eNote 28

Gauss' Divergence Theorem

In this eNote we will use flow curves of vector fields to investigate how the surface of a spatial region is deformed with the flow and thus reflects a change in the volume of the region that is bounded by the surface. We shall need to know the analysis of vector fields from eNote 26 together with surface and volume integrals from eNote 25. We shall see that the divergence is the local 'motor' for volume change. Therefore if we integrate the divergence across the whole spatial region we get the total instantaneous volumetric expansion (with sign). If the vector field is everywhere exploding (divergence positive) then the volume of every spatial region increases; if the vector field is everywhere imploding (divergence negative) then the volume of every spatial region flowing with a vector field decreases.
Alternatively one can keep an eye on whether the surface of a spatial region is locally expanding or locally contracting with respect to the spatial region. This is exactly what we do with the orthogonal surface integral of the vector field with respect to the surface – an integral which is also called the flux.
By this we have two possibilities for computation of the expansion or contraction of a given spatial region when it flows along the flow curves of the vector field. And they yield the exact same result; this is the content of Gauss' divergence theorem.

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28.1 The Orthogonal Surface Integral, the Flux

Let $\mathbf{V}(x,y,z)$ be a smooth vector field in (x,y,z) space, see eNote 26, and let $F_\mathbf{r}$ denote a smooth parameterized surface:

$$F_\mathbf{r} \quad : \quad \mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \quad , \quad (u,v) \in [a,b] \times [c,d] \quad . \tag{28-1}$$

As with the construction of the line integrals in eNote 27 we then have at every point on the surface *two* well-defined vectors, on the one hand the value of the vector field at the point, $\mathbf{V}(\mathbf{r}(u,v))$, and on the other hand the normal vector $\mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)$ to the surface at the point. It is by using these two vectors that we construct the flux of the vector field through the surface.

The *orthogonal surface integral* of $\mathbf{V}(x,y,z)$ along a given parameterized surface $F_\mathbf{r}$ – also called the *flux* of $\mathbf{V}(x,y,z)$ through $F_\mathbf{r}$ – is the surface integral of the projection (with sign) of $\mathbf{V}(\mathbf{r}(u,v))$ on the normal to the surface represented by the standard unit vector $\mathbf{n}_F$ that is proportional to and has the same direction as the cross product $\mathbf{N}_F(u,v) = \mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)$.

|||| Definition 28.1 The Flux of a Vector Field

The orthogonal surface integral of the vector field $\mathbf{V}(x,y,z)$ along a parameterized surface $F_\mathbf{r}$, i.e. the flux of the vector field through the surface, is defined by

$$\mathrm{Flux}(\mathbf{V}, F_\mathbf{r}) = \int_{F_\mathbf{r}} \mathbf{V} \cdot \mathbf{n}_F \, \mathrm{d}\mu \quad . \tag{28-2}$$

The integrand in the surface integral that gives the flux is thus given by the scalar product

$$f(\mathbf{r}(u,v)) = \mathbf{V}(\mathbf{r}(u,v)) \cdot \mathbf{n}_F(u,v) \quad , \tag{28-3}$$

where $\mathbf{n}_F(u,v)$ is defined by

$$\mathbf{n}_F(u,v) = \begin{cases} \mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v) / \|\mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)\| & \text{as long as} \quad \mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v) \neq \mathbf{0} \\ \mathbf{0} \quad \text{if} \quad \mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v) = \mathbf{0} \end{cases} \tag{28-4}$$

Therefore the flux of $\mathbf{V}(x,y,z)$ through $F_\mathbf{r}$ in the direction $\mathbf{n}_F$ is relatively simple to compute - we need not first find the length of $\mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)$ (cf. the reduction of the tangential line integral):

$$
\begin{aligned}
\text{Flux}(\mathbf{V}, F_{\mathbf{r}}) &= \int_{F_{\mathbf{r}}} \mathbf{V} \cdot \mathbf{n}_F \, \mathrm{d}\mu \\
&= \int_c^d \int_a^b (\mathbf{V}(\mathbf{r}(u,v)) \cdot \mathbf{n}_F(u,v)) \; \text{Jacobian}_{\mathbf{r}}(u,v) \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_c^d \int_a^b (\mathbf{V}(\mathbf{r}(u,v)) \cdot \mathbf{n}_F(u,v)) \; \|\mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)\| \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_c^d \int_a^b \mathbf{V}(\mathbf{r}(u,v)) \cdot (\mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)) \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_c^d \int_a^b \mathbf{V}(\mathbf{r}(u,v)) \cdot \mathbf{N}_F(u,v) \; \mathrm{d}u \, \mathrm{d}v \quad .
\end{aligned}
\tag{28-5}
$$

Note that the last integrand in (28-5) is continuous and thus integrable, even though this is not evident from the definition, because the vector field $\mathbf{n}_F(u,v)$ is not necessarily continuous - unless $\mathbf{r}(u,v)$ is a regular parametric representation.

We thus have a simple expression for flux calculations:

|||| Theorem 28.2 The Flux, the Orthogonal Surface Integral

The orthogonal surface integral of $\mathbf{V}(x,y,z)$ over the surface $F_{\mathbf{r}}$, that is the flux of $\mathbf{V}(x,y,z)$ through $F_{\mathbf{r}}$, is calculated like this:

$$
\begin{aligned}
\text{Flux}(\mathbf{V}, F_{\mathbf{r}}) &= \int_c^d \int_a^b \mathbf{V}(\mathbf{r}(u,v)) \cdot (\mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)) \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_c^d \int_a^b \mathbf{V}(\mathbf{r}(u,v)) \cdot \mathbf{N}_F(u,v) \; \mathrm{d}u \, \mathrm{d}v \quad .
\end{aligned}
\tag{28-6}
$$

Note that if a surface $F_{\mathbf{r}}$ is parameterized by another parametric representation $\widehat{F}_{\widehat{\mathbf{r}}}$ that gives the opposite standard unit normal vector $\mathbf{n}_{\widehat{F}}$ at every point on the surface: $\mathbf{n}_{\widehat{F}}(\widehat{u}, \widehat{v}) = -\mathbf{n}_F(u,v)$, where $\mathbf{r}(u,v) = \widehat{\mathbf{r}}(\widehat{u}, \widehat{v})$, then the flux changes sign:

$$
\text{Flux}(\mathbf{V}, \widehat{F}_{\widehat{\mathbf{r}}}) = -\,\text{Flux}(\mathbf{V}, F_{\mathbf{r}}) \quad . \tag{28-7}
$$

The orthogonal line integral is briefly introduced as a dual concept in relation to the more natural tangential line integral in eNote 27. The dual concept in relation to the natural orthogonal surface integral introduced above is the *tangential surface integral* of a given vector field over a given surface.

|||| Definition 28.3 The Tangential Surface Integral

In analogy to the orthogonal surface integral, the flux, we define the *tangential* surface integral, which we will denote $\text{Tan}(\mathbf{V}, F_{\mathbf{r}})$ of $\mathbf{V}$ over the surface $F_{\mathbf{r}}$, by projecting $\mathbf{V}(\mathbf{r}(u,v))$ perpendicularly onto the tangent plane to $F_{\mathbf{r}}$ (spanned by $\mathbf{r}'_u(u,v)$ and $\mathbf{r}'_v(u,v)$ at the point $\mathbf{r}(u,v)$) and then finding the surface integral of the length of this projection (as a function of (u,v)).

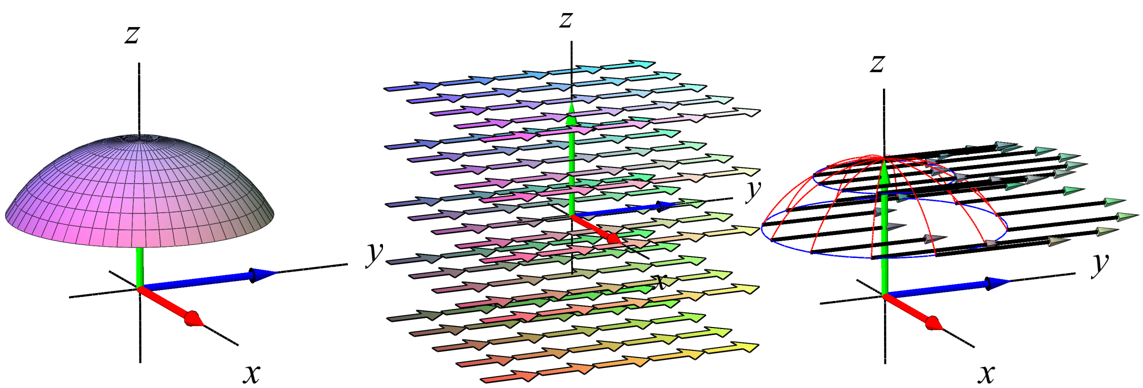

Figure 28.1: This segment of a spherical surface is given by the parametric representation $\mathbf{r}(u,v) = (\sin(u)\cos(v), \sin(u)\sin(v), \cos(u))$, $u \in [0, \frac{\pi}{3}]$, $v \in [-\pi, \pi]$. The vector field is given by $\mathbf{V}(x,y,z) = (0,1,0)$.

|||| Exercise 28.4

Concerning the figures 28.1 and 28.2:

1. Determine the tangential surface integral of each of the vector fields $\mathbf{V}(x,y,z) = (0,1,0)$ and $\mathbf{V}(x,y,z) = (1/\sqrt{5}, 0, -2/\sqrt{5})$ along the spherical segment.
2. Determine the respective orthogonal surface integrals (the fluxes) of each of the vector fields through the spherical segment.

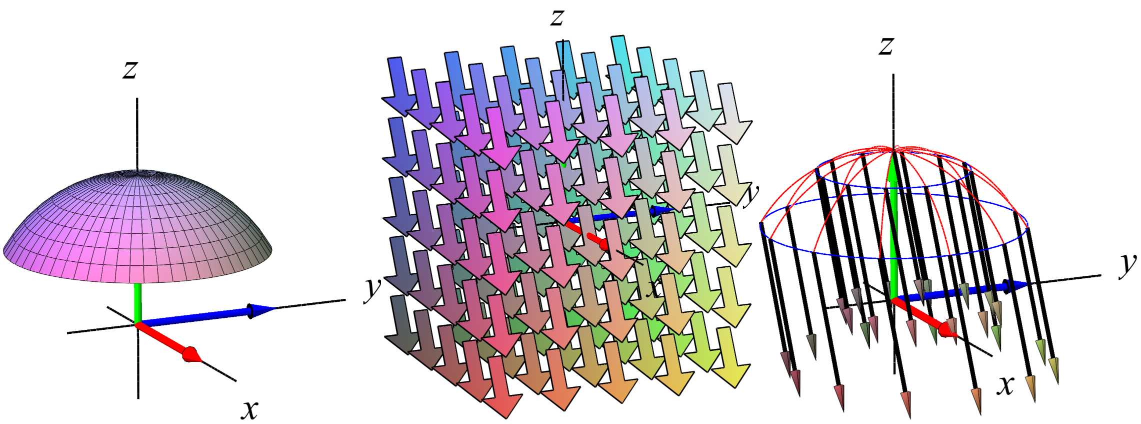

Figure 28.2: Same spherical segment as in Figure 28.1. The vector field is here given by $\mathbf{V}(x, y, z) = (1, 0, -2)/\sqrt{5})$.

|||| Exercise 28.5 A Sun Collector Problem

A sun collector roof has the form of part of the cylinder with the equation $x^2 + \left(z - \frac{1}{2}\right)^2 = 1$, viz. the part that lies above the $(x, y)-$plane and that is bounded by $-1 \leq y \leq 1$ (as usual we assume that the $(x, y)-$plane is horizontal and that 'above' means $z \geq 0$), see Figure 28.3.

