eNote 29

Stokes' Theorem

In this eNote we will use Gauss' Divergence Theorem from eNote 28 in order to motivate, prove and illustrate Stokes' Theorem that expresses a precise relation between curl *and* circulation *of a given vector field: For every smooth parameterized surface $F_{\mathbf{r}}$ the flux of the curl of a smooth vector field $\mathbf{V}(x, y, z)$ through $F_{\mathbf{r}}$ is equal to the circulation of the vector field along the boundary curve ∂F. As here indicated we shall need both volume, surface and line integrals together with knowledge about vector fields and their restrictions to surfaces and curves. I.e. the basis material for this eNote is to be found in a series of eNotes: eNote 24 (about plane and line integrals), eNote 25 (about surface integrals), eNote 26 (about vector fields), eNote 27 (about tangential line integrals and circulations), and eNote 28 (about flux calculations and Gauss' Theorem).*

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29.1 Stokes' Theorem

Stokes' theorem is one of the most elegant and most utilized results from the analysis of vector fields in 3D space. Like Gauss' divergence theorem, the theorem has numerous applications - e.g. in fluid mechanics and in electromagnetism.

Here is a quote that tells a bit of the history behind the result:

Figure 29.1: George Gabriel Stokes. See Biography.

The history of Stokes' Theorem is clear but very complicated. It was first given by Stokes without proof - as was necessary since it was given as an examination question for the Smith's Prize Examination of that year [1854, at Cambridge]! Among the candidates for the prize was Maxwell, who later traced to Stokes the origin of the theorem, which by 1870 was frequently used. On this see George Gabriel Stokes, *Mathematical and Physical Papers*, vol. V (Cambridge, England, 1905), 320–321. See also the important historical footnote which indicates that Kelvin in a letter of 1850 was the first who actually stated the theorem, although others as Ampère had employed "the same kind of analysis ... in particular cases."

M. J. Crowe, [A History of Vector Analysis, 1967], p. 147.

As indicated Stokes' result is as important as is Gauss' divergence theorem. They are both members of a 'family' of results, where the main idea is to rewrite an integral over a region to another integral over the boundary of the region - that is, so to speak, to 'push the integration out on the boundary'. The oldest member of that family is the Fundamental Theorem of Calculus, cf. Theorem 23.13 in eNote 23 which we repeat here in the following formulation:

Figure 29.2: James Clerk Maxwell (Biography), William Thomson (Lord Kelvin) (Biography), and André-Marie Ampère (Biography).

|||| Theorem 29.1 The Fundamental Theorem of Calculus

Let $f(u)$ denote a continuous function on $\mathbb{R}$. Then the following function is differentiable

$$A(x) = \int_0^x f(u)\,\mathrm{d}u \qquad \text{with} \qquad A'(x) = f(x) \quad . \tag{29-1}$$

If $F(x)$ is another function that fulfills $F'(x) = f(x)$ then

$$\int_a^b f(u)\,\mathrm{d}u = F(b) - F(a) \quad . \tag{29-2}$$

This is Stokes' Theorem:

|||| Theorem 29.2 Stokes' Theorem

Let $F_{\mathbf{r}}$ denote a smooth parameterized surface with the boundary curve $\partial F_{\mathbf{r}}$ and a unit normal vector field $\mathbf{n}_F$ and let $\mathbf{V}$ be a smooth vector field in $\mathbb{R}^3$. Then it applies that

$$\int_F \mathbf{Curl}(\mathbf{V}) \cdot \mathbf{n}_F \,\mathrm{d}\mu = \int_{\partial F} \mathbf{V} \cdot \mathbf{e}_{\partial F}\,\mathrm{d}\mu \quad . \tag{29-3}$$

In the calculation of the right-hand side the orientation of the boundary curve (given by the direction of the unit tangential vector field $\mathbf{e}_{\partial F}$) must be chosen such that the cross product $\mathbf{e}_{\partial F} \times \mathbf{n}_F$ points away from the surface along the boundary.

Therefore Stokes' Theorem says – like (29-2) – that an inner integral (here on a surface segment) can be expressed as a boundary integral (along the boundary of the surface segment).
Stokes' Theorem can also be formulated as follows, cf. the definitions of flux and circulation in eNote 28 and eNote 27, respectively:

$$\mathrm{Flux}(\mathbf{Curl}(\mathbf{V}), F_{\mathbf{r}}) = \mathrm{Cirk}(\mathbf{V}, \partial F) \quad . \tag{29-4}$$

I.e.: The flux of the *curl* of the vector field $\mathbf{V}$ through the surface $F_{\mathbf{r}}$ is equal to the *circulation* of the vector field along the closed boundary curve of the surface segment ∂F.

Note that if $F_{\mathbf{r}}$ is the *whole* surface of a spatial region then $\partial F = \emptyset$ and therefore it follows from Stokes' theorem that

$$\mathrm{Flux}(\mathbf{Curl}(\mathbf{V}), F_{\mathbf{r}}) = 0 \tag{29-5}$$

in accordance with Corollary 28.22.

29.2 Motivation and Proof of Stokes' Theorem

We shall need the following result that is also mentioned in Section 26.5:

|||| Theorem 29.3

Let $\mathbf{V}(x, y, z)$ and $\mathbf{W}(x, y, z)$ denote two vector fields in $\mathbb{R}^3$. Then the following identity holds

$$\mathrm{Div}(\mathbf{V} \times \mathbf{W}) \;=\; \mathbf{Curl}(\mathbf{V}) \cdot \mathbf{W} - \mathbf{V} \cdot \mathbf{Curl}(\mathbf{W}) \quad . \tag{29-6}$$

Therefore in particular we have: If $\mathbf{W}$ is a gradient vector field of a function $\psi(x, y, z)$ in $\mathbb{R}^3$, i.e. $\mathbf{W} = \boldsymbol{\nabla}(\psi)$, then we get from $\mathbf{Curl}(\boldsymbol{\nabla}(\psi)) = \mathbf{0}$:

$$\mathrm{Div}(\mathbf{V} \times \boldsymbol{\nabla}(\psi)) = \mathbf{Curl}(\mathbf{V}) \cdot \boldsymbol{\nabla}(\psi) \quad . \tag{29-7}$$

By using Gauss' divergence theorem in connection with (29-7) we directly get an interesting result about integrals over spatial regions like this:

|||| Theorem 29.4 A Consequence of Gauss' Divergence Theorem

Let $\psi(x,y,z)$ denote a smooth function and $\mathbf{V}(x,y,z)$ a smooth vector field in $\mathbb{R}^3$. Let Ω be a spatial region with the boundary $\partial\Omega$ and outward-pointing unit normal vector field $\mathbf{n}_{\partial\Omega}$ on $\partial\Omega$.

Then we have directly from Gauss' Theorem 28.15:

$$\int_{\Omega} \mathrm{Div}(\mathbf{V} \times \boldsymbol{\nabla}(\psi))\, \mathrm{d}\mu = \int_{\partial\Omega} (\mathbf{V} \times \boldsymbol{\nabla}(\psi)) \cdot \mathbf{n}_{\partial\Omega}\, \mathrm{d}\mu \quad . \tag{29-8}$$

Using (29-7) we thus get

$$\int_{\Omega} \mathbf{Curl}(\mathbf{V}) \cdot \boldsymbol{\nabla}(\psi)\, d\mu = \int_{\partial\Omega} (\mathbf{n}_{\partial\Omega} \times \mathbf{V}) \cdot \boldsymbol{\nabla}(\psi)\, \mathrm{d}\mu \quad . \tag{29-9}$$

On the right-hand side in equation (29-9) we have used that the scalar triple product $[\mathbf{abc}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ of three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfies the following identities (of which we have used the first one):

$$[\mathbf{abc}] = [\mathbf{cab}] = [\mathbf{bca}] = -[\mathbf{bac}] = -[\mathbf{cba}] = -[\mathbf{acb}] \quad . \tag{29-10}$$

In particular we get the total curl of the vector field $\mathbf{V}$ in Ω like this:

|||| Corollary 29.5 Total Curl in a Spatial Region

Let Ω be a spatial region with the boundary $\partial\Omega$ and outward-directed unit normal vector field $\mathbf{n}_{\partial\Omega}$ on $\partial\Omega$. Then for every smooth vector field $\mathbf{V}(x,y,z)$ it applies that:

$$\int_{\Omega} \mathbf{Curl}(\mathbf{V})\, \mathrm{d}\mu = \int_{\partial\Omega} \mathbf{n}_{\partial\Omega} \times \mathbf{V}\, \mathrm{d}\mu \quad . \tag{29-11}$$

|||| Proof

This follows directly from (29-9) by choosing alternatingly $\psi(x,y,z) = x$, $\psi(x,y,z) = y$ and $\psi(x,y,z) = z$, such that $\boldsymbol{\nabla}(\psi)$ is $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, respectively.

