambda \\ 7 = 3\lambda \end{cases}$$

so there is no solution and therefore $\mathbf{v}_2$ is not an eigenvector.

3 (a) It's the area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, so we can use the formula

$$|\det[\mathbf{v}_1, \mathbf{v}_2]| = \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = 10.$$

(b) We can use $f(A) = f(\mathbf{v}_1) = (9,3)$ and $f(C) = f(\mathbf{v}_2) = (1,7)$, and the area of the parallelogram spanned by these is: $\left|\det\left(\begin{bmatrix} 9 & 1 \\ 3 & 7 \end{bmatrix}\right)\right| = 60.$

4 We have

$$_eF_v = [\,_ef(\mathbf{v}_1),\, _ef(\mathbf{v}_2)] = \begin{bmatrix} 9 & 1 \\ 3 & 7 \end{bmatrix},$$

and the mapping matrix in the standard basis is:

$$_eF_e = {_eF_v}\, {_vM_e} = \frac{1}{5}\begin{bmatrix} 13 & 6 \\ 1 & 12 \end{bmatrix}.$$

5 Working in the v basis, we find the eigenvalues and eigenvectors of $_vF_v$ to be:

$$\lambda_1 = 3, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 2, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Here $\mathbf{u}_1$ and $\mathbf{u}_2$ are coordinates for these vectors in the $\mathbf{v}$ basis (since we used $_vF_v$), so the meaning is that the mapping matrix is diagonal, $\text{diag}(3,2)$ in the basis:

$$\mathbf{u}_1 = \mathbf{v}_1, \qquad \mathbf{u}_2 = -\mathbf{v}_1 + \mathbf{v}_2.$$

Note: comparing with the answer we got in the first version,

$$_e\mathbf{u}_1 = {_e\mathbf{v}_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \mathbf{w}_1, \qquad _e\mathbf{u}_2 = {_e(-\mathbf{v}_1 + \mathbf{v}_2)} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 2\mathbf{w}_1,$$

so the basis is the same up to a scaling of $\mathbf{w}_1$.