- Let us – to simplify somewhat – assume that the sun radiates from a cloud-free sky onto the sun collector roof at a given point in time t along the unit vector field in 3D space that at time t is parallel to the vector $\mathbf{V} = \mathbf{V}(t) = (0, -\cos(t), -\sin(t))$ where $t \in [0, \pi]$.

- The sun rises at time $t = 0$ and at this point in time sends horizontal rays parallel to the y-axis in the direction $(0, -1, 0)$. At noon ($t = \frac{\pi}{2}$) the rays are vertical and parallel to the z-axis in the direction $(0, 0, -1)$. At time $t = \pi$ the sun settles but just before this happens it radiates (almost) horizontal rays parallel to the y-axis in the direction $(0, 1, 0)$.

- The energy uptake of the sun collector per area unit and per time unit for a given position of the sun collector is assumed to be equal to the scalar product $\mathbf{V} \cdot \mathbf{n}$ between the sun radiation vector field $\mathbf{V}$ and the roof's *inward* unit normal vector to the surface of the roof $\mathbf{n}$ at the position. Note that the inward normal field $\mathbf{n}$ is not necessarily equal to $\mathbf{n}_F$, this of course depending on the chosen parametric representation of the roof.

- Question A:

1. Argue the assumption that the energy uptake is equal to the scalar product $\mathbf{V} \cdot \mathbf{n}$, and note that energy uptake naturally only happens whenever this scalar product is positive.
2. What is the energy uptake of the sun collector per unit time at a given point in time, t , during a day?
3. What is the total energy uptake of the sun collector during a day?

- Question B: Suppose that we rotate the cylindrical roof $\pi/2$ counter-clockwise (or clockwise) about the $z-$axis.

  1. What is the rotated sun collector's energy uptake per unit time at a given point in time, t , of the day?
  2. What is the total energy uptake of the rotated sun collector in one 'day'?

- Question C: Suppose we rotate the original cylindric roof from question A the angle θ counter-clockwise (or clockwise) about the $z-$axis, where $\theta \in [0, \pi/2]$.

  1. Which of these sun collector roofs gives the largest total energy uptake per day?

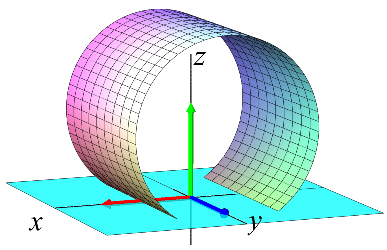

Figure 28.3: The sun collector roof that is referred to in Exercise 28.5.

28.2 Motivation for the Flux via Flow Expansion

The integral curves, the flow curves, of a given vector field $\mathbf{V}(x, y, z)$ can be used to 'push' a given surface $F_{\mathbf{r}}$ in the direction of the vector field and with a local speed that is given by the length of the vector field at every point. In other words: every point on the surface flows for a certain time along the flow curves of the vector field.

In this way the surface runs through – the surface 'sweeps' through – a spatial region $\Omega_{\mathbf{r}}(t)$ which at any instant in time t (the time which we let the surface flow along the vector field) has a volume $\text{Vol}_{\pm}(\Omega_{\mathbf{r}}(t)) = \text{Vol}_{\pm}(t)$. This volume is clearly 0 for $t = 0$ and thereafter it is small if the vector field is almost tangential to the surface and large if the vector field is perpendicular to the surface in the same direction as the standard normal vector field to the surface. If the vector field points in the opposite direction of the standard normal vector field we will calculate the volume with the corresponding local contribution to the volume as being negative – hence the designation $\text{Vol}_{\pm}(\Omega_{\mathbf{r}}(t))$.

See the Figures 28.4, 28.5 and 28.7 together with the Examples 28.7 and 28.8 where it is illustrated how the Jacobian function (without numerical sign) can be applied for the calculation of local contributions to the volume (with sign).

The following fundamental relation between the flux and the derivative of the volume function at the time $t = 0$ applies:

|||| Theorem 28.6 The Flux as the Derivative of the Volume in Surface Flows

Let $\Omega_{\mathbf{r}}(t)$ designate the spatial region that is swept through when the surface segment $F_{\mathbf{r}}$ flows for time t with the flow curves through the point of the surface. Then the following relation applies, where $\text{Vol}_{\pm}(\Omega_{\mathbf{r}}(t))$ denotes the signed calculated volume of the region, that is, in relation to the chosen standard normal vector $\mathbf{n}_F(u,v)$ for $F_{\mathbf{r}}$.

$$\text{Flux}(\mathbf{V}, F_{\mathbf{r}}) = \frac{d}{dt}_{|_{t=0}} \text{Vol}_{\pm}(\Omega_{\mathbf{r}}(t)) = \text{Vol}'_{\pm}(0) \quad . \tag{28-8}$$

It is this property (in Theorem 28.6) that motivates the *name* flux: Locally the flux is a measure for the (with sign calculated) volume growth rate that is instantaneously generated by the flow of the surface along the integral curves of the vector field through the points of the surface. The sign is positive where the vector field forms an acute angle with the standard normal vector field $\mathbf{n}_F(u,v)$ and negative where that angle is obtuse. See Figures 28.7 and 28.4.

|||| Proof

We assume as usual that the surface $F_{\mathbf{r}}$ is given by a smooth parametric representation:

$$F_{\mathbf{r}} \quad : \quad \mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \quad , \quad (u,v) \in [a,b] \times [c,d] \quad . \tag{28-9}$$

The flow curve of $\mathbf{V}(x,y,z)$ through the surface point $\mathbf{r}(u,v)$ is called $\tilde{\mathbf{r}}(u,v,t)$. The region $\Omega_{F_{\mathbf{r}}}(t)$ in (x,y,z) space that is swept by the surface while it flows with the flow curves until the time t is therefore given by the parametric representation:

$$\Omega_{F_{\mathbf{r}}}(t) \quad : \quad \tilde{\mathbf{r}}(u,v,w) \quad , \quad w \in [0,t] \ , \ u \in [a,b] \ , \ v \in [c,d] \quad . \tag{28-10}$$

For the determination of the volume (signed) of this region we use Taylor's limit formula to the first order with the development point $w = 0$ for each of the smooth vector functions $\tilde{\mathbf{r}}'_u(u,v,w)$, $\tilde{\mathbf{r}}'_v(u,v,w)$ and $\tilde{\mathbf{r}}'_w(u,v,w)$ considered to be functions of w for every fixed (u,v):

$$\begin{aligned}
\tilde{\mathbf{r}}'_u(u,v,w) &= \tilde{\mathbf{r}}'_u(u,v,0) + \boldsymbol{\varepsilon}_{(u,v)}(w) = \mathbf{r}'_u(u,v) + \boldsymbol{\varepsilon}_{(u,v)}(w) \\
\tilde{\mathbf{r}}'_v(u,v,w) &= \tilde{\mathbf{r}}'_v(u,v,0) + \boldsymbol{\varepsilon}_{(u,v)}(w) = \mathbf{r}'_v(u,v) + \boldsymbol{\varepsilon}_{(u,v)}(w) \\
\tilde{\mathbf{r}}'_w(u,v,w) &= \tilde{\mathbf{r}}'_w(u,v,0) + \boldsymbol{\varepsilon}_{(u,v)}(w) = \mathbf{V}(\mathbf{r}(u,v)) + \boldsymbol{\varepsilon}_{(u,v)}(w) \quad ,
\end{aligned} \tag{28-11}$$

so that the Jacobian function for the volume calculation – signed – looks like this:

$$\begin{aligned}
\text{Jacobian}_{\tilde{\mathbf{r}}}(u,v,w) &= \left(\tilde{\mathbf{r}}'_u(u,v,w) \times \tilde{\mathbf{r}}'_v(u,v,w)\right) \cdot \tilde{\mathbf{r}}'_w(u,v,w) \\
&= \left(\mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)\right) \cdot \mathbf{V}(\mathbf{r}(u,v)) + \varepsilon_{(u,v)}(w) \\
&= \mathbf{N}_F(u,v) \cdot \mathbf{V}(\mathbf{r}(u,v)) + \varepsilon_{(u,v)}(w) \quad .
\end{aligned} \tag{28-12}$$

Here we see that when w tends towards 0 we obtain precisely the wanted sign of the Jacobian function, which is the local contribution l (the integrand) to the volume computation: The Jacobian function is positive *close to the surface* when the vector field $\mathbf{V}(\mathbf{r}(u,v))$ *on the surface* points in the same direction as the standard normal vector $\mathbf{N}_F(u,v)$, and the Jacobian function is negative *close to the surface* when the vector field $\mathbf{V}(\mathbf{r}(u,v))$ *on the surface* points in the opposite direction of the normal vector $\mathbf{N}_F(u,v)$.

The volume $\text{Vol}_{\pm}(\Omega_{\mathbf{r}}(t))$ (computed with sign) is now for sufficiently small flow times t given by:

$$\begin{aligned}
\text{Vol}_{\pm}(\Omega_{\mathbf{r}}(t)) &= \int_0^t \int_c^d \int_a^b \text{Jacobian}_{\tilde{\mathbf{r}}}(u,v,w) \ \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w \\
&= \int_0^t \int_c^d \int_a^b \left(\mathbf{N}_F(u,v) \cdot \mathbf{V}(\mathbf{r}(u,v)) + \varepsilon_{(u,v)}(w)\right) \ \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w \\
&= \int_0^t \left(\left(\int_c^d \int_a^b \mathbf{N}_F(u,v) \cdot \mathbf{V}(\mathbf{r}(u,v)) \ \mathrm{d}u \, \mathrm{d}v \right) + \left(\int_c^d \int_a^b \varepsilon_{(u,v)}(w) \ \mathrm{d}u \, \mathrm{d}v \right) \right) \ \mathrm{d}w \\
&= \int_0^t \text{Flux}(\mathbf{V}, F_{\mathbf{r}}) \ \mathrm{d}w + \int_0^t \varepsilon_{(u,v)}(w) \ \mathrm{d}w \quad ,
\end{aligned} \tag{28-13}$$

from which we read:

$$\frac{d}{dt}\,\mathrm{Vol}_{\pm}(\Omega_{\mathbf{r}}(t)) = \mathrm{Flux}(\mathbf{V}, F_{\mathbf{r}}) + \varepsilon(t) \tag{28-14}$$

and thus

$$\mathrm{Vol}'_{\pm}(0) = \mathrm{Flux}(\mathbf{V}, F_{\mathbf{r}}) \quad . \tag{28-15}$$

■

|||| Example 28.7 Flow of a Circular Disc by Parallel Displacement

We look at a circular disc $F_{\mathbf{r}}$ in the (x, y) plane. The radius of the disc is 1 and the centre is at $(0, 0, 0)$:

$$F_{\mathbf{r}} \quad : \quad \mathbf{r}(u, v) = (u \cdot \cos(v), u \cdot \sin(v), 0) \quad , \quad (u, v) \in [0, 1] \times [-\pi, \pi] \quad . \tag{28-16}$$