■

The right-hand side of the equation (29-11) we will call the *total torsion* of the surface $\partial\Omega$ with the vector field $\mathbf{V}$ with respect to the normal vector field $\mathbf{n}_{\partial\Omega}$ and furthermore denote this vector by $\mathbf{Tors}(\mathbf{V}, \partial\Omega)$, like this:

|||| Definition 29.6 Torsion

Let $F_{\mathbf{r}}$ denote a smooth surface in (x,y,z) space with standard unit normal vector field $\mathbf{n}_F$, and let $\mathbf{V}(x,y,z)$ be a smooth vector field. Then we define the torsion of $F_{\mathbf{r}}$ with the vector field $\mathbf{V}(x,y,z)$ in the following way:

$$\mathbf{Tors}(\mathbf{V}, F_{\mathbf{r}}) = \int_{F_{\mathbf{r}}} \mathbf{n}_F \times \mathbf{V} \, \mathrm{d}\mu \quad . \tag{29-12}$$

Generally the total curl of a given vector field in a spatial region (also called the 'total vorticity' in fluid mechanics) is also – via 29.5 – equal to the torsion of the surface of that region by the vector field. This is very similar to the corresponding Gauss identity for the total divergence of a smooth vector field, Theorem 28.15: The total divergence of the vector field in a certain region is equal to the flux of the vector field out through the surface of that region.

In correspondence with Theorem 28.13 we also have the following interpretation and construction of the rotation vector at a given point:

|||| Theorem 29.7 Curl-Torsion Relation

The curl of a vector field expresses *the volume-relative local torsion of the surface* for the vector field, or to be exact: Let K_ρ denote a solid sphere with radius ρ and centre at the point (x_0, y_0, z_0). Then:

$$\lim_{\rho \to 0} \left(\frac{1}{\mathrm{Vol}(K_\rho)} \, \mathbf{Tors}(\mathbf{V}, \partial K_\rho) \right) = \mathbf{Curl}(\mathbf{V})(x_0, y_0, z_0) \quad . \tag{29-13}$$

Theorem 29.7 is proved in the same way as the analogous identity for the divergence. The vector field is expanded to the first order with the development point p and is inserted in the torsion expression, following which the following identities are used:

$$\begin{aligned}
&\int_{\partial K_\rho} x \, d\mu = \int_{\partial K_\rho} y \, d\mu = \int_{\partial K_\rho} z \, d\mu = 0 \quad , \\
&\int_{\partial K_\rho} x^2 \, d\mu = \int_{\partial K_\rho} y^2 \, d\mu = \int_{\partial K_\rho} z^2 \, d\mu = (4\,\pi/3)\,\rho^4 = \rho \, \mathrm{Vol}(K_\rho) \quad , \\
&\int_{\partial K_\rho} x\,y \, d\mu = \int_{\partial K_\rho} x\,z \, d\mu = \int_{\partial K_\rho} z\,y \, d\mu = 0 \quad .
\end{aligned} \tag{29-14}$$

The only partial derivatives of the coordinate functions of the vector field that are not eliminated through integration are precisely those appearing in the rotation vector $\mathbf{Curl}(\mathbf{V})$ when this is evaluated at p. Try this!

|||| Example 29.8 Local Torsion of the Spherical Surface Yielding the Curl

We look at a rotating vector field $\mathbf{V}(x,y,z) = (-y, x, 1)$. This field has the constant $\mathbf{Curl}(\mathbf{V}) = (0,0,2)$. We will show how $\mathbf{Tors}(\mathbf{V}, \partial K_\rho)$ gives precisely this curl at the point $(0,0,0)$. So let K_ρ denote the spherical surface that has radius ρ, centre at $(0,0,0)$ and surface ∂K_ρ with outward-directed normal vector field $\mathbf{n} = \frac{1}{\rho}(x,y,z)$. Then

$$\begin{aligned}
\mathbf{Tors}(\mathbf{V}, \partial K_\rho) &= \int_{\partial K_\rho} \mathbf{n}_{\partial \mathbf{K}} \times \mathbf{V} \, d\mu \\
&= \frac{1}{\rho} \int_{\partial K_\rho} (x,y,z) \times (-y, x, 1) \, d\mu \\
&= \frac{1}{\rho} \int_{\partial K_\rho} (\, y - x\,z\,,\; -x - y\,z\,,\; x^2 + y^2 \,) \, d\mu \\
&= \quad (0,\, 0,\, 2 \, \mathrm{Vol}(K_\rho)\,) \\
&= \quad \mathrm{Vol}(K_\rho) \, \mathbf{Rot}(\mathbf{V})(0,0,0) \quad ,
\end{aligned}$$

in compliance with Theorem 29.7.

In more complicated cases the total torsion of a surface with a given vector field can be computed with Maple. If the actual surface is the boundary surface of a given spatial region the torsion can also be computed as the integral of the corresponding rotational vector field over the spatial region. By this it is also possible to show the applicability of Equation (29-11).

29.2.1 The Surface, the Boundary and the Normal Vector Field

The surfaces that are considered in Stokes' Theorem are parameterized in the ordinary way:

$$F_{\mathbf{r}}: \qquad \mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3 \quad , \tag{29-15}$$

where

$$(u,v) \in D = [a,b] \times [c,d] \subset \mathbb{R}^2 \quad . \tag{29-16}$$

The boundary $\partial F_{\mathbf{r}}$ of $F_{\mathbf{r}}$ appears through the use of the vector function $\mathbf{r}$ on the four straight line segments constituting the boundary ∂D of the rectangular parametric region $D = [a,b] \times [c,d]$. We parameterize all of ∂D at once by using a parameter θ and a vector function $\mathbf{d}$:

$$\partial D: \qquad \mathbf{d}(\theta) = (u(\theta), v(\theta)) \in \partial D \subset \mathbb{R}^2 \ , \ \theta \in I \subset \mathbb{R} \quad ,$$

where $u(\theta)$ and $v(\theta)$ only are piecewise differentiable functions of θ. E.g. they can be chosen as linear functions of θ for each of the four line segments that constitute ∂D.

The boundary of $F_{\mathbf{r}}$ is then

$$\partial F_{\mathbf{r}}: \qquad \mathbf{b}(\theta) = \mathbf{r}(\mathbf{d}(\theta)) = \mathbf{r}(u(\theta), v(\theta)) \in \mathbb{R}^3 \ , \ \theta \in I \subset \mathbb{R} \quad .$$

The Jacobian functions of $\mathbf{r}$ and $\mathbf{b}$ are:

$$\text{Jacobian}_{\mathbf{r}}(u,v) = ||\mathbf{r}'_u \times \mathbf{r}'_v|| \quad , \quad \text{and} \tag{29-17}$$

$$\text{Jacobian}_{\mathbf{b}}(\theta) = ||\mathbf{b}'_\theta|| \quad , \tag{29-18}$$

respectively. The regularity of $\mathbf{r}$ gives

$$\text{Jacobian}_{\mathbf{r}}(u,v) > 0 \quad \text{for all} \quad (u,v) \in D \quad .$$

The unit normal vector field along $F_{\mathbf{r}}$ is correspondingly $\mathbf{n}_F = \mathbf{n}(u,v)$:

$$\mathbf{n}(u,v) = \frac{\mathbf{r}'_u \times \mathbf{r}'_v}{||\mathbf{r}'_u \times \mathbf{r}'_v||} \quad \text{for all} \quad (u,v) \in D \quad . \tag{29-19}$$

29.2.2 Tubular Shells and a Distance Function

We define the tubular shell with thickness t from $F_{\mathbf{r}}$ as the following region in $\mathbb{R}^3$:

$$\Omega_t : \quad \mathbf{R}(u,v,w) = \mathbf{r}(u,v) + w\,\mathbf{n}(u,v) \ , \ (u,v) \in D \ , \ w \in [0,t] \ . \tag{29-20}$$

In particular the *surface* $F_{\mathbf{r}}$ is exactly the base surface of the shell and is found by the use of the map $\mathbf{R}$ on D (where $w = 0$):

$$F_0 = F_{\mathbf{R}}(0) = F_{\mathbf{r}} \quad : \quad \mathbf{r}(u,v) = \mathbf{R}(u,v,0) \ , \ (u,v) \in D \quad . \tag{29-21}$$

Correspondingly we get for $w = t$, the top surface of the shell. This is parameterized like this:

$$F_t = F_{\mathbf{R}}(t) = F_{\widehat{\mathbf{r}}} \quad : \quad \widehat{\mathbf{r}}(u,v) = \mathbf{R}(u,v,t) \ , \ (u,v) \in D \quad . \tag{29-22}$$

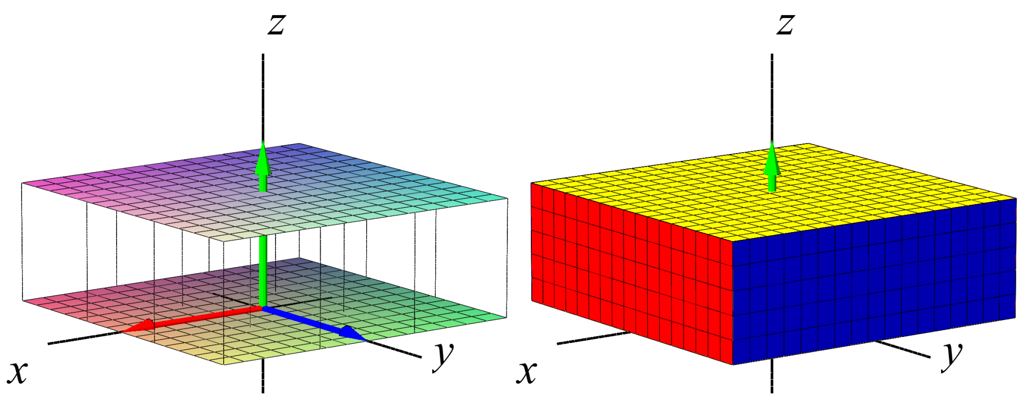

Figure 29.3: A tubular shell that is defined by flow along the *unit normal vector field* $\mathbf{n}_F$ from a square.

