eNote 27

Vector Fields Along Curves

In eNote 26 the introductory considerations about vector fields are given. In this eNote we shall look at the values of the vector fields along curves and use methods from eNote 24 about line integration in order to state the so-called tangential line integrals and use them to investigate whether or not a given vector field is a gradient vector field. If a vector field is a gradient vector field of a function $f(x,y,z)$ then we can construct all such indefinite integrals by use of tangential line integration of the vector field. And as we shall see it applies vice versa that if the tangential line integral of a given vector field over all closed curves is 0, then the vector field is a gradient vector field. The tangential line integral of a vector field over a closed curve is called the circulation of the vector field over the curve. The name alone makes it no surprise that a general circulation 0 is equivalent to the vector field itself having the rotation vector field $\mathbf{0}$.

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27.1 The Tangential Line Integral

Let $\mathbf{V}(x,y,z)$ be a smooth vector field in (x,y,z) space, see eNote 26, and let $K_{\mathbf{r}}$ denote a smooth parametrized curve:

$$K_{\mathbf{r}} \quad : \quad \mathbf{r}(u) = (x(u), y(u), z(u)) \quad , \quad u \in [a,b] \quad . \tag{27-1}$$

Along the curve $K_{\mathbf{r}}$ we then have – at every point $\mathbf{r}(u)$ on the curve – *two* vectors, on the one hand the value of the vector field in the point, $\mathbf{V}(\mathbf{r}(u))$, and on the other the tangent vector $\mathbf{r}'(u)$ to the curve at the point. By using these two vectors we can construct a smooth function on the curve, which thereafter can be integrated over the curve:

The *tangential line integral* of $\mathbf{V}(x, y, z)$ along a given parametrized curve $K_{\mathbf{r}}$ is the line integral of the length of the *projection* (signed) of $\mathbf{V}(\mathbf{r}(u))$ on the tangent to the curve that is represented by $\mathbf{r}'(u)$.

The integral we seek is also defined like this:

|||| Definition 27.1

The tangential line integral of $\mathbf{V}(x, y, z)$ along $K_{\mathbf{r}}$ is defined by:

$$\operatorname{Tan}(\mathbf{V}, K_{\mathbf{r}}) = \int_{K_{\mathbf{r}}} \mathbf{V} \cdot \mathbf{e} \, d\mu \quad . \tag{27-2}$$

Accordingly, in this case the integrand is given by the scalar product:

$$f(\mathbf{r}(u)) = \mathbf{V}(\mathbf{r}(u)) \cdot \mathbf{e}(u) \quad , \tag{27-3}$$

where $\mathbf{e}(u)$ is defined by

$$\mathbf{e}(u) = \begin{cases} \mathbf{r}'(u)/\|\mathbf{r}'(u)\| & \text{if} \quad \mathbf{r}'(u) \neq \mathbf{0} \\ \mathbf{0} \quad \text{if} \quad \mathbf{r}'(u) = \mathbf{0} \quad . \end{cases} \tag{27-4}$$

Note that we then have for all u:

$$\mathbf{e}(u) \, \|\mathbf{r}'(u)\| = \mathbf{r}'(u) \quad . \tag{27-5}$$

The tangential line integral $\operatorname{Tan}(\mathbf{V}, K_{\mathbf{r}})$ of $\mathbf{V}$ along $K_{\mathbf{r}}$ is therefore relatively simple to *compute* - in fact we need not first find the Jacobian function $\operatorname{Jacobian}_{\mathbf{r}}(u)$, that is the *length* of $\mathbf{r}'(u)$:

$$\begin{aligned} \operatorname{Tan}(\mathbf{V}, K_{\mathbf{r}}) &= \int_{K_{\mathbf{r}}} \mathbf{V} \cdot \mathbf{e} \, d\mu \\ &= \int_a^b (\mathbf{V}(\mathbf{r}(u)) \cdot \mathbf{e}(u)) \operatorname{Jacobian}_{\mathbf{r}}(u) \, du \\ &= \int_a^b \mathbf{V}(\mathbf{r}(u)) \cdot (\mathbf{e}(u) \, \|\mathbf{r}'(u)\|) \, du \\ &= \int_a^b \mathbf{V}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) \, du \quad . \end{aligned} \tag{27-6}$$