The standard normal vector to the circular disc is given by the parametric representation:

$$\begin{aligned} \mathbf{N}_F(u, v) &= \mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v) \\ &= (\cos(v), \sin(v), 0) \times (-u \cdot \sin(v), u \cdot \cos(v), 0) \\ &= (0, 0, u) \quad . \end{aligned} \tag{28-17}$$

The unit normal vector field of the circular disc with this circular representation is the constant vector:

$$\mathbf{n}_F = (0, 0, 1) \quad . \tag{28-18}$$

Now let $\mathbf{V}(x, y, z)$ denote the vector field:

$$\mathbf{V}(x, y, z) = (\alpha, \beta, \gamma) \quad , \tag{28-19}$$

where α, β and γ are constants. Then in this case the flux is given by:

$$\begin{aligned} \mathrm{Flux}(\mathbf{V}, F_{\mathbf{r}}) &= \int_{-\pi}^{\pi} \int_0^1 \mathbf{V}(\mathbf{r}(u, v)) \cdot \mathbf{N}_F(u, v)\; \mathrm{d}u\, \mathrm{d}v \\ &= \int_{-\pi}^{\pi} \int_0^1 (\alpha, \beta, \gamma) \cdot (0, 0, u)\; \mathrm{d}u\, \mathrm{d}v \\ &= \int_{-\pi}^{\pi} \int_0^1 u \cdot \gamma\; \mathrm{d}u\, \mathrm{d}v \\ &= \gamma \cdot \pi \quad , \end{aligned} \tag{28-20}$$

Note that the flux is negative for $\gamma < 0$ and positive for $\gamma > 0$, that is, the sign depends on whether the vector field points in the direction of the standard normal vector of the circular disc or in the opposite direction.

Now we will in a similar way determine the volume of the spatial region that the flow of the circular disc 'sweeps through' by following the integral curves, thus demonstrating the content – and validity – of Theorem 28.6:

The flow curve $\bar{\mathbf{r}}(u,v,t)$ of $\mathbf{V}(x,y,z)$ through the point $\mathbf{r}(u,v)$ is a straight line through the point, viz. the one that has the constant tangent vector (α,β,γ):

$$\begin{aligned}\bar{\mathbf{r}}(u,v,t) &= \mathbf{r}(u,v) + t\cdot(\alpha,\beta,\gamma) \\ &= (u\cdot\cos(v), u\cdot\sin(v), 0) + t\cdot(\alpha,\beta,\gamma) \quad , \quad t\in[0,\infty[\quad .\end{aligned} \tag{28-21}$$

It is seen that

$$\bar{\mathbf{r}}_t'(u,v,t) = (\alpha,\beta,\gamma) = \mathbf{V}(\bar{\mathbf{r}}(u,v,t)) \quad , \tag{28-22}$$

such that the integral curve condition is exactly fulfilled.

The spatial region $\Omega_{F_\mathbf{r}}(t)$, which the circular disc forms by flowing with the flow curves of $\mathbf{V}(x,y,z)$ until time t, is thereby given by a parametric representation that is readily seen from the parametric representation of the flow curve through the points of the surface:

$$\Omega_{F_\mathbf{r}}(t) \quad : \quad \bar{\mathbf{r}} = \bar{\mathbf{r}}(u,v,w) \quad , \quad w\in[0,t] \quad , \quad (u,v)\in[0,1]\times[-\pi,\pi] \quad . \tag{28-23}$$

The volume of this region can be found by the standard method via the Jacobian function, which we use here *with sign* in order to determine the volume *with sign* in relation to the normal vector:

$$\begin{aligned}\text{Jacobian}_{\bar{\mathbf{r}}}(u,v,w) &= (\bar{\mathbf{r}}_u(u,v,w)\times\bar{\mathbf{r}}_v(u,v,w))\bullet\bar{\mathbf{r}}_w(u,v,w) \\ &= \mathbf{N}_F(u,v)\bullet\bar{\mathbf{r}}_w(u,v,w) \\ &= (\cos(v),\sin(v),0)\times(-u\cdot\sin(v), u\cdot\cos(v),0)\bullet(\alpha,\beta,\gamma) \\ &= (0,0,u)\bullet(\alpha,\beta,\gamma) \\ &= u\cdot\gamma \quad ,\end{aligned} \tag{28-24}$$

where we have used that in this simple actual case, where the vector field moves the circular disc in the direction of the vector field, it applies that: $(\bar{\mathbf{r}}_u(u,v,w)\times\bar{\mathbf{r}}_v(u,v,w)) = \mathbf{N}_F(u,v)$, which is independent of w. The volume *with sign* of $\Omega_{F_\mathbf{r}}(t)$ into the normal vector field $(0,0,1)$ of the circular disc is therefore

$$\text{Vol}_\pm(t) = \int_0^t\int_{-\pi}^{\pi}\int_0^1 u\cdot\gamma\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = t\cdot\gamma\cdot\pi \quad . \tag{28-25}$$

From this we get

$$\text{Vol}_\pm'(0) = \gamma\cdot\pi \quad , \tag{28-26}$$

that is the same value as the $\text{Flux}(\mathbf{V}, F_\mathbf{r})$ found above. Thus we have verified Theorem 28.6.

If the vector field (α, β, γ) is in the same direction as the standard normal to the surface (i.e. if $\gamma > 0$), both the flux, the volume $\text{Vol}_{\pm}(t)$, and the volume derivative $\text{Vol}'_{\pm}(0)$ are positive; if the vector field (α, β, γ) is in the opposite direction in relation to the standard normal (i.e. if $\gamma < 0$), then both the flux, the volume (with sign) $\text{Vol}_{\pm}(t)$ and the volume derivative $\text{Vol}'_{\pm}(0)$ are negative.

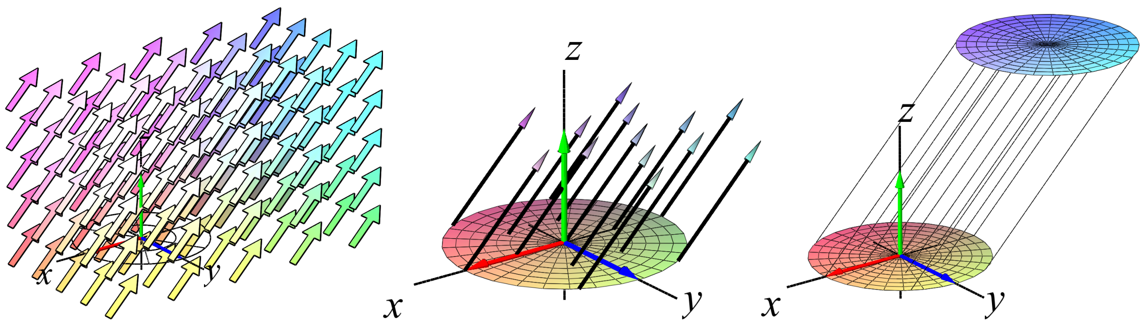

Figure 28.4: A simple, constant vector field and the corresponding short-time flow of a circular disc. In relation to the normal $(0, 0, 1)$ of the circular disc the flux and the volumetric increase are positive.

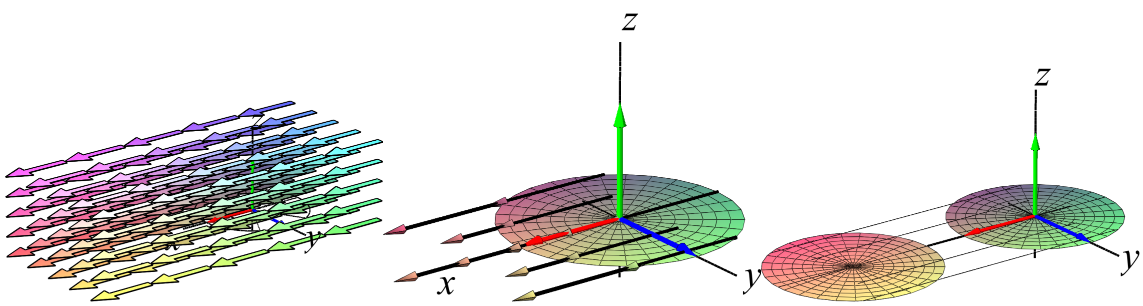

Figure 28.5: A simple, constant vector field and the corresponding short-time flow of a circular disc. The flux and the volumetric increase are 0.

|||| Example 28.8 Flow of a Circular Disc with Rotation

We consider as in Example 28.7 the circular disc $F_{\mathbf{r}}$ in the (x, y) plane. The disc has radius 1 and centre at $(0, 0, 0)$:

$$F_{\mathbf{r}} \quad : \quad \mathbf{r}(u, v) = (u \cdot \cos(v), u \cdot \sin(v), 0) \quad , \quad (u, v) \in [0, 1] \times [-\pi, \pi] \quad . \tag{28-27}$$

Here we will let the circular disc flow with the flow curves of the following rotating vector field:

$$\mathbf{V}(x,y,z) = (-z, 0, x) \quad . \tag{28-28}$$

The flow curve $\widetilde{\mathbf{r}}(u,v,t) = (x(t), y(t), z(t))$ of this vector field through a circular disc point $(x_0, y_0, z_0) = \mathbf{r}(u,v)$ is given as the solution to the system of first-order differential equations:

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = (\mathbf{V}(x(t), y(t), z(t)))^\top = \begin{bmatrix} -z(t) \\ 0 \\ x(t) \end{bmatrix} \tag{28-29}$$

with the initial condition $(x(0), y(0), z(0) = (x_0, y_0, z_0) = \mathbf{r}(u,v)$. The system of coupled differential equations has the general solution, see the solution methods in eNote 17:

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1 \cdot \cos(t) - c_3 \cdot \sin(t) \\ c_2 \\ c_1 \cdot \sin(t) + c_3 \cdot \cos(t) \end{bmatrix} \tag{28-30}$$

where c_1, c_2 and c_3, are arbitrary constants. The special solutions, the flow curves $\widetilde{\mathbf{r}}(u,v,t)$ through the circular disc points $(x_0, y_0, z_0) = \mathbf{r}(u,v) = (u \cdot \cos(v), u \cdot \sin(v), 0)$ are then given by the following parameterized circles in (x,y,z) space:

$$(\widetilde{\mathbf{r}}(u,v,t))^\top = \begin{bmatrix} x_0 \cdot \cos(t) - z_0 \cdot \sin(t) \\ y_0 \\ x_0 \cdot \sin(t) + y_0 \cdot \cos(t) \end{bmatrix} = \begin{bmatrix} u \cdot \cos(v) \cdot \cos(t) \\ u \cdot \sin(v) \\ u \cdot \cos(v) \cdot \sin(t) \end{bmatrix} \quad . \tag{28-31}$$

The swept spatial region $\Omega_{F_\mathbf{r}}(t)$ is already parameterized in this way:

$$\Omega_{F_\mathbf{r}}(t) \quad : \quad \widetilde{\mathbf{r}}(u,v,w) = (u \cdot \cos(v) \cdot \cos(w),\ u \cdot \sin(v),\ u \cdot \cos(v) \cdot \sin(w)) \quad , \tag{28-32}$$

where $w \in [0,t]$, $u \in [0,1]$, and $v \in [-\pi, \pi]$. Cf. Figure 28.6.