The Jacobian function of $\mathbf{R}$ is

$$\begin{aligned} \text{Jacobian}_{\mathbf{R}}(u,v,w) &= |\left(\mathbf{R}'_u \times \mathbf{R}'_v\right) \cdot \mathbf{R}'_w| \\ &= |\left((\mathbf{r}'_u + w\,\mathbf{n}'_u) \times (\mathbf{r}'_v + w\,\mathbf{n}'_v)\right) \cdot \mathbf{n}_F| \quad . \end{aligned} \tag{29-23}$$

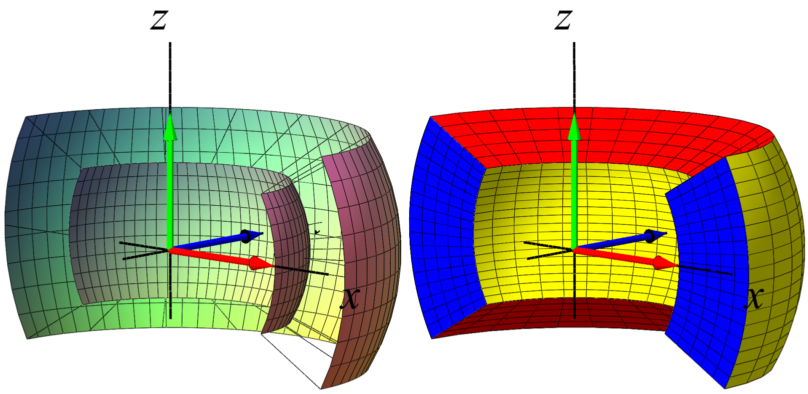

Figure 29.4: A tubular shell defined by a flow along the *unit normal vector field* $\mathbf{n}_F$ from a surface segment of a spherical surface.

Since $\mathbf{n}_F$ is a unit vector field that is parallel to $\mathbf{r}'_u \times \mathbf{r}'_v$ along $F_\mathbf{r}$ we obtain especially (for $w = 0$):

$$\begin{aligned} \text{Jacobian}_\mathbf{R}(u,v,0) &= |\,(\mathbf{r}'_u \times \mathbf{r}'_v) \cdot \mathbf{n}_F| \\ &= ||\mathbf{r}'_u \times \mathbf{r}'_v|| \\ &= \text{Jacobian}_\mathbf{r}(u,v) > 0 \quad . \end{aligned} \tag{29-24}$$

The map $\mathbf{R}$ is regular and bijective on $D \times [0,t]$ - if only t is sufficiently small. This follows partly from $\text{Jacobian}_\mathbf{R}(u,v,0) > 0$ and partly from the continuity of $\text{Jacobian}_\mathbf{R}(u,v,w)$.

The value of w considered as a function of $\Omega_t \subset \mathbb{R}^3$ is a differentiable function of the three variables (x,y,z). Even though this appears to be quite evident, intuitively, it is in fact a result that is best formulated and argued in the so-called *inverse function theorem*, a subject covered in the course Matematik 3:

|||| Theorem 29.9 Inverse Function Theorem

Let Q denote an open set in $\mathbb{R}^3$ and let $\mathbf{f} : Q \to \mathbb{R}^3$ denote a differentiable bijective map with $\text{Jacobian}_\mathbf{f}(\mathbf{x}) > 0$ for all $\mathbf{x} \in Q$.

Then the inverse map $\mathbf{f}^{\circ -1} : \mathbf{f}(Q) \to Q$ is also differentiable with $\text{Jacobian}_{\mathbf{f}^{\circ -1}}(\mathbf{y}) > 0$ for all $\mathbf{y} \in \mathbf{f}(Q)$.

Accordingly, in this case the intuition is precise and correct. When t is sufficiently small then w is a differentiable function of the three variables, (x, y, z); let us call that function $h(x, y, z)$, $(x, y, z) \in \Omega_t$. That function then has a proper gradient $\boldsymbol{\nabla}(h)(x, y, z)$ that is perpendicular to the level surfaces of h. In particular $\boldsymbol{\nabla}(h)$ is perpendicular to the top surface $F_t = F_{\mathbf{R}}(t)$ of the shell Ω_t, where $h = t$ and it is correspondingly perpendicular to the base surface $F_0 = F_{\mathbf{R}}(0) = F_{\mathbf{r}}$, where $h = 0$.

The gradient $\boldsymbol{\nabla}(h)$ is actually precisely equal to $\mathbf{n}_F$ on the base surface. To realize this we only need to show that the length of the gradient is equal to 1. The direction is certainly the same. Therefore let (u_0, v_0) denote a point in D and consider the restriction of h to the straight line $\mathbf{r}(u_0, v_0) + w\,\mathbf{n}(u_0, v_0)$, where $w \in [0, t]$. Let us use the notation $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$ and $\mathbf{n}_0 = \mathbf{n}(u_0, v_0)$. Then the chain rule gives

$$\begin{aligned} 1 &= \left| \frac{d}{dw} h(\mathbf{r}_0 + w\,\mathbf{n}_0) \right| \\ &= \left| \mathbf{n}_0 \cdot \boldsymbol{\nabla}(h)(\mathbf{r}_0 + w\,\mathbf{n}_0) \right| \\ &= \| \boldsymbol{\nabla}(h)(\mathbf{r}_0 + w\,\mathbf{n}_0) \| \quad , \end{aligned} \tag{29-25}$$

such that we on the surface $F_{\mathbf{r}}$, where $w = 0$, have $\|\boldsymbol{\nabla}(h)(\mathbf{r}_0)\| = 1$. This was what we should prove. Therefore we have:

$$\boldsymbol{\nabla}(h)_{|_F} = \mathbf{n}_F \quad . \tag{29-26}$$

The function $h(x, y, z)$ is the *Euclidian distance* from the point (x, y, z) to the surface $F_{\mathbf{r}}$. Locally, i.e. sufficiently close to (and in the direction of the normal of) the surface $F_{\mathbf{r}}$, $\boldsymbol{\nabla} h(x, y, z)$ is an *extension* of the unit normal vector field $\mathbf{n}_F$ to the surface. The vector field $\boldsymbol{\nabla} h(x, y, z)$ is in itself a unit vector field. By letting $F_{\mathbf{r}}$ flow for time t along the flow curves of $\boldsymbol{\nabla} h(x, y, z)$ the spatial region, the solid shell, Ω_t, is formed (swept), see Figures 29.3, 29.4.

29.2.3 Integration in the Shell

For a function $f(x, y, z)$ that is defined in Ω_t we get the integral of f over the region like this:

$$\begin{aligned} &\int_{\Omega_t} f \, \mathrm{d}\mu \\ &= \int_0^t \left(\int_D f(\mathbf{R}(u, v, w)) \mathrm{Jacobian}_{\mathbf{R}}(u, v, w) \, \mathrm{d}u \, \mathrm{d}v \right) \mathrm{d}w \quad . \end{aligned} \tag{29-27}$$

The t-derivative of this integral is, for $t = 0$, simply the surface integral of f over $F_{\mathbf{r}}$:

|||| Lemma 29.10 The Surface Integral as the t-Derivative of a Volume Integral

$$\left(\frac{d}{dt}\right)_{|t=0} \int_{\Omega_t} f \, \mathrm{d}\mu = \int_{F_{\mathbf{r}}} f \, \mathrm{d}\mu \quad . \tag{29-28}$$

|||| Proof

It follows directly from the Fundamental Theorem 29.1:

$$\begin{aligned} &\left(\frac{d}{dt}\right)_{|t=0} \int_{\Omega_t} f \, \mathrm{d}\mu \\ &= \left(\frac{d}{dt}\right)_{|t=0} \int_0^t \left(\int_D f(\mathbf{R}(u,v,w)) \mathrm{Jacobian}_{\mathbf{R}}(u,v,w) \, \mathrm{d}u \, \mathrm{d}v \right) \mathrm{d}w \\ &= \int_D f(\mathbf{R}(u,v,0)) \mathrm{Jacobian}_{\mathbf{R}}(u,v,0) \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_D f(\mathbf{r}(u,v)) \mathrm{Jacobian}_{\mathbf{r}}(u,v) \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_{F_{\mathbf{r}}} f \, \mathrm{d}\mu \quad . \end{aligned} \tag{29-29}$$

■

29.2.4 The Wall

The shell Ω_t has a boundary $\partial\Omega_t$ consisting of the top surface F_t, the base surface $F_0 = F_{\mathbf{r}} = F_{\mathbf{R}}(0)$ and a 'wall' W_t with the height t. See Figures 29.3 and 29.4. We get the wall component by restricting the map $\mathbf{R}$ to $\partial D \times [0,t]$ like this:

$$\begin{aligned} W_t : \qquad \mathbf{B}(\theta, w) &= \mathbf{R}(u(\theta), v(\theta), w) \\ &= \mathbf{r}(u(\theta), v(\theta)) + w\,\mathbf{n}(u(\theta), v(\theta)) \\ &= \mathbf{b}(\theta) + w\,\mathbf{n}(\mathbf{d}(\theta)) \; , \; \theta \in I \; , \; w \in [0,t] \quad . \end{aligned} \tag{29-30}$$