Accordingly we have the following simple expression for tangential line integrals:

|||| Theorem 27.2

The tangential line integral of $\mathbf{V}(x, y, z)$ along $K_{\mathbf{r}}$ can be *computed* like this:

$$\mathrm{Tan}(\mathbf{V}, K_{\mathbf{r}}) = \int_{K_{\mathbf{r}}} \mathbf{V} \cdot \mathbf{e} \, d\mu = \int_a^b \mathbf{V}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) \, du \quad . \tag{27-7}$$

Note that the last integrand in (27-6) is continuous when $\mathbf{V}(x, y, z)$ and $\mathbf{r}'(u)$ are continuous even though it does not at first sight appear from the definition (the vector field $\mathbf{e}(u)$ is not necessarily continuous – unless $\mathbf{r}(u)$ is a regular parametric representation).

If the curve is traversed backwards, then $\mathrm{Tan}(\mathbf{V}, K_{\mathbf{r}})$ shifts sign.

Viz. let

$$\overline{K}_{\overline{\mathbf{r}}} \quad : \quad \overline{\mathbf{r}}(u) = \mathbf{r}(b + a - u) \quad , \quad u \in [a, b] \quad . \tag{27-8}$$

Then $\overline{\mathbf{r}}'(u) = -\mathbf{r}'(u)$ and $\overline{\mathbf{e}}(u) = -\mathbf{e}(u)$ such that

$$\mathrm{Tan}(\mathbf{V}, \overline{K}_{\overline{\mathbf{r}}}) = -\,\mathrm{Tan}(\mathbf{V}, K_{\mathbf{r}}) \quad . \tag{27-9}$$

|||| Definition 27.3

Analogously with the tangential line integral we define the *orthogonal* line integral $\mathrm{Ort}(\mathbf{V}, K_{\mathbf{r}})$ of $\mathbf{V}$ along $K_{\mathbf{r}}$ by projecting $\mathbf{V}(\mathbf{r}(u))$ perpendicularly onto the plane in (x, y, z) space that in itself is perpendicular to $\mathbf{r}'(u)$ and then finding the line integral of the length of the projection (as a function of u).

|||| Example 27.4

Let $\mathbf{V}(x,y,z) = (0,z,y)$. We wish to determine the tangential line integral of $\mathbf{V}$ along the following parametrized segment of a helix

$$K_\mathbf{r}: \ \mathbf{r}(u) = (\cos(u),\, \sin(u),\, u)\,, \quad u \in \left[0, \frac{\pi}{2}\right] \quad . \tag{27-10}$$

By substituting into (27-6) we get

$$\begin{aligned} \mathrm{Tan}(\mathbf{V}, K_\mathbf{r}) &= \int_0^{\pi/2} \mathbf{V}(\mathbf{r}(u)) \cdot \mathbf{r}'(u)\, \mathrm{d}u \\ &= \int_0^{\pi/2} (0, u, \sin(u)) \cdot (-\sin(u), \cos(u), 1)\, \mathrm{d}u \\ &= \int_0^{\pi/2} (u\cos(u) + \sin(u))\, \mathrm{d}u \\ &= [u \sin(u)]_0^{\pi/2} = \frac{\pi}{2} \quad . \end{aligned} \tag{27-11}$$

|||| Exercise 27.5

Let $\mathbf{V}(x,y,z) = (0,x,z)$. Determine both the tangential and the orthogonal line integral of $\mathbf{V}$ along the following parametrized segment of a circle

$$K_\mathbf{r}: \ \mathbf{r}(u) = (\cos(u),\, \sin(u),\, 0)\,, \quad u \in \left[0, \frac{\pi}{2}\right] \quad . \tag{27-12}$$