The flux of the vector field through the circular disc is expected to be 0 because the region that the circular disc sweeps through during the rotation has the *volume* 0 when the volume is signed: One half of the region is seen to be *above the circular disc* (in the direction of $(0,0,1)$) and the other half *below the circular disc* (in the direction of $(0,0,-1)$); the two halves of the swept region have volumes with opposite signs and therefore they cancel each other. We compute the flux of the vector field through the circular disc:

$$\begin{aligned} \text{Flux}(\mathbf{V}, F_\mathbf{r}) &= \int_{-\pi}^{\pi} \int_0^1 \mathbf{V}(\mathbf{r}(u,v)) \bullet \mathbf{N}_F(u,v)\ \mathrm{d}u\,\mathrm{d}v \\ &= \int_{-\pi}^{\pi} \int_0^1 (0,0,u \cdot \cos(v)) \bullet (0,0,u)\ \mathrm{d}u\,\mathrm{d}v \\ &= \int_{-\pi}^{\pi} \int_0^1 u^2 \cdot \cos(v)\ \mathrm{d}u\,\mathrm{d}v \\ &= \int_{-\pi}^{\pi} \frac{1}{3} \cdot \cos(v)\ \mathrm{d}v \\ &= \frac{1}{3} \cdot \int_{-\pi}^{\pi} \cos(v)\ \mathrm{d}v \\ &= \frac{1}{3} \cdot [\sin(v)]_{-\pi}^{\pi} \\ &= 0 \quad . \end{aligned} \tag{28-33}$$

In order to illustrate once more the general volume calculation (with sign) and again verify Theorem 28.6 through concrete computations, we will show that the signed calculated volume of the region $\Omega_{F_r}(t)$ being swept by the circular disc by the flow really is 0. The signed calculated Jacobian function is determined by:

$$\text{Jacobian}_{\tilde{\mathbf{r}}}(u,v,w) = u^2 \cdot \cos(v) \quad . \tag{28-34}$$

such that the signed volume of $\Omega_{F_r}(t)$ is:

$$\begin{aligned} \text{Vol}_{\pm}(\Omega_{F_r}(t)) &= \int_0^t \int_{-\pi}^{\pi} \int_0^1 u^2 \cdot \cos(v)\; \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}w \\ &= \frac{1}{3} \cdot \int_0^t \int_{-\pi}^{\pi} \cdot \cos(v)\; \mathrm{d}v\, \mathrm{d}w \\ &= \frac{1}{3} \cdot t \cdot [\sin(v)]_{-\pi}^{\pi} \\ &= 0 \quad . \end{aligned} \tag{28-35}$$

Note that the ordinary volume $\text{Vol}(\Omega_{F_r}(t))$ of $\Omega_{F_r}(t)$ is of course not 0, but:

$$\begin{aligned} \text{Vol}(\Omega_{F_r}(t)) &= \int_0^t \int_{-\pi}^{\pi} \int_0^1 u^2 \cdot |\cos(v)|\; \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}w \\ &= \frac{1}{3} \cdot \int_0^t \int_{-\pi}^{\pi} |\cos(v)|\; \mathrm{d}v\, \mathrm{d}w \\ &= \frac{1}{3} \cdot t \cdot 4 \cdot [\sin(v)]_0^{\pi/2} \\ &= \frac{4}{3} \cdot t \quad . \end{aligned} \tag{28-36}$$

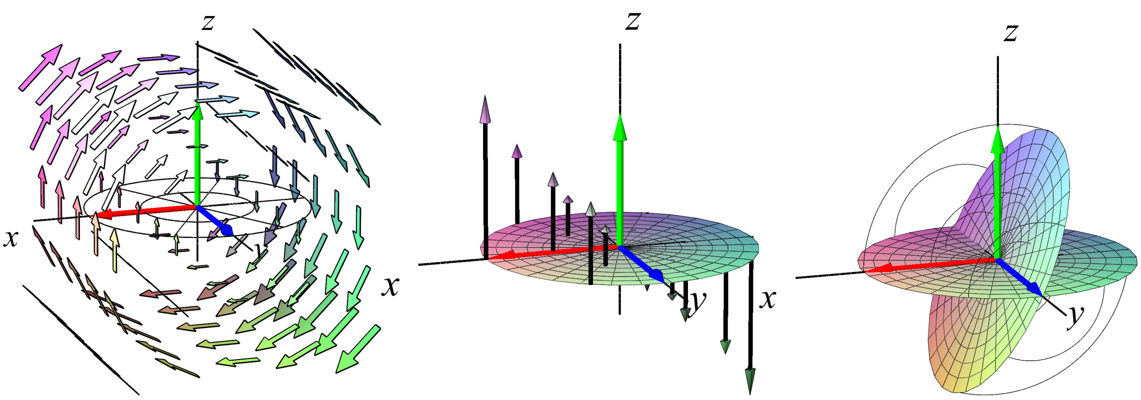

Figure 28.6: A rotating vector field and corresponding short-time flow of a circular disc. The flux and the volume increase are both 0.

|||| Example 28.9 Flux Through an Elliptic Paraboloid

A vector field $\mathbf{V}(x, y, z)$ and an elliptic paraboloid $F_{\mathbf{r}}$ are given by

$$\mathbf{r}(u, v) = \left(2 \cdot u \cdot \cos(v), 2 \cdot u \cdot \sin(v), 2 \cdot u^2 \cdot (\cos^2(v) + \frac{1}{9} \cdot \sin^2(v))\right) \quad , \tag{28-37}$$

where $(u, v) \in [0, 1/2] \times [-\pi, \pi]$ and $\mathbf{V}(x, y, z) = (-y, x, 1)$.

For the determination of the flux $\mathrm{Flux}(\mathbf{V}, F_{\mathbf{r}})$ of $\mathbf{V}(x, y, z)$ through $F_{\mathbf{r}}$ we have

$$\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v) = \left(-8 \cdot u^2 \cdot \cos(v), -\frac{8}{9} \cdot u^2 \cdot \sin(v), 4 \cdot u\right) \tag{28-38}$$

$$\mathbf{V}(\mathbf{r}(u, v)) = (-2 \cdot u \cdot \sin(v), 2 \cdot u \cdot \cos(v), 1) \quad . \tag{28-39}$$

such that:

$$\begin{aligned} \mathrm{Flux}(\mathbf{V}, F_{\mathbf{r}}) &= \int_{-\pi}^{\pi} \int_0^{1/2} \mathbf{V}(\mathbf{r}(u, v)) \bullet (\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)) \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_{-\pi}^{\pi} \int_0^{1/2} \frac{4}{9} \cdot u \cdot \left(32 \cdot u^2 \cdot \cos(v) \cdot \sin(v) + 9\right) \, \mathrm{d}u \, \mathrm{d}v \\ &= \cdots \\ &= \pi \quad . \end{aligned} \tag{28-40}$$

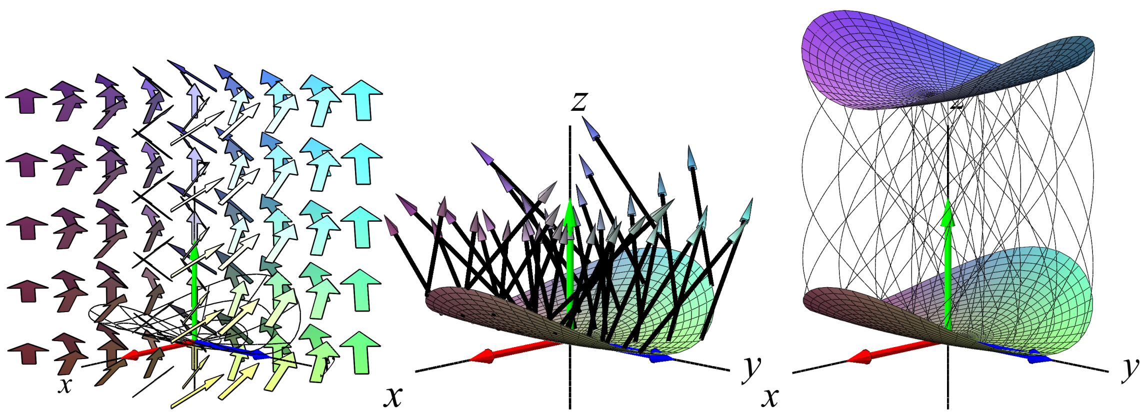

Figure 28.7: A rotating vector field, and corresponding short-time flow of an elliptic paraboloid. In relation to the normal to the circular disc the flux and the volume increase are positive. See Example 28.9.

As we shall see in the next paragraph the total flux of a given vector field through the *surface* of a given spatial region $\Omega_\mathbf{r}$ is particularly interesting. The surface may well be composed of finitely many smooth surface segments, e.g. like the surface of a polyhedron. The only convention that we shall stick to is that it is always the *outward-pointing normal* (in relation to the spatial region) we shall use everywhere on the surface when computing the flux.

|||| Example 28.10 Flux Through the Surface of a Solid Cylinder

We will illustrate the flux calculation through a total surface of a spatial region by computing the flux of the simple vector field

$$\mathbf{V}(x,y,z) = (f(x),0,0) \quad , \quad (x,y,z) \in \mathbb{R}^3 \quad , \tag{28-41}$$

where $f(x)$ is a smooth function of x. We also choose a very simple region Ω in (x,y,z) space, viz. the solid cylinder with radius $1/2$ and the x-axis as an axis of symmetry together with the corresponding x-interval $x \in [0,1]$. See Figure 28.8.

The surface, the boundary, of the cylinder, which we will denote by $F = \partial\Omega$, consists of three parts: One is the cylindrical curved surface and the two others are the two circular discs at its ends. Since the vector field is parallel to the cylindrical curved part of the surface there will be no flux contribution from this (!). E.g. the only contribution to the flux through the total surface of the solid cylinder derives from the circular discs at the two ends.

Since the vector field is perpendicular to both of these discs the flux through the end circular disc at $x = 0$ is given by: $-f(0) \cdot \frac{\pi}{4}$, since the outward pointing unit normal for that circular disc is $(-1,0,0)$ and the flux through the end circular disc at $x = 1$ is similarly given by: $f(1) \cdot \frac{\pi}{4}$, since the unit normal there is $(1,0,0)$.

The total flux of $\mathbf{V}(x,y,z)$ out through the surface $\partial\Omega$ of the solid cylinder is therefore:

$$\text{Flux}(\mathbf{V},\partial\Omega) = (f(1) - f(0)) \cdot \frac{\pi}{4} \quad . \tag{28-42}$$

I.e.: If $f(x)$ (and thus the vector field) is larger at $x = 0$ than at $x = 1$, then the total flux *out* through the cylindrical surface is negative.