The Jacobian function of this map is therefore

$$\begin{aligned} \text{Jacobian}_{\mathbf{B}}(\theta, w) &= \|\mathbf{B}'_\theta \times \mathbf{B}'_w\| \\ &= \|(\mathbf{b}'_\theta + w\,(\mathbf{n} \circ \mathbf{d})'_\theta) \times \mathbf{n}_F\| \quad . \end{aligned} \tag{29-31}$$

Since $\mathbf{n}_F$ is perpendicular to the surface $F_\mathbf{r}$ and therefore also to the boundary $\partial F_\mathbf{r}$ (parameterized via $\mathbf{b}$), we get:

$$\text{Jacobian}_{\mathbf{B}}(\theta, 0) = \|\mathbf{b}'_\theta \times \mathbf{n}_F\| = \|\mathbf{b}'_\theta\| = \text{Jacobian}_{\mathbf{b}}(\theta) \quad . \tag{29-32}$$

29.2.5 Integration along the Wall

For a given function $g(x, y, z)$ defined on W_t the integral of g over this surface:

$$\int_{W_t} g \, \mathrm{d}\mu = \int_0^t \left(\int_I g(\mathbf{B}(\theta, w)) \, \text{Jacobian}_{\mathbf{B}}(\theta, w) \, \mathrm{d}\theta \right) dw \quad . \tag{29-33}$$

The t-derivative of this integral is, for $t = 0$, the line integral of g over ∂F:

|||| Lemma 29.11 The Line Integral as the t-Derivative of the Surface Integral

$$\left(\frac{d}{dt}\right)_{|t=0} \int_{W_t} g \, \mathrm{d}\mu = \int_{\partial F} g \, \mathrm{d}\mu \quad . \tag{29-34}$$

|||| Proof

Again this follows from the Fundamental Theorem, Theorem 29.1:

$$\begin{aligned} &\left(\frac{d}{dt}\right)_{|t=0} \int_{W_t} g \, \mathrm{d}\mu \\ &= \left(\frac{d}{dt}\right)_{|t=0} \int_0^t \left(\int_I g(\mathbf{B}(\theta, w)) \text{Jacobian}_{\mathbf{B}}(\theta, w) \, \mathrm{d}\theta \right) dw \\ &= \int_I g(\mathbf{B}(\theta, 0)) \text{Jacobian}_{\mathbf{B}}(\theta, 0) \, \mathrm{d}\theta \\ &= \int_I g(\mathbf{b}(\theta)) \text{Jacobian}_{\mathbf{b}}(\theta) \, \mathrm{d}\theta \\ &= \int_{\partial F} g \, \mathrm{d}\mu \quad . \end{aligned} \tag{29-35}$$

■

29.2.6 Proof of Stokes' Theorem

We are now ready to prove Theorem 29.2.

|||| Proof

The function $h(x,y,z)$ from the previous Section 29.2.2 is substituted instead of $\psi(x,y,z)$ in Theorem 29.4, Equation (29-9). With the integration region $\Omega = \Omega_t$ we then get:

$$\begin{aligned}\int_{\Omega_t} \mathbf{Curl}(\mathbf{V}) \cdot \boldsymbol{\nabla}(h)\, \mathrm{d}\mu \\ &= \int_{\partial\Omega_t} (\mathbf{n}_{\partial\Omega_t} \times \mathbf{V}) \cdot \boldsymbol{\nabla}(h)\, \mathrm{d}\mu \\ &= \int_{F_t} (\mathbf{n}_{F_t} \times \mathbf{V}) \cdot \boldsymbol{\nabla}(h)\, \mathrm{d}\mu \\ &- \int_{F_0} (\mathbf{n}_{F_0} \times \mathbf{V}) \cdot \boldsymbol{\nabla}(h)\, \mathrm{d}\mu \\ &+ \int_{W_t} (\mathbf{n}_{W_t} \times \mathbf{V}) \cdot \boldsymbol{\nabla}(h)\, \mathrm{d}\mu \quad .\end{aligned} \tag{29-36}$$

But in Equation (29-36) we have

$$\begin{aligned}\int_{F_t} (\mathbf{n}_{F_t} \times \mathbf{V}) \cdot \boldsymbol{\nabla}(h)\, \mathrm{d}\mu &= 0 \quad \text{and} \\ \int_{F_0} (\mathbf{n}_{F_0} \times \mathbf{V}) \cdot \boldsymbol{\nabla}(h)\, \mathrm{d}\mu &= 0 \quad ,\end{aligned} \tag{29-37}$$

since $\boldsymbol{\nabla}(h)$ is perpendicular to both surfaces F_t and F_0 such that $\boldsymbol{\nabla}(h)$ is proportional to $\mathbf{n}_{F_t}$ and $\mathbf{n}_{F_0}$ on the respective surfaces.

Along $\partial F \subset W_t$ we have $\mathbf{n}_{W_t} = \mathbf{e}_{\partial F} \times \mathbf{n}_F$ and therefore $\mathbf{e}_{\partial F} = \mathbf{n}_F \times \mathbf{n}_{W_t}$ - in accordance with the orientation rule in Theorem 29.2. If we take the derivative with respect to t on both sides of the reduced equation (29-36) we get:

$$\begin{aligned}&\left(\frac{d}{dt}\right)_{|t=0} \int_{\Omega_t} \mathbf{Curl}(\mathbf{V}) \cdot \boldsymbol{\nabla}(h)\, \mathrm{d}\mu \\ &= \left(\frac{d}{dt}\right)_{|t=0} \int_{W_t} (\mathbf{n}_{W_t} \times \mathbf{V}) \cdot \boldsymbol{\nabla}(h)\, \mathrm{d}\mu \quad ,\end{aligned} \tag{29-38}$$

such that we finally have - by virtue of Equations (29-28) and (29-34):

$$\begin{aligned} &\int_F \mathbf{Curl}(\mathbf{V}) \cdot \mathbf{n}_F \, \mathrm{d}\mu \\ &= \int_{\partial F} (\mathbf{n}_{W_t} \times \mathbf{V}) \cdot \mathbf{n}_F \, \mathrm{d}\mu \\ &= \int_{\partial F} \mathbf{V} \cdot (\mathbf{n}_F \times \mathbf{n}_{W_t}) \, \mathrm{d}\mu \\ &= \int_{\partial F} \mathbf{V} \cdot \mathbf{e}_{\partial F} \, \mathrm{d}\mu \quad . \end{aligned} \tag{29-39}$$

Stokes' Theorem is hereby proven.

■

29.3 Verification of Stokes' Theorem in Concrete Examples

Like Gauss' Divergence Theorem, Stokes' Theorem can be verified by – in actual cases – calculating *both sides* of the identity (29-3) in Stokes' Theorem.

|||| Example 29.12 Stokes' Theorem with a Circular Disc

A standard circular disc in the (x, y) plane is given by its ordinary parametric representation:

$$F_{\mathbf{r}} \quad : \quad \mathbf{r}(u,v) = (u \cdot \cos(v), u \cdot \sin(v), 0) \quad , \quad (u,v) \in [0,1] \times [-\pi, \pi] \tag{29-40}$$

with the Jacobian function

$$\text{Jacobian}_{\mathbf{r}}(u,v) = u \quad , \tag{29-41}$$

the standard unit normal vector field to the surface

$$\mathbf{n}_F = (0,0,1) \quad , \tag{29-42}$$

and the standard unit tangential vector field along the *circular* boundary ∂F:

$$\mathbf{e}_{\partial F} = \mathbf{r}'_v(1,v) = (-\sin(v)\,,\, \cos(v)\,,\, 0) \quad . \tag{29-43}$$

Note that it is $\mathbf{r}(1,v)$, $v \in [-\pi, \pi]$ that gives the *whole boundary*, that $\mathbf{r}'_v(1,v)$ is a unit vector for all v, and that $\mathbf{r}'_v(1,v) \times \mathbf{n}_F$ points away from the circular disc along the whole boundary curve.

A smooth vector field in (x, y, z) space is given by its coordinate functions like this:

$$\mathbf{V}(x, y, z) = (x \cdot y \, , \, x \, , \, x^2) \quad , \tag{29-44}$$

and has the corresponding curl vector field

$$\mathbf{Curl}(\mathbf{V})(x, y, z) = (0 \, , \, -2 \cdot x \, , \, 1 - x) \quad . \tag{29-45}$$

We will verify Stokes' Theorem by computing both sides of equation (29-3) in this concrete case.

