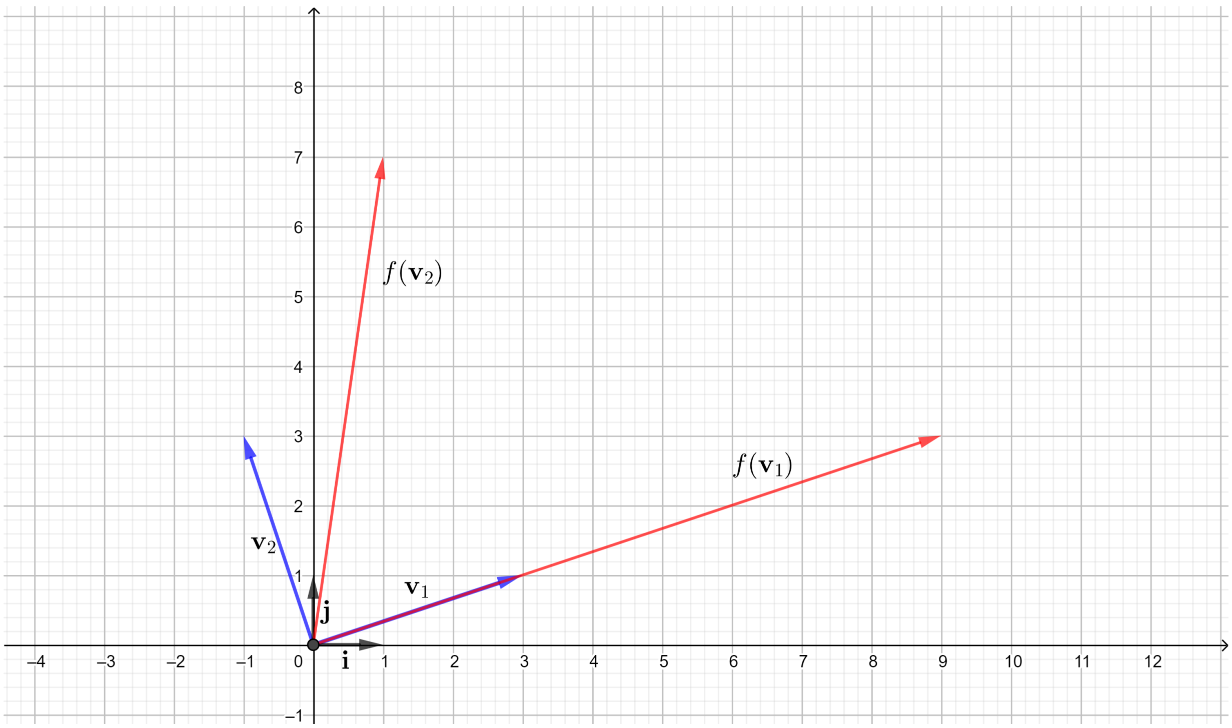
TECHNICAL UNIVERSITY OF DENMARK
Written exam on the Fall syllabus, 5 December 2021.
Course Name: Advanced Engineering Mathematics 1.
Course Number. 01006
Aids Allowed: All aids allowed by DTU may be used. The rules in general are described in "Syllabus & Rules", under "Agendas" on the course homepage.

Essay Question

A standard right-angle $(O,\mathbf{i},\mathbf{j})$ coordinate system is given in the plane. We consider the vectors space $G2$ of geometric vectors in the plane, with initial point at the origin O. Two vectors, $\mathbf{v}_1 = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{v}_2 = -\mathbf{i}+3\mathbf{j}$ form a basis $v = (\mathbf{v}_1,\mathbf{v}_2)$ for $G2$. We consider a linear map $f : G2 \to G2$, as shown in the figure. (NB: the coordinates of all vectors shown are integers).



(1) Show that $\mathbf{v}_1$ is an eigenvector for f, but $\mathbf{v}_2$ is not.

(2) The images $f(\mathbf{v}_1)$ and $f(\mathbf{v}_2)$ can be written as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$f(\mathbf{v}_1) = a\mathbf{v}_1 + b\mathbf{v}_2 \quad \text{and} \quad f(\mathbf{v}_2) = c\mathbf{v}_1 + d\mathbf{v}_2\,.$$

Find the numbers a, b, c and d and write down the mapping matrix ${}_v\mathbf{F}_v$ for f with respect to the basis v. Determine the coordinate vector ${}_vf(\mathbf{v}1+\mathbf{v}2)$.

(3) Let A, B and C denote the endpoints of $\mathbf{v}_1$, $\mathbf{v}_1+\mathbf{v}_2$ and $\mathbf{v}_2$ respectively.
a) Determine the area of the parallelogram that has corners O, A, B and C.
b) Determine the area of the parallelogram that has corners O, $f(A)$, $f(B)$ and $f(C)$.

(4) Determine the mapping matrix ${}_e\mathbf{F}_e$ for f with respect to the basis $e = (\mathbf{i},\mathbf{j})$.

(5) Determine a new basis for $G2$ with respect to which the mapping matrix for f becomes a diagonal matrix.

(6) If we now modify f so that $f(\mathbf{v}_1) = k\mathbf{v}_1$, where $k \in \mathbb{R}$, and $f(\mathbf{v}_2) = \mathbf{v}_1 + 7\mathbf{v}_2$, are there values of k for which f is not diagonalizable?

End of Exam