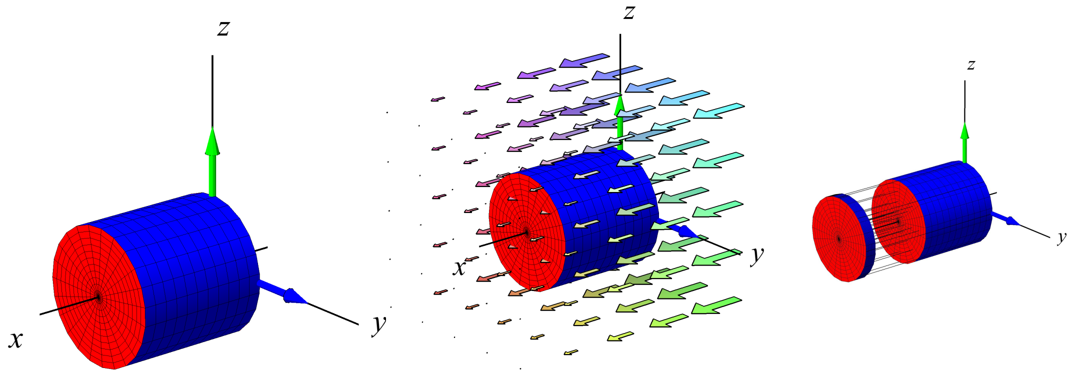

Figure 28.8: A vector field that is parallel to the x-axis and an image after a short time flow. After very long time of flowing the whole cylinder squished together to a circular disc at $x = 2$. See Example 28.10.

The flux through the two end discs on the cylinder in Figure 28.8 depends on the value of the vector field at these end points – see Example 28.10. Since the volume of the cylinder is indicated to reduced by the flow this inspection gives rise to the supposition that the total flux *out* through the two end discs is negative. This is in accordance with the fact that the vector field for the case shown is $\mathbf{V}(x,y,z) = (f(x),0,0) = (2-x,0,0)$ which according to the computation in example 28.10 gives the total flux:

$$\text{Flux}(\mathbf{V},\partial\Omega) = (1-2)\cdot\frac{\pi}{4} = -\frac{\pi}{4} \quad . \tag{28-43}$$

This means that there is a larger volume deformation *into the cylinder* at $x = 0$ than there is volume deformation *out of* the cylinder at $x = 1$. The time derivative of the volume by the flow is negative such that the cylinder actually gets smaller when all points follow their respective flow curves. This is in fact true not only for small t-values but for all $t > 0$, such that the cylinder ends up collapsed, totally compressed to a flat circular disc at $x = 2$ at the time $t = \infty$.

28.3 Motivation for the Divergence via Flow Expansion

Let us consider a solid sphere K_ρ in (x,y,z) space with radius ρ and centre at $(0,0,0)$ and let us expand this sphere by letting all points flow with the flow curves of the explosion vector field $\mathbf{V}(x,y,z) = (x,y,z)$.

The solid sphere has the parametric representation:

$$K_\rho \quad : \quad \mathbf{r}(u,v,w) = (u \cdot \sin(v) \cdot \cos(w), u \cdot \sin(v) \cdot \sin(w), u \cdot \cos(v)) \quad , \qquad \text{(28-44)}$$

where $(u,v,w) \in [0,\rho] \times [0,\pi] \times [-\pi,\pi]$.

According to the flow solutions to this explosion vector field the radius of the sphere grows by the factor e^T, where T is the flow time. We see this in the following way:

The general flow curves of the vector field are readily found by solving the system of differential equations:

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = (\mathbf{V}(x(t),y(t),z(t)))^\top = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \quad . \qquad \text{(28-45)}$$

The general solution is here:

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1 \cdot \mathrm{e}^t \\ c_2 \cdot \mathrm{e}^t \\ c_3 \cdot \mathrm{e}^t \end{bmatrix} \qquad \text{(28-46)}$$

where c_1, c_2, og c_3 are arbitrary constants.

The special solutions, the flow curves $\widetilde{\mathbf{r}}(u,v,w,t)$ through points of the sphere $(x_0,y_0,z_0) = \mathbf{r}(u,v,w) = (u \cdot \sin(v) \cdot \cos(w), u \cdot \sin(v) \cdot \sin(w), u \cdot \cos(v))$, are then given by the following parameterized lines in (x,y,z) space:

$$(\widetilde{\mathbf{r}}(u,v,w,t))^\top = \begin{bmatrix} x_0 \cdot \mathrm{e}^t \\ y_0 \cdot \mathrm{e}^t \\ z_0 \cdot \mathrm{e}^t \end{bmatrix} = \begin{bmatrix} \mathrm{e}^t \cdot (u \cdot \sin(v) \cdot \cos(w)) \\ \mathrm{e}^t \cdot (u \cdot \sin(v) \cdot \sin(w)) \\ \mathrm{e}^t \cdot (u \cdot \cos(v)) \end{bmatrix} \quad . \qquad \text{(28-47)}$$

These lines and their parametric representations give the flow curve through *every* point in the solid sphere.

The spatial surface $\Omega_{F_\mathbf{r}}(t)$ that after time t is *added to* the solid sphere, is thus already parameterized: We only have to put $u = \rho$ in the flow line parameterizations above. I.e., again we find out how the flow of the surface of the sphere contributes to the volume expansion (calculated with sign):

$$\Omega_{F_\mathbf{r}}(T) \quad : \quad \widetilde{\mathbf{r}}(\rho,v,w,t) = \mathrm{e}^t \cdot \rho \cdot (\cos(v) \cdot \cos(w),\ \sin(v),\ \cos(v) \cdot \sin(w)) \quad , \qquad \text{(28-48)}$$

where $t \in [0, T]$, $v \in [0, \pi]$ and $w \in [-\pi, \pi]$. It is seen that $\widetilde{\mathbf{r}}(\rho, v, w, T)$ is a parametric representation of the exploding sphere's surface after time T and that it is a sphere with radius $\mathrm{e}^T \cdot \rho$. By flowing with the flow curves of the explosion vector field the original sphere (with radius ρ at the time $t = 0$) expands to a sphere that has the radius $\mathrm{e}^T \cdot \rho$ as conjectured.

The volume of an exploding sphere at time T is the *sum* of the two volumes $\mathrm{Vol}_{\pm}(\Omega_{F_{\mathbf{r}}}(T))$ and $\mathrm{Vol}(K_\rho)$. The first of these is the signed volume of the region that the sphere's *surface* sweeps through during the time T.

The volume of the sphere as a function of the flow time t is thus given by: $\mathrm{Vol}(\Omega_{F_{\mathbf{r}}}(t)) + \mathrm{Vol}(K_\rho) = \mathrm{Vol}(t) = (4\,\pi/3)\,\rho^3\,e^{3t}$. Therefore the volume of the sphere grows (at time $t = 0$) with the differential quotient

$$\frac{d}{dt}\mathrm{Vol}(t)_{|_{t=0}} = 4\,\pi\,\rho^3 \quad . \tag{28-49}$$

This growth in volume we have essentially found by keeping an eye on the expansion of the solid sphere's *surface* ∂K_ρ when all the sphere's points flow along the flow curves of the vector fields – exactly what we have been going into in the first half of this eNote.

From this we get intuitively that if the scalar product between the vector field $\mathbf{V}$ and the *outward-pointing* unit normal vector $\mathbf{n}$ at a position of the surface is large, then the local contribution to the volume expansion will be correspondingly large, because the surface at this position is pushed quickly outwards when it flows with the flow curves of the vector field.

This might of course mitigated by the fact of the scalar product at other positions on the surface being negative, such that the surface is pushed inwards at these positions.

In short we see again that the total outward-pointing flux of the vector field through the surface of the spatial region gives the (signed) volume expansion.

Therefore the corresponding extension of Theorem 28.6 is:

|||| Theorem 28.11 Total Flux Out through the Surface of a Spatial Field

Let $\Omega_{\mathbf{r}}$ denote an arbitrary parameterized region in (x, y, z) space with the *outward-pointing* unit normal vector field $\mathbf{n}_{\partial\Omega}$ along the surface $\partial\Omega_{\mathbf{r}}$.

The surface might be composed of smooth parameterized surface segments as e.g. the surface of a polyhedron.

Let $\mathbf{V}(x, y, z)$ denote an arbitrary smooth vector field in (x, y, z) space.

We let all points in $\Omega_{\mathbf{r}}$ flow for time t along the flow curves of the vector field $\mathbf{V}(x, y, z)$.

The signed (in relation to $\mathbf{n}_{\partial\Omega}$) volume $\mathrm{Vol}_{\pm}(t)$ of the region after this deformation then has the following differential quotient at time $t = 0$:

$$\frac{d}{dt}\mathrm{Vol}_{\pm}(t)_{|_{t=0}} = \int_{\partial\Omega_{\mathbf{r}}} \mathbf{V}\cdot\mathbf{n}_{\partial\Omega}\,\mathrm{d}\mu = \mathrm{Flux}(\mathbf{V}, \partial\Omega_{\mathbf{r}}) \quad . \tag{28-50}$$

|||| Example 28.12 Explosion of a Solid Sphere

We check the theorem in the case of the exploding sphere: The flux of the explosion vector field out through the surface of the sphere is simply the area of the surface $4\,\pi\,\rho^2$ multiplied by the scalar product $\mathbf{V}\cdot\mathbf{n}$, because that scalar product in this special case is constant: $\mathbf{V}\cdot\mathbf{n} = \|\mathbf{V}\| = \sqrt{x^2+y^2+z^2} = \rho$. Therefore the total flux $4\,\pi\,\rho^3$ is precisely the differential quotient (at time $t = 0$) of the volume as a function of the flow time t. Thus we have verified the theorem for this special case.

The parameterized spatial regions which we consider usually have a boundary consisting of six surface segments. This means that the computation of the volume increase of the region by the deformation along the flow curves requires the computation of six outward-pointing flux contributions – one contribution for each surface segment.

If two of the six surface segments by a standard parameterization of a spatial region coincide or coincide partly, then the standard unit normals for the one surface segment are precisely opposite to the standard unit normals for the other surface segment where the surface segments coincide, such that corresponding flux contributions *cancel*!

For an arbitrary smooth vector field in 3D space we can investigate the volume expansion (by flow along the flow curves of the vector field) of a small solid sphere K_ρ, that has its center at a given point, e.g. $p = (x_0, y_0, z_0)$, and radius ρ. We Taylor-expand the vector field's coordinate functions V_1, V_2 and V_3, to the first order with the point of development (x_0, y_0, z_0) and find the outward-pointing flux of the vector field through the small sphere's surface ∂K_ρ. This flux divided by the volume $(4\,\pi/3)\,\rho^3$ of the solid sphere K_ρ has a limit value when the radius ρ tends towards 0. See the sketch for this computation below.

It appears (see below) that this boundary value precisely is the *divergence of the vector field* at the point considered! In the light of Theorem 28.11 and equation (28-50) we have therefore motivated the following interpretation of the divergence:

|||| Theorem 28.13

The divergence of a vector field expresses *the volume-relative local flux out through the surface* for the vector field and thereby also *the relative local volume growth* by the deformation along the flow curves of the vector field:

$$\begin{aligned} \operatorname{Div}(\mathbf{V})(x_0, y_0, z_0) &= \lim_{\rho \to 0} \left(\frac{1}{\operatorname{Vol}(K_\rho)} \operatorname{Flux}(\mathbf{V}, \partial K_\rho) \right) \\ &= \lim_{\rho \to 0} \left(\frac{1}{\operatorname{Vol}(K_\rho)} \frac{d}{dt} \operatorname{Vol}_\pm(t)_{|_{t=0}} \right) \quad . \end{aligned} \tag{28-51}$$

|||| Proof

We will only sketch how the divergence appears with the suggested method. We simplify the presentation in two ways: On the one hand we choose a coordinate system such that $p = (0,0,0)$, and on the other hand we only include the linearization of $\mathbf{V}(x,y,z)$ about $(0,0,0)$ (ε-terms from Taylors limit formula for the 3 coordinate functions are not taken into account). On the other hand the computations are exact for vector fields of the first degree or less.