The total flux of the vector field $\mathbf{Curl}(\mathbf{V})(x, y, z)$ through $F_{\mathbf{r}}$ in the direction of the standard unit normal $\mathbf{n}_F = (0, 0, 1)$ is

$$\begin{aligned} \mathrm{Flux}(\mathbf{Curl}(\mathbf{V}), F_{\mathbf{r}}) &= \int_{F_{\mathbf{r}}} \mathbf{Curl}(\mathbf{V}) \bullet \mathbf{n}_F \, \mathrm{d}\mu \\ &= \int_{F_{\mathbf{r}}} (1 - x) \, \mathrm{d}\mu \\ &= \int_{-\pi}^{\pi} \int_0^1 (1 - u \cdot \cos(v)) \cdot u \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_{-\pi}^{\pi} \left(\frac{1}{2} - \frac{1}{3} \cdot \cos(v) \right) \mathrm{d}v \\ &= \pi \quad . \end{aligned} \tag{29-46}$$

The circulation of the vector field $\mathbf{V}(x, y, z)$ along $\partial F_{\mathbf{r}}$ in the direction of the standard unit tangential vector field $\mathbf{e}_{\partial F} = (-\sin(v), \cos(v) \, , \, 0)$ is

$$\begin{aligned} \mathrm{Cirk}(\mathbf{V}, \partial F) &= \int_{\partial F} \mathbf{V} \bullet \mathbf{e}_{\partial F} \, \mathrm{d}\mu \\ &= \int_{-\pi}^{\pi} \mathbf{V}(\mathbf{r}(1, v)) \bullet (-\sin(v), \cos(v) \, , \, 0) \, \mathrm{d}v \\ &= \int_{-\pi}^{\pi} (\cos(v) \cdot \sin(v) \, , \, \cos(v) \, , \, *) \bullet (-\sin(v), \cos(v) \, , \, 0) \, \mathrm{d}v \\ &= \int_{-\pi}^{\pi} (-\cos(v) \cdot \sin^2(v) + \cos^2(v)) \, \mathrm{d}v \\ &= \int_{-\pi}^{\pi} \cos^2(v) \, \mathrm{d}v \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \left(\cos^2(v) + \sin^2(v) \right) \, \mathrm{d}v \\ &= \frac{1}{2} \int_{-\pi}^{\pi} 1 \, \mathrm{d}v \\ &= \pi \quad . \end{aligned} \tag{29-47}$$

Therefore both sides in Equation (29-3) give π in this example – and thus we have verified Stokes' theorem.

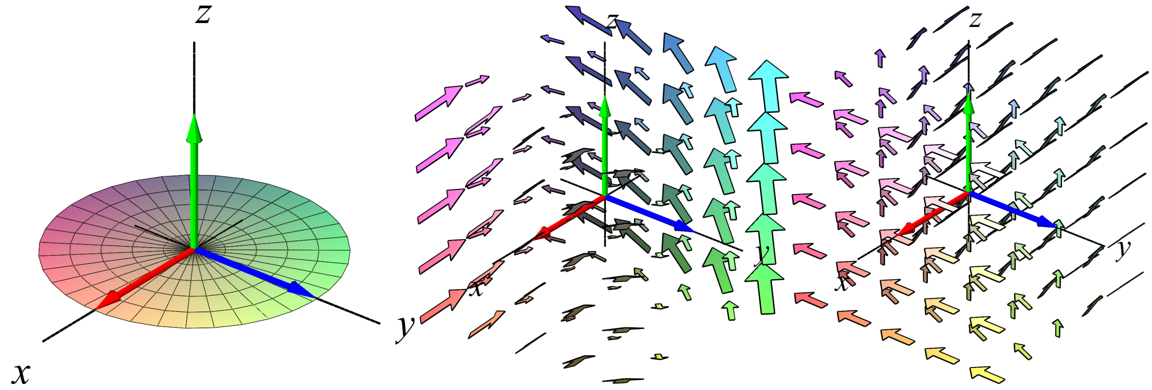

Figure 29.5: A circular disc in a vector field $\mathbf{V}(x,y,z) = (x \cdot y,\ x,\ x^2)$ (in the middle) and vector field $\mathbf{Curl}(\mathbf{V})(x,y,z) = (0,\ -2 \cdot x,\ 1 - x)$.

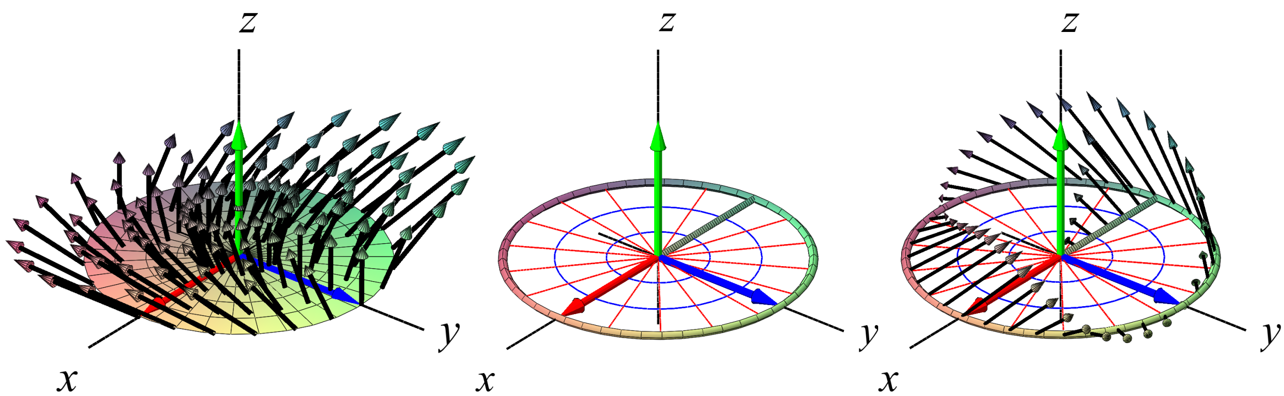

Figure 29.6: The restriction of the vector field $\mathbf{Curl}(\mathbf{V})(x,y,z) = (0,\ -2 \cdot x,\ 1 - x)$ to a circular disc, the boundary of the circular disc, and the vector field $\mathbf{V}(x,y,z) = (x \cdot y,\ x,\ x^2)$ bounded to the surface.

|||| Example 29.13 Stokes' Theorem with a Paraboloid

A segment of a paraboloid is given by a parametric representation like this:

$$F_{\mathbf{r}} \quad : \quad \mathbf{r}(u,v) = (2 \cdot u \cdot \cos(v), 2 \cdot u \cdot \sin(v), u^2) \quad , \tag{29-48}$$

where

$$(u,v) \in [1/2, 1] \times [0, \pi] \tag{29-49}$$

The parametric representation has the Jacobian function

$$\text{Jacobian}_{\mathbf{r}}(u,v) = 4 \cdot u \cot \sqrt{1 + u^2} \quad , \tag{29-50}$$

and the standard normal vector field (not necessarily consisting of unit vectors) to the surface is given by

$$\mathbf{N}_F(u,v) = \mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v) = (-4 \cdot u^2 \cdot \cos(v),\, -4 \cdot u^2 \cdot \sin(v),\, 4 \cdot u) \quad . \tag{29-51}$$

A smooth vector field in (x, y, z) space is given by its coordinate functions like this:

$$\mathbf{V}(x,y,z) = (z, y, -x) \quad , \tag{29-52}$$

with the corresponding rotation vector field

$$\mathbf{Curl}(\mathbf{V})(x,y,z) = (0\,,\,2\,,\,0) \quad . \tag{29-53}$$

We will verify Stokes' Theorem by computing both sides of Equation (29-3) in this concrete case.

The total flux of the vector field $\mathbf{Curl}(\mathbf{V})(x,y,z)$ through $F_{\mathbf{r}}$ in the direction of the standard unit normal is, using the normal vector field $\mathbf{N}_F(u,v)$ directly:

$$\begin{aligned} \text{Flux}(\mathbf{Curl}(\mathbf{V}), F_{\mathbf{r}}) &= \int_{F_{\mathbf{r}}} \mathbf{Curl}(\mathbf{V}) \cdot \mathbf{n}_F \, d\mu \\ &= \int_0^{\pi} \int_{1/2}^{1} \mathbf{Curl}(\mathbf{V}) \cdot \mathbf{N}_F(u,v) \, du \, dv \\ &= \int_0^{\pi} \int_{1/2}^{1} \left(-8 \cdot u^2 \cdot \sin(v)\right) \, du \, dv \\ &= \int_0^{\pi} \left(\frac{-7}{3} \cdot \sin(v)\right) dv \\ &= \frac{-14}{3} \quad . \end{aligned} \tag{29-54}$$

The circulation of the vector field $\mathbf{V}(x,y,z)$ along $\partial F_{\mathbf{r}}$ in the direction of the correctly oriented boundary curve consists of four components. We get these from the parametric representation of the surface $\mathbf{r}(u,v)$ partly by fixing v to be 0 and π, respectively, and partly by fixing u to be $1/2$ and 1, respectively, while in each case at the same time seeing to it that the free parameter runs the 'right way', i.e. such that all four boundary components get the right orientation in relation to the normal vector field of the surface:

$$\begin{aligned} \partial_1 F \quad : \quad \mathbf{r}_1(u) &= \mathbf{r}(u,0) = (2 \cdot u,\, 0,\, u^2) \quad , \quad u \in [1/2,\, 1] \\ \mathbf{r}'_1(u) &= (2,\, 0,\, 2 \cdot u) \\ \mathbf{V}(\mathbf{r}_1(u)) &= (u^2,\, 0,\, -2 \cdot u) \\ \mathbf{r}'_1(u) \cdot \mathbf{V}(\mathbf{r}_1(u)) &= -2 \cdot u^2 \quad , \end{aligned} \tag{29-55}$$

$$\begin{aligned}\partial_3 F \quad : \quad \mathbf{r}_3(u) &= \mathbf{r}(1-u+1/2, v) = (-3+2\cdot u,\, 0,\, (-3/2+u)^2) \quad , \quad u \in [1/2,\, 1] \\ \mathbf{r}_3'(u) &= (2,\, 0,\, 2\cdot u - 3) \\ \mathbf{V}(\mathbf{r}_3(u)) &= ((-3/2+u)^2,\, 0,\, 3-2\cdot u) \\ \mathbf{r}_3'(u)\cdot\mathbf{V}(\mathbf{r}_3(u)) &= \frac{-9}{2} + 6\cdot u - 2\cdot u^2 \quad ,\end{aligned} \tag{29-56}$$