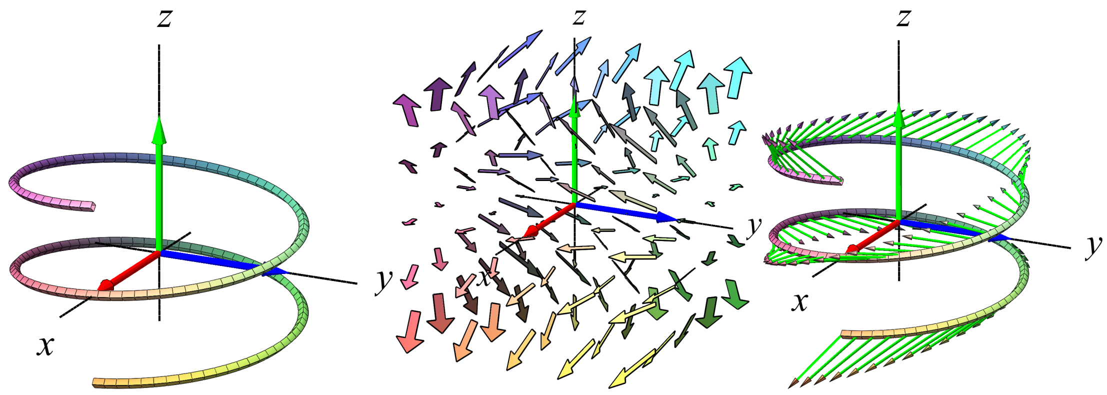

Figure 27.1: The helix $\mathbf{r}(u) = \left(\cos(u), \sin(u), \frac{1}{10}u\right)$, $u \in [-2\pi, 2\pi]$, and the vector field $\mathbf{V}(x,y,z) = (x, -(x+y), 2z)$ are indicated here – both in (x,y,z) space and along the helix.

|||| Method 27.6 Tangential Line Integrals to a Variable Point

Let $\mathbf{V}(x,y,z)$ be a given vector field in (x,y,z) space. We will now construct a function $F^*(x,y,z)$ in (x,y,z) space, which at the point (x_0,y_0,z_0) is defined as the line integral of $\mathbf{V}(x,y,z)$ along the straight line segment from $(0,0,0)$ to the point (x_0,y_0,z_0), i.e. F^* is defined like this:

$$F^*(x_0,y_0,z_0) = \text{Tan}(\mathbf{V}, K_{\mathbf{r}}) \quad , \tag{27-13}$$

where the curve is given by the simplest possible curve from $(0,0,0)$ to (x_0,y_0,z_0):

$$K_{\mathbf{r}} : \ \mathbf{r}(u) = (u \cdot x_0, u \cdot y_0, u \cdot z_0) = u \cdot (x_0,y_0,z_0)\,, \ \ u \in [0,1] \quad . \tag{27-14}$$

Hereby we then have the following function values of the (star) function corresponding to $\mathbf{V}(x,y,z)$:

$$\begin{aligned} F^*(x_0,y_0,z_0) &= \text{Tan}(\mathbf{V}, K_{\mathbf{r}}) \\ &= \int_0^1 \mathbf{V}(\mathbf{r}(u)) \cdot \mathbf{r}'(u)\, \mathrm{d}u \\ &= (x_0,y_0,z_0) \cdot \int_0^1 \mathbf{V}(u \cdot x_0, u \cdot y_0, u \cdot z_0)\, \mathrm{d}u \quad . \end{aligned} \tag{27-15}$$

The last integral in (27-15) should be understood as an integral of each of the three coordinate functions of $\mathbf{V}(u \cdot x_0, u \cdot y_0, u \cdot z_0)$. Therefore the integral gives a vector with three components so that the scalar product with the position vector (x_0,y_0,z_0) then yields a numerical value that accordingly is the value of the $*$-function $F^*(x,y,z)$ at (x_0,y_0,z_0).

|||| Example 27.7 Construction of Tangential Line Integral to a Variable Point

Let $\mathbf{V}(x,y,z)$ denote the following vector field in (x,y,z) space:

$$\mathbf{V}(x,y,z) = (x, 2y, 3z) \quad . \tag{27-16}$$

Then the corresponding $*$-function is:

$$\begin{aligned} F^*(x_0,y_0,z_0) &= (x_0,y_0,z_0) \bullet \int_0^1 \mathbf{V}(u \cdot x_0, u \cdot y_0, u \cdot z_0)\, \mathrm{d}u \\ &= (x_0,y_0,z_0) \bullet \int_0^1 (u \cdot x_0, 2u \cdot y_0, 3u \cdot z_0)\, \mathrm{d}u \\ &= \frac{1}{2} \cdot (x_0,y_0,z_0) \bullet (x_0, 2 \cdot y_0, 3 \cdot z_0) \\ &= \frac{1}{2} \cdot (x_0^2 + 2 \cdot y_0^2 + 3 \cdot z_0^2) \\ &= \frac{1}{2} \cdot x_0^2 + y_0^2 + \frac{3}{2} \cdot z_0^2 \quad . \end{aligned} \tag{27-17}$$

Note that the (star) function

$$F^*(x,y,z) = \frac{1}{2} \cdot x^2 + y^2 + \frac{3}{2} \cdot x^2$$

that is constructed in Example 27.7 has the following gradient that is exactly the vector field with which we started:

$$\boldsymbol{\nabla} F^*(x,y,z) = (x, 2y, 3z) = \mathbf{V}(x,y,z) \quad . \tag{27-18}$$

This is not a coincidence! The vector field has the rotation vector field $\mathbf{Rot} = \mathbf{0}$, and is therefore a gradient vector field, as we shall see in Theorem 27.14 below.