The task is to rediscover $\mathrm{Div}(\mathbf{V})$ at the point $(0,0,0)$ by use of a flux computation. We start by using the Taylor's limit formula on $\mathbf{V}(x,y,z)$. It is understood that V_i and the partial derivatives of V_i are evaluated at the point of development $(0,0,0)$ unless otherwise stated.

$$\begin{aligned}\mathbf{V}(x,y,z) = (\ & V_1 + x\,\frac{\partial V_1}{\partial x} + y\,\frac{\partial V_1}{\partial y} + z\,\frac{\partial V_1}{\partial z}, \\ & V_2 + x\,\frac{\partial V_2}{\partial x} + y\,\frac{\partial V_2}{\partial y} + z\,\frac{\partial V_2}{\partial z}, \\ & V_3 + x\,\frac{\partial V_3}{\partial x} + y\,\frac{\partial V_3}{\partial y} + z\,\frac{\partial V_3}{\partial z}\,) \quad .\end{aligned} \tag{28-52}$$

Since the unit normal vector field on the surface of the small solid ρ-sphere K_ρ with centre $(0,0,0)$ is given by $\mathbf{n} = (x/\rho,\ y/\rho,\ z/\rho)$ it follows that the integrand in the flux calculation is the following:

$$\begin{aligned}\mathbf{V}(x,y,z)\cdot\mathbf{n} = \left(\frac{1}{\rho}\right)(\ & x\,V_1 + x^2\,\frac{\partial V_1}{\partial x} + xy\,\frac{\partial V_1}{\partial y} + xz\,\frac{\partial V_1}{\partial z} + \\ & y\,V_2 + yx\,\frac{\partial V_2}{\partial x} + y^2\,\frac{\partial V_2}{\partial y} + yz\,\frac{\partial V_2}{\partial z} + \\ & z\,V_3 + zx\,\frac{\partial V_3}{\partial x} + zy\,\frac{\partial V_3}{\partial y} + z^2\,\frac{\partial V_3}{\partial z}\,) \quad .\end{aligned} \tag{28-53}$$

Now we only have to integrate this expression over the surface of the sphere ∂K_ρ and then divide the result by the volume of the sphere $\mathrm{Vol}(K_\rho)$. Even though this may look complicated it is in fact fairly simple considering the following identities:

$$\begin{aligned}&\int_{\partial K_\rho} x\ \mathrm{d}\mu = \int_{\partial K_\rho} y\ \mathrm{d}\mu = \int_{\partial K_\rho} z\ \mathrm{d}\mu = 0 \quad , \\ &\int_{\partial K_\rho} x^2\ \mathrm{d}\mu = \int_{\partial K_\rho} y^2\ \mathrm{d}\mu = \int_{\partial K_\rho} z^2\ \mathrm{d}\mu = (4\,\pi/3)\,\rho^4 = \rho\ \mathrm{Vol}(K_\rho) \quad , \\ &\int_{\partial K_\rho} x\,y\ \mathrm{d}\mu = \int_{\partial K_\rho} x\,z\ \mathrm{d}\mu = \int_{\partial K_\rho} z\,y\ \mathrm{d}\mu = 0 \quad .\end{aligned} \tag{28-54}$$

From Equation (28-54) now follows e.g. the following contribution to the integral over the spherical surface ∂K_ρ of the right-hand side in Equation (28-53). (Note that $\frac{\partial V_1}{\partial x}(0,0,0)$ is a constant that can be placed outside the integral symbol.):

$$\int_{\partial K_\rho} \left(\frac{1}{\rho}\right) x^2 \frac{\partial V_1}{\partial x}(0,0,0)\, \mathrm{d}\mu = \mathrm{Vol}(K_\rho) \frac{\partial V_1}{\partial x}(0,0,0) \quad . \tag{28-55}$$

Naturally we get a total of three such contributions to the integral over the spherical surface ∂K_ρ of the right-hand side in (28-53), and since the integral over ∂K_ρ of the left-hand side in Equation (28-53) is exactly the flux $\mathrm{Flux}(\mathbf{V}, \partial K_\rho)$ we therefore have the following identity in this simplified case:

$$\begin{aligned} \frac{1}{\mathrm{Vol}(K_\rho)} \mathrm{Flux}(\mathbf{V}, \partial K_\rho) &= \frac{1}{\mathrm{Vol}(K_\rho)} \int_{\partial K_\rho} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}\mu \\ &= \frac{\partial V_1}{\partial x}(0,0,0) + \frac{\partial V_2}{\partial y}(0,0,0) + \frac{\partial V_3}{\partial z}(0,0,0) \\ &= \mathrm{Div}(\mathbf{V})(0,0,0) \quad . \end{aligned} \tag{28-56}$$

For general vector fields the corresponding identity only applies in the limit where ρ is very small (i.e. for $\rho \to 0$), such that the above use of Taylor's limit formula for $\mathbf{V}(x, y, z)$ to the first order exactly becomes the dominant representative for the vector field inside the sphere K_ρ. For vector fields of the first degree or less, though, the identity (28-56) applies as stated for all values of ρ. Naturally this is due to the fact that regardless of the value of the radius ρ the vector field is in this special case represented exactly in all of K_ρ by its Taylor's limit formula to the first order with the development point at the centre.

By this we have concluded the sketch of the proof for the local determination of divergence of a vector field and shown the geometrical interpretation in Theorem 28.13.

■

|||| Exercise 28.14

Show the identities in Equation (28-54). In the cases where the integral is 0 this can be shown by sign and symmetry considerations.

This exposition of the divergence now gives rise to a natural idea pointing in the opposite direction:

Since the local increase in volume at any point through deformation of a spatial region along the flow curves of a given vector field is determined by the divergence of the vector field, then the following is a reasonable supposition: If we integrate the divergence over a region in 3D space, then we would rather expect the result to be comparable to the total volume growth of the whole region. That is, the sum of the local volume increases should be roughly equal to the total volume increase.

This is precisely the content of Gauss' theorem, which we shall now formulate in combination with Theorem 28.11:

28.4 Gauss' Divergence Theorem

Figure 28.9: Carl Friedrich Gauss. See Biography.

|||| Theorem 28.15 Gauss' Divergence Theorem

Let $\Omega_{\mathbf{r}}$ denote a spatial region with the bounding surface $\partial\Omega_{\mathbf{r}}$ and the *outward-pointing* unit normal vector field $\mathbf{n}_{\partial\Omega}$ on the boundary surface. For every smooth vector field $\mathbf{V}$ in (x, y, z) space the following applies:

$$\frac{d}{dt}\text{Vol}_{\pm}(t)_{|t=0} = \int_{\Omega_{\mathbf{r}}} \text{Div}(\mathbf{V})\, \mathrm{d}\mu = \int_{\partial\Omega_{\mathbf{r}}} \mathbf{V} \cdot \mathbf{n}_{\partial\Omega}\, \mathrm{d}\mu = \text{Flux}(\mathbf{V}, \partial\Omega_{\mathbf{r}}) \quad , \tag{28-57}$$

where the flux accordingly must be computed with respect to the *outward-pointing* unit normal vector field on the boundary of the given spatial region.

Both sides of the following equation, the essence of Gauss' theorem, can readily be computed in concrete cases and Gauss' Theorem thereby verified.

$$\int_{\Omega_{\mathbf{r}}} \text{Div}(\mathbf{V})\, \mathrm{d}\mu = \text{Flux}(\mathbf{V}, \partial\Omega_{\mathbf{r}}) \quad . \tag{28-58}$$

Here we will work through some examples of such double computations.

In some cases it is much simpler to compute the divergence integral over a given spatial region than it is to compute the flux of the vector field through the total surface of the region. If asked to compute the latter you of course instead compute the former and refer to Gauss' divergence theorem. Vice versa there are cases where the flux integral is the simpler one to compute – then of course one uses the corresponding alternative strategy.

|||| Example 28.16

If the vector field $\mathbf{V}$ has the divergence 0 at all points in (x, y, z) space, then every spatial region that flows with the vector field keeps its volume. The form can of course be largely changed as time progresses, but the volume is constant. In addition the flux *out* through the surface of every *fixed* spatial region is correspondingly 0.

|||| Example 28.17

Therefore for the explosion vector field $\mathbf{V}(x,y,z) = (x,y,z)$ having the constant divergence $\mathrm{Div}(\mathbf{V}) = 3$, the following applies for an arbitrary spatial region: $3\,\mathrm{Vol}(\Omega) = \mathrm{Flux}(\mathbf{V}, \partial\Omega)$.

|||| Exercise 28.18

Verify (by direct computation) the statement that $\mathbf{V}(x,y,z) = (x,y,z)$ in the above example 28.17 for the spatial region consisting of the solid cylinder in Figure 28.8.

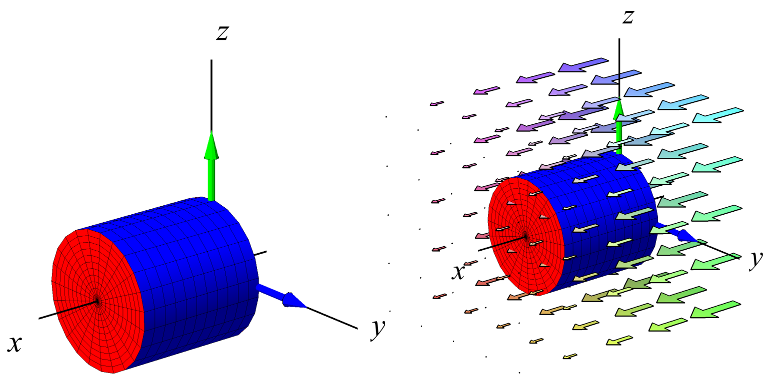

Figure 28.10: A vector field that is parallel to the x-axis and the cylinder from Example 28.10.

|||| Example 28.19 The Divergence and the Flux in the Cylinder Example

We will illustrate Gauss' theorem for the vector field

$$\mathbf{V}(x,y,z) = (f(x),0,0) \quad , \quad (x,y,z) \in \mathbb{R}^3 \quad , \tag{28-59}$$

where $f(x)$ is a smooth function of x, by computing the integral of the divergence of the vector field over the solid cylinder with radius $1/2$, the x-axis as the axis of symmetry, and the x-interval $x \in [0,1]$, which we studied in Example 28.10. In that example we found the total flux of the vector field out through the surface of the cylinder:

$$\mathrm{Flux}(\mathbf{V},\partial\Omega) = (f(1) - f(0)) \cdot \frac{\pi}{4} \quad . \tag{28-60}$$