$$\begin{aligned}\partial_2 F \quad : \quad \mathbf{r}_2(v) &= \mathbf{r}(1, v) = (2\cdot\cos(v),\, 2\cdot\sin(v),\, 1) \quad , \quad v \in [0,\, \pi] \\ \mathbf{r}_2'(v) &= (-2\cdot\sin(v),\, 2\cdot\cos(v),\, 0) \\ \mathbf{V}(\mathbf{r}_2(v)) &= (1,\, 2\cdot\sin(v),\, -2\cdot\cos(v)) \\ \mathbf{r}_2'(v)\cdot\mathbf{V}(\mathbf{r}_2(v)) &= 2\cdot\sin(v)\cdot(2\cdot\cos(v)-1) \quad ,\end{aligned} \tag{29-57}$$

$$\begin{aligned}\partial_4 F \quad : \quad \mathbf{r}_4(v) &= \mathbf{r}(1/2,\, \pi - v) = (-\cos(v), \sin(v), 1/4) \quad , \quad v \in [0,\, \pi] \\ \mathbf{r}_4'(v) &= (\sin(v),\, \cos(v),\, 0) \\ \mathbf{V}(\mathbf{r}_4(v)) &= (\frac{1}{4},\, \sin(v),\, \cos(v)) \\ \mathbf{r}_4'(v)\cdot\mathbf{V}(\mathbf{r}_4(v)) &= \frac{1}{4}\cdot\sin(v)\cdot(\cos(v)+1) \quad .\end{aligned} \tag{29-58}$$

Thus we have the following total circulation of $\mathbf{V}(x, y, z)$ along the boundary:

$$\begin{aligned}\mathrm{Cirk}(\mathbf{V}, \partial F) &= \sum_{i=1}^{4} \mathrm{Cirk}(\mathbf{V}, \partial_i F) \\ &= \sum_{i=1}^{4} \int_{\partial_i F} \mathbf{V}(\mathbf{r}_i(t)) \cdot \mathbf{r}_i'(t)\; \mathrm{d}t \\ &= \int_{1/2}^{1} \left(-2\cdot u^2\right)\; \mathrm{d}u \\ &\quad + \int_{1/2}^{1} \left(\frac{-9}{2} + 6\cdot u - 2\cdot u^2\right)\; \mathrm{d}u \\ &\quad\quad + \int_0^{\pi} 2\cdot\sin(v)\cdot(2\cdot\cos(v)-1)\; \mathrm{d}v \\ &\quad\quad\quad + \int_0^{\pi} \frac{1}{4}\cdot\sin(v)\cdot(\cos(v)+1)\; \mathrm{d}v \\ &= -\frac{7}{12} - \frac{7}{12} - 4 + \frac{1}{2} = -\frac{14}{3} \quad .\end{aligned} \tag{29-59}$$

Therefore both sides in the equation (29-3) yield $-14/3$ in this example, and again we have verified Stokes' Theorem.

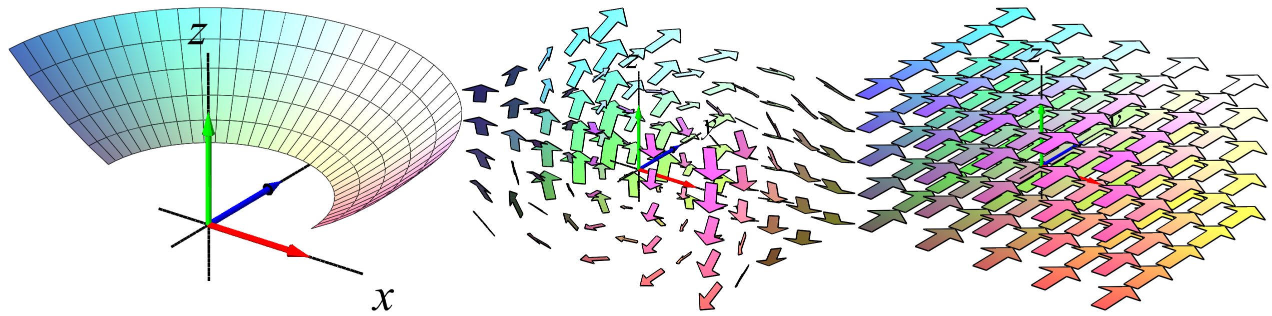

Figure 29.7: A segment of a paraboloid in a vector field $\mathbf{V}(x,y,z) = (z,\, y,\, -x)$ (in the middle) and the vector field $\mathbf{Curl}(\mathbf{V})(x,y,z) = (0,\, 2,\, 0)$.

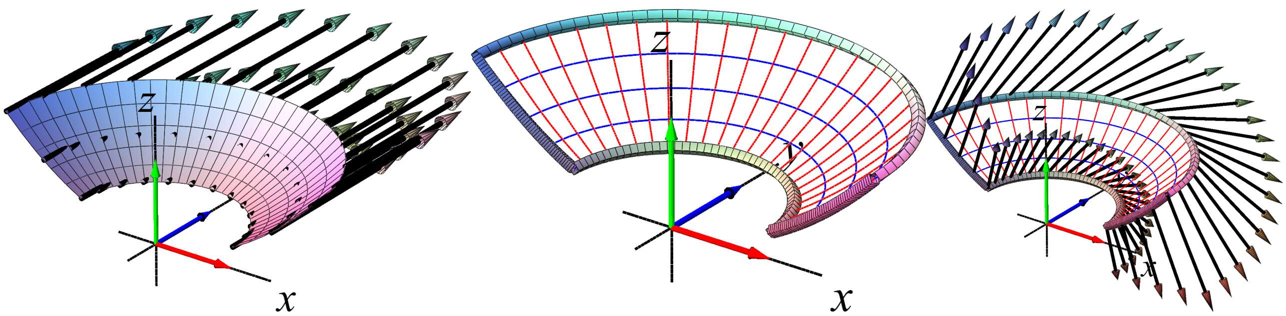

Figure 29.8: The restriction of the vector field $\mathbf{Curl}(\mathbf{V})(x,y,z) = (0,\, 2,\, 0)$ to a segment of a paraboloid, the boundary of the surface segment and the vector field $\mathbf{V}(x,y,z) = (z,\, y,\, -x)$ restricted to the boundary.

|||| Exercise 29.14

Let $F_\mathbf{r}$ denote the surface having the following parametric representation and parametric region:

$$F_\mathbf{r} \,:\, \mathbf{r}(u,v) = ((1+u^2)\cos(v),\, (1+u^2)\sin(v),\, \sin(u)) \quad , \tag{29-60}$$

where

$$(u,v) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\pi, \frac{\pi}{2}] \quad . \tag{29-61}$$

Verify that Stokes' Theorem is fulfilled for the vector field $\mathbf{V} = (yx\,,\, yz\,,\, xz)$ over the surface $F_\mathbf{r}$ with the given boundary $\partial F_\mathbf{r}$.

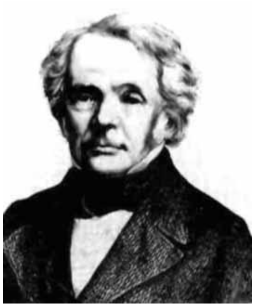

Figure 29.9: August Ferdinand Möbius, see Biography.

|||| Exercise 29.15

Let $F_{\mathbf{r}}$ denote the surface – a so-called Möbius strip – that has the following parametric representation and parameter region:

$$\begin{aligned} F_{\mathbf{r}} \, : & \mathbf{r}(u,v) \\ & = (2\cos(u) + v\cos(u/2)\cos(u)\,,\, 2\sin(u) + v\cos(u/2)\sin(u)\,,\, v\sin(u/2)) \quad , \end{aligned} \tag{29-62}$$

where $(u,v) \in [-\pi,\pi] \times [-1,1]$. Verify that Stokes' Theorem is fulfilled for the vector field $\mathbf{V} = (-y\,,\,x\,,\,1)$ over the surface $F_{\mathbf{r}}$ with the given boundary $\partial F_{\mathbf{r}}$, see Figure 29.10: Show explicitly that:

$$\int_F \mathbf{Curl}(\mathbf{V}) \cdot \mathbf{n}_F \, \mathrm{d}\mu = \int_{\partial F} \mathbf{V} \cdot \mathbf{e}_{\partial F} \, \mathrm{d}\mu = -32 \quad . \tag{29-63}$$

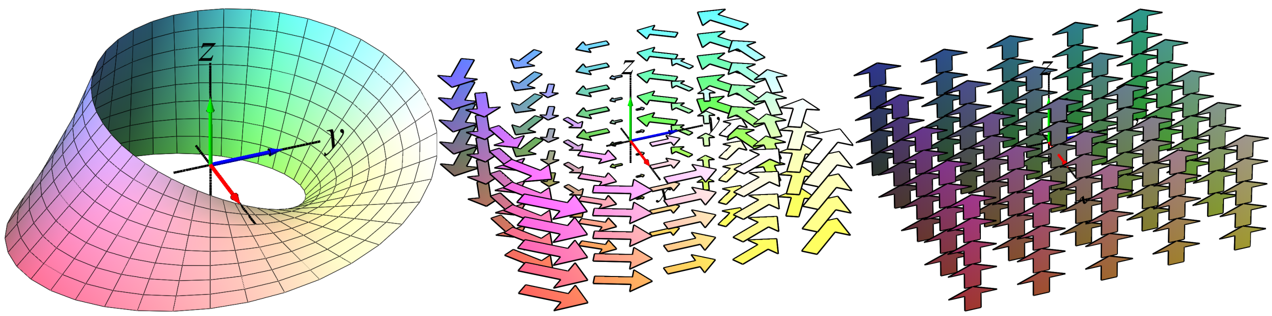

Figure 29.10: A Möbius strip in a vector field $\mathbf{V}(x,y,z) = (-y,x,1)$ (in the middle) and the vector field's rotation vector field $\mathbf{Curl}(\mathbf{V})(x,y,z) = (0,0,2)$.