27.2 The Indefinite Integral of a Gradient Vector Field

|||| Definition 27.8 The Indefinite Integral of a Gradient Vector Field

Let $\mathbf{V}(x,y,z)$ be a smooth vector field in (x,y,z) space. If a smooth function $f(x,y,z)$ exists such that

$$\nabla f(x,y,z) = \mathbf{V}(x,y,z) \quad , \tag{27-19}$$

then $f(x,y,z)$ is said to be an *indefinite integral* of the vector field $\mathbf{V}(x,y,z)$.

If we know one indefinite integral, we then know all of them – apart from an arbitrary constant! As we shall see, Method 27.6 gives a construction of all indefinite integrals of a given vector field. If an indefinite integral of the vector field does *not* exist this method nevertheless gets us a $*$-function in (x,y,z) space – this function is, however, not an indefinite integral. This can and should be decided by *testing*, that is, by directly computing the gradient of the $*$-function and comparison with the given vector field.

|||| Exercise 27.9

Let $\mathbf{V}(x,y,z)$ denote the vector field:

$$\mathbf{V}(x,y,z) = (x, 2y, 3z) \quad . \tag{27-20}$$

Determine the function values of the (star) function $F_p^*(x,y,z)$ from the general point $p = (a,b,c)$ corresponding to $\mathbf{V}(x,y,z)$.

Hint: The straight line segment from the point (a,b,c) to (x_0,y_0,z_0) can be parametrized like this:

$$K_{\mathbf{r}}: \ \mathbf{r}(u) \ = \ (a + u\cdot(x_0 - a), b + u\cdot(y_0 - b), c + u\cdot(z_0 - c)) \, , \quad u \in [0,1] \quad . \tag{27-21}$$

such that

$$\mathbf{r}'(u) \ = \ (x_0 - a), y_0 - b, z_0 - c) \quad . \tag{27-22}$$

Is it still valid that

$$\nabla F_p^*(x,y,z) = \mathbf{V}(x,y,z) \quad ? \tag{27-23}$$

|||| Theorem 27.10 Tangential Line Integral of a Gradient Vector Field

Let $f(x,y,z)$ denote a smooth function of three variables in (x,y,z) space and let $\mathbf{V}(x,y,z) = \boldsymbol{\nabla} f(x,y,z)$ denote the gradient vector field of $f(x,y,z)$. Let $K_{\mathbf{r}}$ be a smooth parametrized curve from a point p to a point q in this space. The curve needs not necessarily be straight.
Then the following applies: The tangential line integral of $\boldsymbol{\nabla} f(x,y,z)$ along $K_{\mathbf{r}}$ only depends on p and q and is independent of the curve:

$$\operatorname{Tan}(\mathbf{V}, K_{\mathbf{r}}) = f(q) - f(p) \quad . \tag{27-24}$$

|||| Proof

We use the chain rule from eNote 19 on the composite function $h(u) = f(\mathbf{r}(u))$ where $\mathbf{r}(u)$, $u \in]u_0, u_1[$, is an arbitrary differentiable curve from $p = (x_0, y_0, z_0) = \mathbf{r}(u_0)$ to $q = (x_1, y_1, z_1) = \mathbf{r}(u_1)$ and thereby get:

$$h'(u) = \mathbf{r}'(u) \cdot \boldsymbol{\nabla} f(\mathbf{r}(u)) \quad . \tag{27-25}$$

From this it follows that $h(u)$ is an indefinite integral of the function on the right-hand side of the above equation:

$$h(u_1) - h(u_0) = \int_{u_0}^{u_1} \mathbf{r}'(u) \cdot \boldsymbol{\nabla} f(\mathbf{r}(u)) \, \mathrm{d}u \quad . \tag{27-26}$$

But since

$$\begin{aligned} h(u_0) &= f(\mathbf{r}(u_0)) = f(p) \quad , \\ h(u_1) &= f(\mathbf{r}(u_1)) = f(q) \quad , \end{aligned} \tag{27-27}$$

we thereby get that

$$\begin{aligned} f(q) &= f(p) + \int_{u_0}^{u_1} \mathbf{r}'(u) \cdot \boldsymbol{\nabla} f(\mathbf{r}(u)) \, \mathrm{d}u \\ &= f(p) + \operatorname{Tan}(\mathbf{V}, K_{\mathbf{r}}) \quad , \end{aligned} \tag{27-28}$$

and this it what we should prove.