The total divergence of the vector field in the solid cylinder is as easy to calculate: The local divergence of $\mathbf{V}(x,y,z) = (f(x),0,0)$ at an arbitrary point (x,y,z) is:

$$\mathrm{Div}(\mathbf{V})(x,y,z) = f'(x) \quad . \tag{28-61}$$

The solid cylinder has a parametric representation:

$$\Omega_{\mathbf{r}} \quad : \quad \mathbf{r}(u,v,w) = (w, u \cdot \cos(v), u \cdot \sin(v)) \quad , \tag{28-62}$$

where $(u,v,w) \in [0,1/2] \times [-\pi,\pi] \times [0,1]$. The Jacobian function of the parameterization is then

$$\text{Jacobian}_{\mathbf{r}}(u,v,w) = u \quad , \tag{28-63}$$

such that the total divergence of the vector field is

$$\begin{aligned}
\int_{\Omega_{\mathbf{r}}} \text{Div}(\mathbf{V}) \, d\mu &= \int_0^1 \int_{-\pi}^{\pi} \int_0^{1/2} f'(w) \cdot u \, du \, dv \, dw \\
&= \int_0^1 \int_{-\pi}^{\pi} \frac{1}{8} \cdot f'(w) \, dv \, dw \\
&= \frac{1}{8} \cdot \int_0^1 2 \cdot \pi \cdot f'(w) \, dw \\
&= \frac{\pi}{4} \cdot \int_0^1 f'(w) \, dw \\
&= \frac{\pi}{4} \cdot [f(w)]_{w=0}^{w=1} \\
&= (f(1) - f(0)) \cdot \frac{\pi}{4} \quad ,
\end{aligned} \tag{28-64}$$

exactly the same result as by using the flux calculation.

|||| Example 28.20 The Vector Field through an Edged Torus

We consider a subset of a solid sphere with radius 1/2, see Figure 28.11. A parametric representation of the solid region is given by:

$$\Omega_{\mathbf{r}} \quad : \quad \mathbf{r}(u,v,w) = (u \cdot \sin(v) \cdot \cos(w), \, u \cdot \sin(v) \cdot \sin(w), \, u \cdot \cos(v)) \quad , \tag{28-65}$$

where the parameters run through the following bounding intervals:

$$u \in \left[\frac{1}{2}, 1\right] \quad , \quad v \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \quad , \quad w \in [-\pi, \pi] \quad . \tag{28-66}$$

A vector field in (x,y,z) space is given like this:

$$\mathbf{V}(x,y,z) = (-z, y, x \cdot z) \quad . \tag{28-67}$$

The task is to determine the total flux of the vector field out through the surface of $\Omega_{\mathbf{r}}$. This is likely to be complicated – the surface has four surface segments that all contribute to the flux computation. Instead we will compute the integral of the divergence of the vector field over the spatial region and finally apply Gauss' divergence theorem.

The Jacobian function of the stated parameterization of the solid spherical region is:

$$\text{Jacobian}_{\mathbf{r}}(u,v,w) = u^2 \cdot \sin(v) \quad , \tag{28-68}$$

and the divergence of the vector field is

$$\begin{aligned} \text{Div}(\mathbf{V})(x,y,z) &= 1 + x \quad , \quad \text{such that} \\ \text{Div}(\mathbf{V})(\mathbf{r}(u,v,w)) &= 1 + u \cdot \sin(v) \cdot \cos(w) \quad . \end{aligned} \tag{28-69}$$

The divergence integral over $\Omega_{\mathbf{r}}$ is therefore:

$$\begin{aligned} \int_{\Omega_{\mathbf{r}}} \text{Div}(\mathbf{V})\ d\mu &= \int_{-\pi}^{\pi} \int_{\pi/3}^{2\pi/3} \int_{1/2}^{1} (1 + u \cdot \sin(v) \cdot \cos(w)) \cdot u^2 \cdot \sin(v)\ du\, dv\, dw \\ &= \cdots \\ &= \frac{7\pi}{12} \quad , \end{aligned} \tag{28-70}$$

which therefore – in accordance with Gauss' theorem – also is the sought total flux out through the surface:

$$\text{Flux}(\mathbf{V}, \partial\Omega_{\mathbf{r}}) = \frac{7\pi}{12} \quad . \tag{28-71}$$

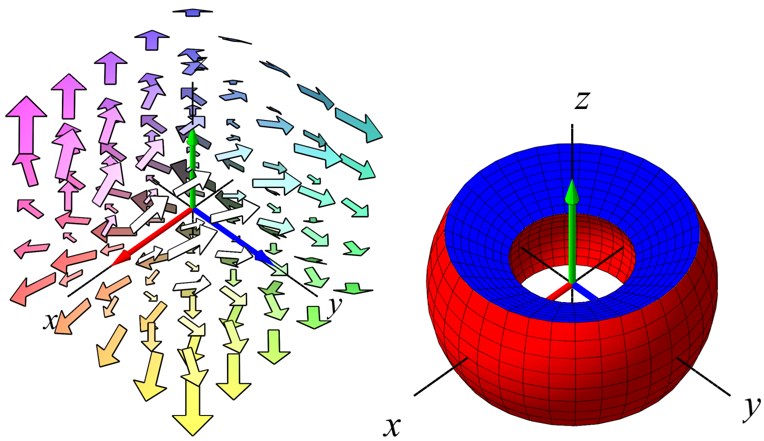

Figure 28.11: A vector field around and through an edged torus.

28.5 A Consequence for Rotational Fields

Here we mention an observation that follows directly from Gauss' divergence theorem in combination with the following local information about the rotational vector fields:

|||| Theorem 28.21 The Divergence of the Curl of a Vector Field Is 0

Let $\mathbf{V}(x,y,z)$ denote a smooth vector field that in itself is the curl of a vector field $\mathbf{W}(x,y,z)$ in (x,y,z) space. Then

$$\mathrm{Div}(\mathbf{V})(x,y,z) = 0 \quad . \tag{28-72}$$

The curl of vector fields is thus divergence-free:

$$\mathrm{Div}(\mathbf{Curl}(\mathbf{W}))(x,y,z) = 0 \quad . \tag{28-73}$$

|||| Proof

We have that

$$\mathbf{V}(x,y,z) = (\frac{\partial W_3}{\partial y} - \frac{\partial W_2}{\partial z}, \frac{\partial W_1}{\partial z} - \frac{\partial W_3}{\partial x}, \frac{\partial W_2}{\partial x} - \frac{\partial W_1}{\partial y}) \quad , \tag{28-74}$$

such that

$$\mathrm{Div}(\mathbf{V})(x,y,z) = \left(\frac{\partial^2 W_3}{\partial y \partial x} - \frac{\partial^2 W_2}{\partial z \partial x}\right) + \left(\frac{\partial^2 W_1}{\partial z \partial y} - \frac{\partial^2 W_3}{\partial x \partial y}\right) + \left(\frac{\partial^2 W_2}{\partial x \partial z} - \frac{\partial^2 W_1}{\partial y \partial z}\right) = 0 \, , \tag{28-75}$$

where we have used that the order of differentiation can be switched, e.g.:

$$\frac{\partial^2 W_3}{\partial y \partial x} = \frac{\partial^2 W_3}{\partial x \partial y} \quad . \tag{28-76}$$

■

If we use Theorem 28.21 in combination with Gauss' theorem we get

|||| Corollary 28.22 The Total Flux of the Curl of a Vector Field Is 0

Let $\mathbf{W}(x,y,z)$ denote a smooth vector field in (x,y,z) space and let Ω be a region in this space with piecewise smooth surface $\partial\Omega$ with outward-directed unit normal vector field $\mathbf{n}_{\partial\Omega}$.

Then the *total flux* of $\mathbf{Curl}(\mathbf{W})(x,y,z)$ out through the surface $\partial\Omega$ of Ω is equal to 0:

$$\text{Flux}(\mathbf{Curl}(\mathbf{W}),\partial\Omega) = \int_{\partial\Omega} \mathbf{Curl}(\mathbf{W}) \bullet \mathbf{n}_{\partial\Omega}\, d\mu = 0 \quad . \tag{28-77}$$

If the spatial region flows with the flow curves of $\mathbf{Curl}(\mathbf{W}(x,y,z)$ then the volume is constant during the whole flow deformation:

$$\frac{d}{dt}\text{Vol}_{\pm}(t) = 0 \quad \text{for } \textit{all} \quad t \quad . \tag{28-78}$$

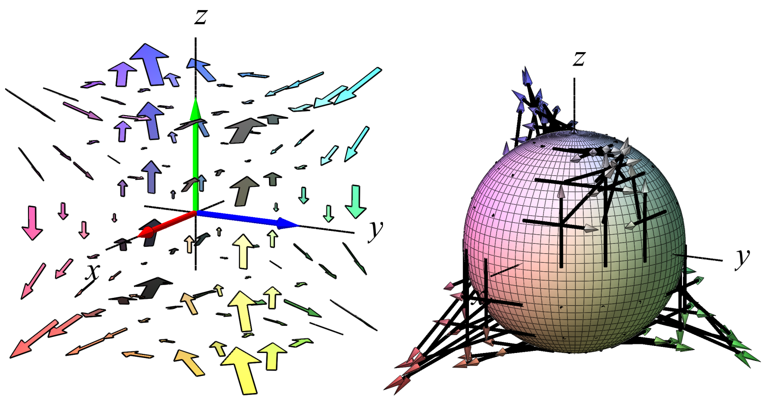

Figure 28.12: A rotational vector field $\mathbf{V}(x,y,z) = \mathbf{Curl}(\mathbf{W})(x,y,z)$ around and through a sphere. The total flux out through the spherical surface is 0, but the local flux through suitably chosen surface segments and the spherical surface is evidently not 0.

|||| Example 28.23 A Rotational Vector Field through a Sphere

We let $\mathbf{W}(x,y,z) = (z^2 \cdot x, x^2 \cdot y, y^2 \cdot z)$.

Then

$$\mathbf{Curl}(\mathbf{W})(x,y,z) = (2\cdot y\cdot z,\, 2\cdot z\cdot x,\, 2\cdot x\cdot y) \quad , \tag{28-79}$$

and evidently

$$\text{Div}(\mathbf{Curl}(\mathbf{W}))(x,y,z) = 0 \quad . \tag{28-80}$$

Now let $F_{\mathbf{r}}$ denote the spherical surface with radius 1 placed with centre at $(0,0,0)$:

$$F_{\mathbf{r}} \quad : \quad \mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)) = (\sin(u)\cdot\cos(v), \sin(u)\cdot\sin(v), \cos(u)) \quad , \tag{28-81}$$

where $u \in [0,\pi]$ and $v \in [-\pi,\pi]$. As is well known the Jacobian function of this parameterization of the spherical surface is given by:

$$\text{Jacobian}_{\mathbf{r}}(u,v) = \sin(u) \quad , \tag{28-82}$$

and the unit normal vector to the spherical surface is in this special case:

$$\mathbf{n}_F(u,v) = (x(u,v), y(u,v), z(u,v)) \quad , \tag{28-83}$$

such that

$$\begin{aligned}\mathbf{Curl}(\mathbf{W})(x,y,z)\bullet\mathbf{n}_F(u,v) &= 2\cdot(y\cdot z,\, y\cdot z,\, y\cdot z)\bullet(x,y,z) \\ &= 6\cdot x(u,v)\cdot y(u,v)\cdot z(u,v) \\ &= 6\cdot\sin^2(u)\cdot\cos(v)\cdot\sin(v)\cdot\cos(u) \quad .\end{aligned} \tag{28-84}$$

The total flux integral of $\mathbf{Rot}(\mathbf{W})(x,y,z)$ out through the spherical surface is therefore:

$$\begin{aligned}\text{Flux}(\mathbf{Curl}(\mathbf{W}), F_{\mathbf{r}}) &= \int_{F_{\mathbf{r}}} \mathbf{Curl}(\mathbf{W})\bullet\mathbf{n}_F\, d\mu \\ &= \int_{-\pi}^{\pi}\int_0^{\pi} (6\cdot\sin^2(u)\cdot\cos(v)\cdot\sin(v)\cdot\cos(u))\cdot\text{Jacobian}_{\mathbf{r}}(u,v)\; du\, dv \\ &= \int_{-\pi}^{\pi}\int_0^{\pi} (6\cdot\sin^2(u)\cdot\cos(v)\cdot\sin(v)\cdot\cos(u))\cdot\sin(u)\; du\, dv \\ &= \int_{-\pi}^{\pi}\int_0^{\pi} 6\cdot\sin^3(u)\cdot\cos(v)\cdot\sin(v)\cdot\cos(u)\; du\, dv\end{aligned} \tag{28-85}$$

and since an indefinite integral of $\sin^3(u)\cdot\cos(u)$ is $\frac{1}{4}\cdot\sin^4(u)$, equal to 0 both for $u = 0$ and for $u = \pi$, then

$$\text{Flux}(\mathbf{Curl}(\mathbf{W}), F_{\mathbf{r}}) = 0 \tag{28-86}$$

in accordance with Corollary 28.22.