|||| Exercise 29.16

Now let $G_{\mathbf{r}}$ denote the same Möbius strip as in Exercise 29.15:

$$\begin{aligned} G_{\mathbf{r}} &: \mathbf{r}(u,v) \\ &= (2\cos(u) + v\cos(u/2)\cos(u)\,,\, 2\sin(u) + v\cos(u/2)\sin(u)\,,\, v\sin(u/2)) \quad , \end{aligned} \tag{29-64}$$

but now with a somewhat modified parametric region:

$$(u,v) \in [0, 2\pi] \times [-1, 1] \quad . \tag{29-65}$$

Verify that Stokes' Theorem is fulfilled for the vector field $\mathbf{V} = (-y\,,\, x\,,\, 1)$ over the surface $G_{\mathbf{r}}$ with the given boundary $\partial G_{\mathbf{r}}$: Show explicitly that:

$$\int_{G_{\mathbf{r}}} \mathbf{Curl}(\mathbf{V}) \cdot \mathbf{n}_G \, d\mu = \int_{\partial G_{\mathbf{r}}} \mathbf{V} \cdot \mathbf{e}_{\partial G} \, d\mu = 0 \quad . \tag{29-66}$$

|||| Exercise 29.17

Explain the difference between the two results (-32 and 0), that are obtained in (29-63) and (29-66), respectively. Hint: Keep an eye on the normal vector field $\mathbf{n}_F$.

Which *values* of the curl flux and circulation are achieved by changing intervals of the parameterization, i.e. only by using the following parametric intervals for different $\alpha \in \mathbb{R}$:

$$(u,v) \in [-\pi + \alpha, \pi + \alpha] \times [-1, 1] \quad ? \tag{29-67}$$

29.4 Stokes' Theorem in the Plane

Plane regions, i.e. regions in the (x,y) plane, can be parameterized like this:

$$F_{\mathbf{r}} : \qquad \mathbf{r}(u,v) = (x(u,v), y(u,v), 0) \in \mathbb{R}^3 \,,\; (u,v) \in D = [a,b] \times [c,d] \subset \mathbb{R}^2 \quad . \tag{29-68}$$

A plane region has a particularly simple unit normal vector field in (x,y,z) space, i.e. when we consider the plane region as a surface in this 3D space, that coincidently lies

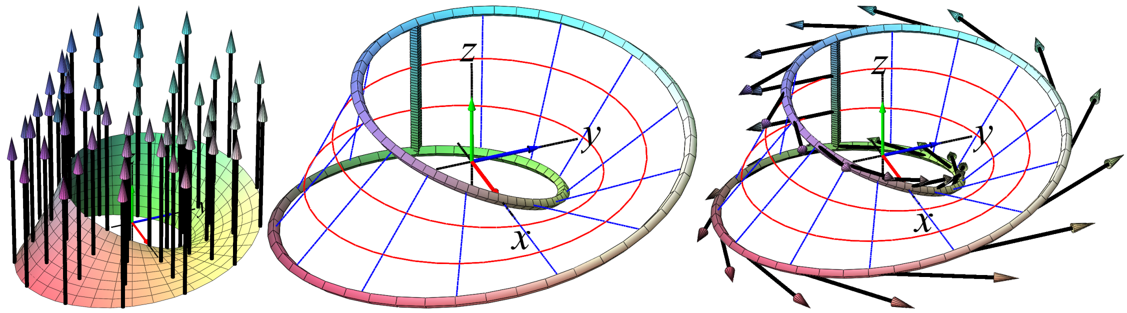

Figure 29.11: The restriction of the vector field $\mathbf{Curl}(\mathbf{V})(x,y,z) = (0,\,0,\,2)$ to a piece of the Möbius strip, the boundary of the strip, and the vector field $\mathbf{V}(x,y,z) = (-y,x,1)$ restricted to the boundary.

entirely in the (x,y) plane: $\mathbf{n}_F = (0,0,1)$.

With respect to the boundary of the plane region we repeat the notation from section 29.2.1: The boundary $\partial F_{\mathbf{r}}$ of $F_{\mathbf{r}}$ appears by using the vector function $\mathbf{r}$ on the four straight line segments constituting the boundary ∂D of the rectangular parametric region $D = [a,b] \times [c,d]$. We parameterize all of ∂D at once by using a parameter θ and a vector function $\mathbf{d}$:

$$\partial D: \qquad \mathbf{d}(\theta) = (u(\theta), v(\theta)) \in \partial D \subset \mathbb{R}^2 \;,\; \theta \in I \subset \mathbb{R} \quad ,$$

where $u(\theta)$ and $v(\theta)$ are only piecewise differentiable functions of θ. E.g. they can be chosen as linear functions of θ for each of the four line segments that constitute ∂D.

The boundary of $F_{\mathbf{r}}$ is then

$$\partial F_{\mathbf{r}}: \qquad \mathbf{b}(\theta) = \mathbf{r}(\mathbf{d}(\theta)) = \mathbf{r}(u(\theta), v(\theta)) \in \mathbb{R}^2 \subset \mathbb{R}^3 \;,\; \theta \in I = [0,T] \subset \mathbb{R} \quad . \tag{29-69}$$

Now let $\mathbf{V}(x,y,z) = (V_1(x,y,z), V_2(x,y,z), V_3(x,y,z))$ denote an arbitrary vector field in (x,y,z) space, then we have the following consequence of Stokes' Theorem:

|||| Theorem 29.18 Stokes' Theorem in the Plane

Using the notation above it applies that:

$$\begin{aligned}
\int_{F_{\mathbf{r}}} \mathbf{Curl}(\mathbf{V}) \cdot \mathbf{n}_F \, \mathrm{d}\mu &= \int_{\partial F} \mathbf{V} \cdot \mathbf{e}_{\partial F} \, \mathrm{d}\mu \\
\int_{F_{\mathbf{r}}} \mathbf{Curl}(\mathbf{V}) \cdot (0,0,1) \, \mathrm{d}\mu &= \int_{\partial F} (V_1, V_2, V_3) \cdot \mathbf{e}_{\partial F} \, \mathrm{d}\mu \\
\int_{F_{\mathbf{r}}} \mathbf{Curl}(\mathbf{V}) \cdot (0,0,1) \, \mathrm{d}\mu &= \int_{I} (V_1, V_2, V_3) \cdot \mathbf{b}'(\theta) \, \mathrm{d}\theta \\
\int_{F_{\mathbf{r}}} \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \, \mathrm{d}\mu &= \int_0^T \left(V_1 b_1'(\theta) + V_2 b_2'(\theta) \right) \, \mathrm{d}\theta \quad ,
\end{aligned} \tag{29-70}$$

and thus

$$\int_c^d \int_a^b \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \mathrm{Jacobian}_{\mathbf{r}}(u,v) \, \mathrm{d}u \, \mathrm{d}v = \int_0^T \left(V_1 b_1'(\theta) + V_2 b_2'(\theta) \right) \, \mathrm{d}\theta \quad . \tag{29-71}$$

|||| Remark 29.19

We see that the third coordinate function $V_3(x,y,z)$ of the vector field does not play a role in the theorem. Therefore the theorem is about vector functions in the *plane*: $\mathbf{V}(x,y) = (V_1(x,y), V_2(x,y))$. We illustrate with a detailed - but simple - example below.

|||| Example 29.20 Stokes' Theorem in the Plane

Let $\mathbf{V}(x,y) = (-y,x)$ and $\mathbf{r}(u,v) = (3u,4v)$, $u \in [a,b]$, $v \in [c,d]$, then we get: $\mathrm{Jacobian}_{\mathbf{r}}(u,v) = 12$ and $\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} = 2$ such that the left-hand side in Equation (29-71) gives:

$$\begin{aligned}
&\int_c^d \int_a^b \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \mathrm{Jacobian}_{\mathbf{r}}(u,v) \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_c^d \int_a^b 2 \cdot 12 \, \mathrm{d}u \, \mathrm{d}v \\
&= 24(b-a)(d-c) \quad .
\end{aligned} \tag{29-72}$$

The right-hand side in Equation (29-71) shall give the same value.

We determine the value of the right-hand side by dividing the boundary of $F_{\mathbf{r}}$ into four straight components and by computing the tangential line integral of $\mathbf{V}$ along every component, while we in each case see to it that the orientation (parameterization) of the component follows the statement that is made explicit in Stokes' Theorem.