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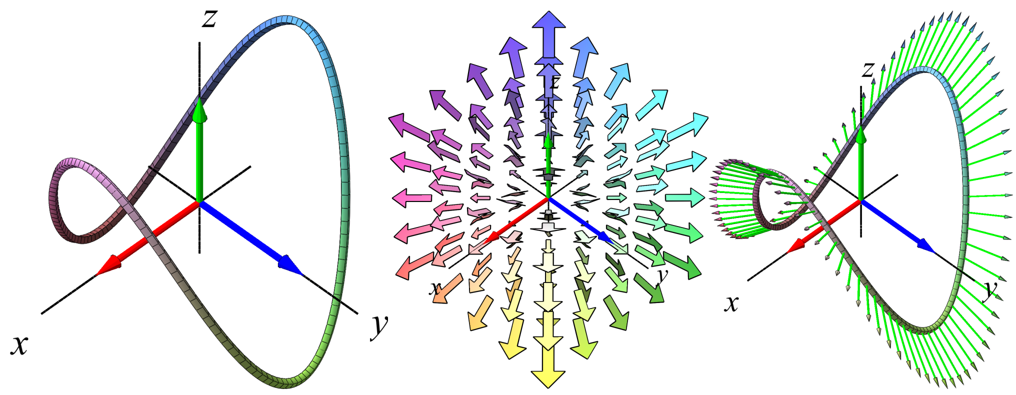

Figure 27.2: The closed curve $\mathbf{r}(u) = (\cos(t), \sin(t), \cos(2 \cdot t))$, $u \in [-\pi, \pi]$ and the gradient vector field $\mathbf{V}(x, y, z) = (x, y, z) = \boldsymbol{\nabla} f(x, y, z)$ of $f(x, y, z) = (x^2 + y^2 + z^2)/2$ are here indicated – both in 3D space and along the curve.

If the curve is closed – generally composed of a finite number of smooth curves like e.g. a polygon with straight edges – we get that the tangential line integral of a gradient vector field over the closed curve is 0.

|||| Definition 27.11 The Circulation of a Vector Field Along a Closed Curve

Let $K_{\mathbf{r}}^{\circ}$ denote a *closed* curve in (x, y, z) space (that generally can be composed of a finite number of smooth curves). Let $\mathbf{V}(x, y, z)$ be a smooth vector field in this space. Then we call the tangential line integral of $\mathbf{V}(x, y, z)$ over $K_{\mathbf{r}}^{\circ}$ the *circulation* of $\mathbf{V}(x, y, z)$ over $K_{\mathbf{r}}^{\circ}$ and write:

$$\mathrm{Circ}(\mathbf{V}, K_{\mathbf{r}}^{\circ}) = \mathrm{Tan}(\mathbf{V}, K_{\mathbf{r}}^{\circ}) \quad . \tag{27-29}$$

The name *circulation* is quite reasonable. If the vector field is a gradient vector field we know that the rotation is $\mathbf{0}$ everywhere, see eNote 26 and this is in accordance with the fact that the circulation in these cases is also 0 for every closed curve.

We have already now understood one half of the following theorem – the other (somewhat more difficult) half is proved below:

|||| Theorem 27.12 The Circulation Theorem

A smooth vector field $\mathbf{V}(x, y, z)$ in (x, y, z) space is a gradient vector field if and only if it applies that:

$$\mathrm{Circ}(\mathbf{V}, K_{\mathbf{r}}^{\circ}) = 0 \tag{27-30}$$

for *all* closed curves $K_{\mathbf{r}}^{\circ}$.