Note that if the integration intervals for u and v in Equation (28-85) in Example 28.23 had been smaller, i.e. if we had considered the flux of the rotational vector field out through *part of* the spherical surface, then the result would not necessarily be 0, which is also evident from Figure 28.12.

The flux of $\mathbf{Curl}(\mathbf{W}(x,y,z)$ through a surface segment can alternatively be calculated as the circulation of $\mathbf{W}(x,y,z)$ along the boundary of the surface segment – this is the content of Stokes' Theorem that is the subject of eNote 27.

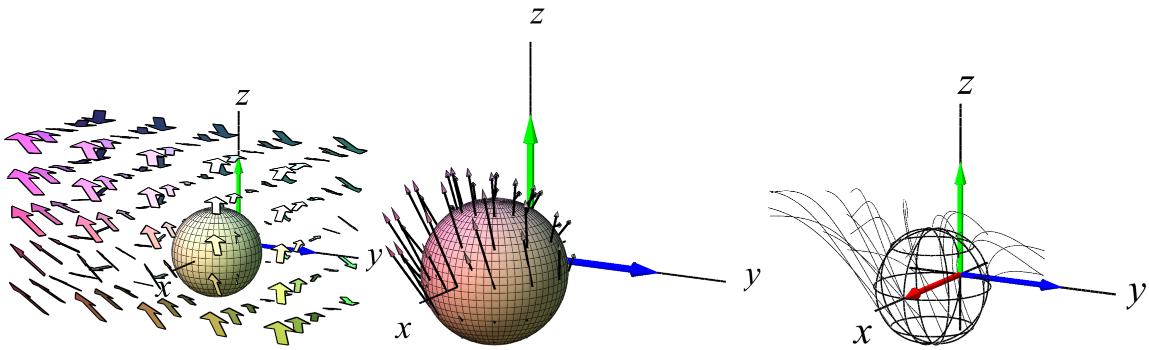

Figure 28.13: A divergence-free vector field $\mathbf{V}(x,y,z) = (-z,(y-x)/2, x-(z/2))$ about and through a sphere.

|||| Example 28.24 A Divergence-Free First-Degree Vector Field

The first-degree vector field

$$\mathbf{V}(x,y,z) = (-z,(y-x)/2, x-(z/2)) \tag{28-87}$$

has the divergence $\mathrm{Div}(\mathbf{V})(x,y,z) = 0$. The total flux of the vector field out through the surface of every spatial region is therefore 0 and the volume of the spatial region is conserved by flows with the flow curves of the vector field.

For first-degree vector fields it applies that any solid sphere deforms through solid *ellipsoids* by flowing with the vector field, see Figure 28.13, Figure 28.14 and Exercise 28.25.

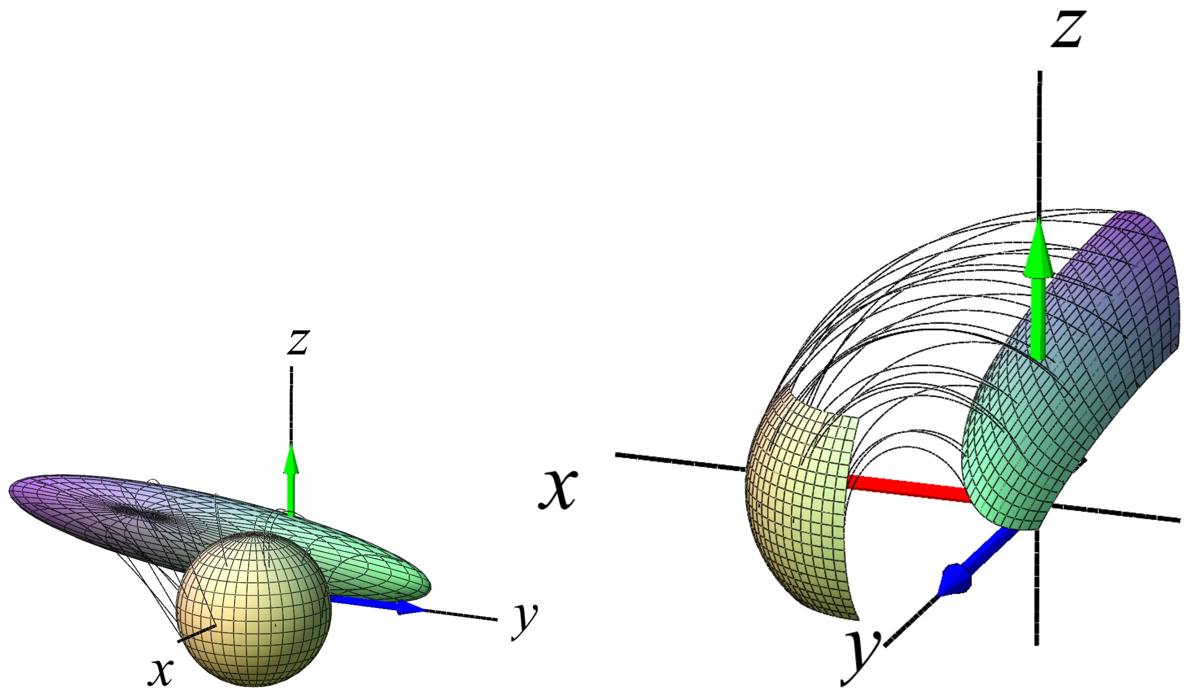

Figure 28.14: The divergence-free vector field $\mathbf{V}(x,y,z) = (-z, (y-x)/2, x-(z/2))$ deforms a solid sphere to a solid ellipsoid by letting all of the points on the sphere flow with the flow curves of the vector field. The volume is conserved. At the right is shown how a segment of the spherical surface flows and forms part of the ellipsoidal surface.

|||| Exercise 28.25 (Advanced)

Let $\mathbf{V}(x,y,z)$ denote an arbitrary first-degree vector field (see eNote 26) and let F_0 denote an arbitrary level surface of a given square polynomial $f_0(x,y,z)$ of x, y and z (see eNote 22). We then let F_0 flow for time t with the flow curves of $\mathbf{V}(x,y,z)$ and by this we get a surface F_t. Show that F_t is also a level surface of a square polynomial $f_t(x,y,z)$ of x, y and z. Hint: See eNote 17.

28.6 Summary

We have introduced the fundamental concept *flux of a vector field through a surface* and related the flux partly to the local volume expansion rate at the surface itself when it flows with the flow curves of the vector field, and partly to the divergence of the vector field inside a spatial region that is bounded by a given surface.

- For a smooth parameterized surface $F_{\mathbf{r}}$ and a smooth vector field $\mathbf{V}(x,y,z)$ the flux of the vector field through the surface is:

$$\begin{aligned}\mathrm{Flux}(\mathbf{V},F_{\mathbf{r}}) &= \int_c^d\int_a^b \mathbf{V}(\mathbf{r}(u,v))\cdot(\mathbf{r}'_u(u,v)\times\mathbf{r}'_v(u,v))\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_c^d\int_a^b \mathbf{V}(\mathbf{r}(u,v))\cdot\mathbf{N}_F(u,v)\;\mathrm{d}u\,\mathrm{d}v \quad .\end{aligned} \tag{28-88}$$

- By letting the surface flow for a period of time t with the flow curves of $\mathbf{V}(x,y,z)$, the surface sweeps through a time-dependent spatial region $\Omega_{\mathbf{r}}(t)$ that locally either lies in the direction of the standard normal of the surface or in the opposite direction. The sign-weighted volume is calculated by use of the Jacobian function of the parametric representation – *without numerical sign*. This volume we call $\mathrm{Vol}_{\pm}(t)$. Then the flux is related to this volume in the following way:

$$\mathrm{Flux}(\mathbf{V},F_{\mathbf{r}}) = \frac{d}{dt}_{|t=0}\mathrm{Vol}_{\pm}(\Omega_{\mathbf{r}}(t)) = \mathrm{Vol}'_{\pm}(0) \quad . \tag{28-89}$$

- The divergence of the vector field is also a measure of local volumetric expansion (where $\mathrm{Div}(\mathbf{V})(x,y,z) > 0$) volume contraction (where $\mathrm{Div}(\mathbf{V})(x,y,z) < 0$) by flow along the flow curves of the vector field. The integral of the divergence over a spatial region $\Omega_{\mathbf{r}}$ is therefore a measure of the total expansion (or contraction) of the volume of the whole region and therefore similarly yielding the value $\mathrm{Vol}'_{\pm}(0)$, where $\mathrm{Vol}_{\pm}(t)$ here is calculated for the volume contribution (with sign) for *all* of the surface by flows along the flow curves in relation to the *outward-directed* normal vector $\mathbf{n}_{\partial\Omega}$ on the surface.

  This is the content of Gauss' divergence theorem:

$$\frac{d}{dt}\mathrm{Vol}_{\pm}(t)_{|t=0} = \int_{\Omega_{\mathbf{r}}}\mathrm{Div}(\mathbf{V})\;\mathrm{d}\mu = \int_{\partial\Omega_{\mathbf{r}}}\mathbf{V}\cdot\mathbf{n}_{\partial\Omega}\;\mathrm{d}\mu = \mathrm{Flux}(\mathbf{V},\partial\Omega_{\mathbf{r}}) \quad . \tag{28-90}$$

- A consequence of Gauss' divergence theorem is the following: If an arbitrary spatial region flows along the flow curves of a divergence-free vector field then the volume of the region is constant in time – even though the form of the region of course in time can change a lot. Every rotational vector field $\mathbf{Curl}(\mathbf{W})(x, y, z)$ is divergence-free. A spatial region that 'rotates' in the very general meaning that it flows along the flow curves of $\mathbf{Curl}(\mathbf{W})(x, y, z)$ therefore conserves its volume.