We start by putting $T = 2(b-a) + 2(d-c)$, i.e. T is the total circumference of the parametric region D. The boundary of D we can then parameterize in the following way:

$$\begin{aligned} \mathbf{d}(\theta) &= (a+\theta\,,\ 0) \quad \text{for} \quad \theta \in [0\,,\ b-a] \\ \mathbf{d}(\theta) &= (b\,,\ c+\theta) \quad \text{for} \quad \theta \in [b-a\,,\ b-a+d-c] \\ \mathbf{d}(\theta) &= (b-\theta\,,\ \mathrm{d}) \quad \text{for} \quad \theta \in [b-a+d-c\,,\ 2(b-a)+d-c] \\ \mathbf{d}(\theta) &= (a\,,\ \mathrm{d}-\theta) \quad \text{for} \quad \theta \in [2(b-a)+d-c\,,\ 2(b-a)+2(d-c)] \quad . \end{aligned} \tag{29-73}$$

The boundary of $F_{\mathbf{r}}$ we then get in the following form:

$$\begin{aligned} \mathbf{b}(\theta) &= (3(a+\theta)\,,\ 0) \quad \text{for} \quad \theta \in [0\,,\ b-a] \\ \mathbf{b}(\theta) &= (3b\,,\ 4(c+\theta)) \quad \text{for} \quad \theta \in [b-a\,,\ b-a+d-c] \\ \mathbf{b}(\theta) &= (3(b-\theta)\,,\ 4d) \quad \text{for} \quad \theta \in [b-a+d-c\,,\ 2(b-a)+d-c] \\ \mathbf{b}(\theta) &= (3a\,,\ 4(d-\theta)) \quad \text{for} \quad \theta \in [2(b-a)+d-c\,,\ 2(b-a)+2(d-c)] \quad , \end{aligned} \tag{29-74}$$

and thus:

$$\begin{aligned} \mathbf{b}'(\theta) &= (3\,,\ 0) \quad \text{for} \quad \theta \in [0\,,\ b-a] \\ \mathbf{b}'(\theta) &= (0\,,\ 4) \quad \text{for} \quad \theta \in [b-a\,,\ b-a+d-c] \\ \mathbf{b}'(\theta) &= (-3\,,\ 0) \quad \text{for} \quad \theta \in [b-a+d-c\,,\ 2(b-a)+d-c] \\ \mathbf{b}'(\theta) &= (0\,,\ -4) \quad \text{for} \quad \theta \in [2(b-a)+d-c\,,\ 2(b-a)+2(d-c)] \quad . \end{aligned} \tag{29-75}$$

Now we can compute the right-hand side in Equation (29-71) as the sum of the four terms that come from the four components of the boundary curve. E.g. for the first line segment in the boundary curve, parallel to the $x-$axis and with $y = 4c$ constant we have (for $\theta \in [0, b-a]$) $b_1'(\theta) = 3$, $b_2'(\theta) = 0$, $V_1(x,y) = y = b_2(u,v) = 4v = 4c$ and similarly for the other 3 contributions (see Figures 29.12 and 29.13):

$$\begin{aligned} \int_0^T V_1 b_1'(\theta) + V_2 b_2'(\theta)\ \mathrm{d}\theta &= \\ &\int_0^{b-a} 12c\ \mathrm{d}\theta \\ &+ \int_{b-a}^{b-a+d-c} 12b\ \mathrm{d}\theta \\ &+ \int_{b-a+d-c}^{2(b-a)+d-c} 12d\ \mathrm{d}\theta \\ &+ \int_{2(b-a)+d-c}^{2(b-a)+2(d-c)} 12a\ \mathrm{d}\theta \\ &= \\ &12c(b-a) \\ &+ 12b(d-c) \\ &+ 12d(b-a) \\ &+ 12a(d-c) \\ &= \\ &24(b-a)(d-c) \quad , \end{aligned} \tag{29-76}$$

in accordance with the result in Equation (29-72), such that the general Equation (29-71) hereby is verified in this case.

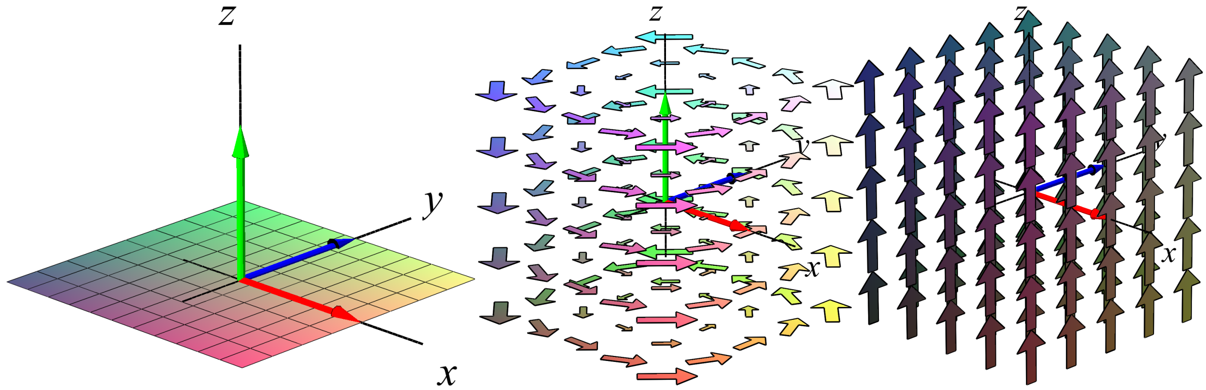

Figure 29.12: A 3D version of Example 29.20. The vector field $\mathbf{V}(x,y) = (-y,x,0)$ has the curl vector field $(0,0,2)$. Here a square is chosen with the parametric representation $\mathbf{r}(u,v) = (3u,\, 4v, 0)$, $u \in [a,b]$, $v \in [c,d]$, $a = -1/3$, $b = 1/3$, $c = -1/4$ and $d = 1/4$.

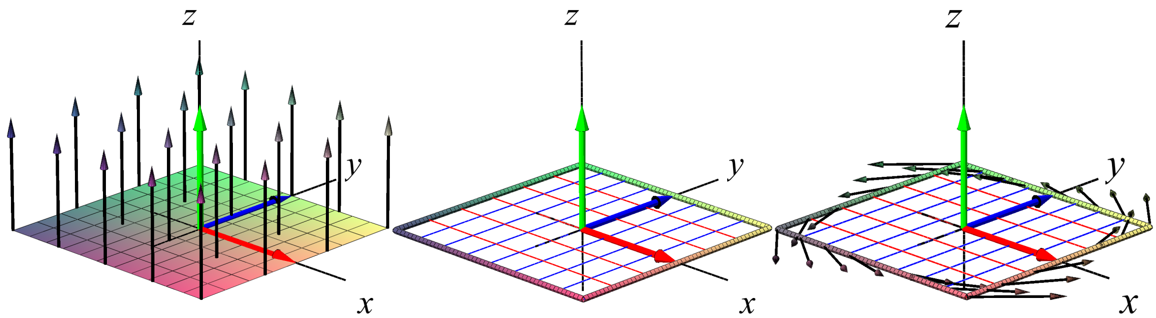

Figure 29.13: Statement of the flux of the vector field $\mathbf{Curl}(\mathbf{V}(x,y,z)) = (0,0,2)$ through the square in the (x,y) plane and corresponding circulation of the vector field $\mathbf{V}(x,y,z) = (-y,x,0)$ along the square boundary curve.

29.5 Summary

We have seen here that the flux of a vector field $\mathbf{Curl}(\mathbf{V})(x,y,z)$ through a surface can be calculated as the circulation of $\mathbf{V}(x,y,z)$ along the boundary curve to the surface – with suitable orientation.

- Let $F_{\mathbf{r}}$ denote a smooth parameterized surface with the boundary curve $\partial F_{\mathbf{r}}$ and the unit normal vector field $\mathbf{n}_F$ and let $\mathbf{V}$ be a smooth vector field in $\mathbb{R}^3$. Then Stokes' Theorem expresses the following identity

$$\int_F \mathbf{Curl}(\mathbf{V}) \cdot \mathbf{n}_F \,\mathrm{d}\mu = \int_{\partial F} \mathbf{V} \cdot \mathbf{e}_{\partial F} \,\mathrm{d}\mu \quad . \tag{29-77}$$

  In computing the right-hand side it is important (in order to get the correct sign) that the orientation of the boundary is chosen such that the cross product $\mathbf{e}_{\partial F} \times \mathbf{n}_F$ points away from the surface along the boundary.

- Alternatively Stokes' Theorem can be expressed like this: The flux of the *curl* of the vector field $\mathbf{V}$ through the surface $F_{\mathbf{r}}$ is equal to the *circulation* of the vector field along the closed boundary curve of the surface segment ∂F:

$$\mathrm{Flux}(\mathbf{Curl}(\mathbf{V}), F_{\mathbf{r}}) = \mathrm{Cirk}(\mathbf{V}, \partial F) \quad . \tag{29-78}$$

- The total curl of a vector field in a spatial region can similarly be computed as a surface integral over the total surface of the region: Let Ω be a spatial region with the boundary $\partial\Omega$ and outward-pointing unit normal vector field $\mathbf{n}_{\partial\Omega}$ on $\partial\Omega$. Then for every smooth vector field $\mathbf{V}(x,y,z)$ it applies that:

$$\int_\Omega \mathbf{Curl}(\mathbf{V}) \,\mathrm{d}\mu = \int_{\partial\Omega} \mathbf{n}_{\partial\Omega} \times \mathbf{V} \,\mathrm{d}\mu = \mathbf{Tors}(\mathbf{V}, \partial\Omega) \quad . \tag{29-79}$$