An indefinite integral can be constructed by the line integral method 27.6.

|||| Proof

As mentioned above we already know that if $\mathbf{V}(x,y,z)$ is a gradient vector field, then the circulation is 0 for every closed curve. To explore the opposite causality we presume that all closed curves give the circulation 0 and from this conclude that then the vector field in question is a gradient vector field. There is in the above exposition only one candidate that possibly could be used as such a function, viz. the $*$-function $F^*(x, y, z)$ corresponding to $\mathbf{V}(x, y, z)$:

$$F^*(x,y,z) = (x,y,z) \cdot \int_0^1 \mathbf{V}(u \cdot x, u \cdot y, u \cdot z)\, \mathrm{d}u \quad . \tag{27-31}$$

since the circulation is 0 along every closed curve we know that this function does not depend on the integration path: The tangential line integral of $\mathbf{V}(x, y, z)$ along any other curve from $(0, 0, 0)$ to (x, y, z) will give the same value as $F^*(x, y, z)$.

We will show that the gradient of $F^*(x, y, z)$ at the point (x_0, y_0, x_0) is $\mathbf{V}(x_0, y_0, z_0)$ so we look at

$$F^*(x,y,z) = F^*(x_0,y_0,z_0) + \mathrm{Tan}(\mathbf{V}, K_{\mathbf{r}}) \quad , \tag{27-32}$$

where $K_{\mathbf{r}}$ is an arbitrary smooth curve from the (development) point (x_0, y_0, z_0) to an arbitrary other point (x, y, z). We *again* choose the straight line segment between the two points:

$$K_{\mathbf{r}} \quad : \quad \mathbf{r}(u) = (x_0,y_0,z_0) + u \cdot ((x,y,z) - (x_0,y_0,z_0)) \quad , \quad u \in [0,1] \quad , \tag{27-33}$$

and we then get:

$$\mathrm{Tan}(\mathbf{V}, K_{\mathbf{r}}) = (x - x_0, y - y_0, z - z_0) \cdot \int_0^1 \mathbf{V}(\mathbf{r}(u))\, \mathrm{d}u \tag{27-34}$$

The first contribution to this scalar product is

$$(x - x_0) \cdot \int_0^1 V_1(\mathbf{r}(u))\, \mathrm{d}u \quad . \tag{27-35}$$

We will now use the observation (see Exercise 27.13) that for every smooth function $g(u)$ on $[0,1]$ it applies that a value ξ between 0 and 1 can be found such that

$$\int_0^1 g(u)\,\mathrm{d}u = g(\xi) \quad . \tag{27-36}$$

By using this integral mean value theorem we get for the substitution into (27-35):

$$\begin{aligned}(x-x_0)\cdot\int_0^1 V_1(\mathbf{r}(u))\,\mathrm{d}u &= (x-x_0)V_1(\mathbf{r}(\xi_1))\\ &= (x-x_0)\cdot\left(V_1(x_0,y_0,z_0)+\varepsilon_1(x-x_0,y-y_0,z-z_0)\right) \quad ,\end{aligned} \tag{27-37}$$

since $V_1(x,y,z) \to V_1(x_0,y_0,z_0)$ for $(x,y,z) \to (x_0,y_0,z_0)$.

Correspondingly we compute the two other contributions to the scalar product in (27-34) such that we have in total:

$$\begin{aligned}\mathrm{Tan}(\mathbf{V},K_{\mathbf{r}}) &= (x-x_0,y-y_0,z-z_0)\bullet\left(\mathbf{V}(x_0,y_0,z_0)+\boldsymbol{\varepsilon}(x-x_0,y-y_0,z-z_0)\right)\\ &= (x-x_0,y-y_0,z-z_0)\bullet\mathbf{V}(x_0,y_0,z_0)\\ &\qquad + \rho_{(x_0,y_0,z_0)}(x,y,z)\cdot\varepsilon(x-x_0,y-y_0,z-z_0)\end{aligned} \tag{27-38}$$

Finally we substitute this into (27-32) and get the following equation

$$\begin{aligned}F^*(x,y,z) &= F^*(x_0,y_0,z_0)+\mathrm{Tan}(\mathbf{V},K_{\mathbf{r}})\\ &= F^*(x_0,y_0,z_0)+(x-x_0,y-y_0,z-z_0)\bullet\mathbf{V}(x_0,y_0,z_0)\\ &\qquad + \rho_{(x_0,y_0,z_0)}(x,y,z)\cdot\varepsilon(x-x_0,y-y_0,z-z_0) \quad .\end{aligned} \tag{27-39}$$

The gradient of the (star) function $F^*(x,y,z)$ at (x_0,y_0,z_0) can thereafter be read by direct inspection – this is the "factor" on $(x-x_0,y-y_0,z-z_0)$ before the epsilon term:

$$\boldsymbol{\nabla} F^*(x_0,y_0,z_0) = \mathbf{V}(x_0,y_0,z_0) \quad , \tag{27-40}$$

which exactly means that $\mathbf{V}(x,y,z)$ is a gradient vector field (with $F^*(x,y,z)$ as an indefinite integral), and this is what we set out to prove.

■

|||| Exercise 27.13

Consider – and realize – the statement we have used in the proof of theorem 27.12:
For every smooth function $g(u)$ on $[0,1]$ it applies that a value ξ between 0 and 1 can be found such that

$$\int_0^1 g(u)\,\mathrm{d}u = g(\xi) \quad . \tag{27-41}$$

In analogy with the circulation theorem 27.12 we have similarly a two-way causation between the gradient vector field property and the *curl* **0**:

|||| Theorem 27.14 The Indefinite Integral Theorem

A smooth vector field $\mathbf{V}(x,y,z)$ in (x,y,z) space is a gradient vector field if and only if:

$$\mathbf{Curl}(\mathbf{V})(x,y,z) = (0,0,0) \quad . \tag{27-42}$$

An indefinite integral of the vector field $\mathbf{V}(x,y,z)$ can be constructed by the line integral method 27.6.

|||| Proof

We only look at the easy part, given that the curl of a gradient vector field is **0**. See also the statement in Theorem 26.29 in eNote 26. Let also $f(x,y,z)$ be a smooth function in (x,y,z) space. Then

$$\boldsymbol{\nabla} f(x,y,z) = \left(f'_x(x,y,z), f'_y(x,y,z), f'_z(x,y,z)\right) \quad . \tag{27-43}$$

Since we only focus on the first coordinate in the rotation vector we get from this:

$$\begin{aligned} \mathbf{Curl}(\boldsymbol{\nabla} f)(x,y,z) &= \left(\frac{\partial}{\partial y} f'_z(x,y,z) - \frac{\partial}{\partial z} f'_y(x,y,z), *, *\right) \\ &= \left(f''_{zy}(x,y,z) - f''_{yz}(x,y,z), *, *\right) \\ &= (0, *, *) \quad , \end{aligned} \tag{27-44}$$

and similarly 0 for the other two coordinates.

■

|||| Example 27.15 The Tangential Line Integral of a Non-Gradient Vector Field

Let $\mathbf{V}(x,y,z) = (-y,x,0)$. Then $\mathbf{V}(x,y,z)$ is *not a gradient vector field* since $\mathbf{Curl}(\mathbf{V})(x,y,z) = (0,0,2) \neq (0,0,0)$, see also Example 26.3 in eNote 26.

If we choose the closed curve:

$$K_{\mathbf{r}} \quad : \quad \mathbf{r}(u) = (\cos(u), \sin(u), 0) \quad , \quad u \in [-\pi, \pi] \quad , \tag{27-45}$$

then in accordance with the circulation theorem we get a circulation that is not 0:

$$\begin{aligned} \mathrm{Tan}(\mathbf{V}, K_{\mathbf{r}}) &= \int_{-\pi}^{\pi} (-\sin(u), \cos(u), 0) \cdot (-\sin(u), \cos(u), 0)\, \mathrm{d}u \\ &= \int_{-\pi}^{\pi} \left(\sin^2(u) + \cos^2(u)\right)\, \mathrm{d}u \\ &= \int_{-\pi}^{\pi} 1\, \mathrm{d}u \\ &= 2 \cdot \pi \quad . \end{aligned} \tag{27-46}$$

If we construct the $*$-function corresponding to $\mathbf{V}(x,y,z) = (-y,x,0)$ we get a well-defined function in (x,y,z) space:

$$\begin{aligned} F^*(x,y,z) &= (x,y,z) \cdot \int_0^1 \mathbf{V}(u \cdot x, u \cdot y, u \cdot z)\, \mathrm{d}u \\ &= (x,y,z) \cdot \int_0^1 (-u \cdot y, u \cdot x, 0)\, \mathrm{d}u \\ &= \frac{1}{2} \cdot ((x,y,z) \cdot (-y,x,0)) \\ &= 0 \quad , \end{aligned} \tag{27-47}$$

and this is clearly *not* an indefinite integral of $\mathbf{V}(x,y,z)$, which is in full accord with the indefinite integral theorem.

27.3 Summary

We have defined the tangential line integral of smooth vector fields along smooth parametrized curves and shown how the tangential line integrals can be used to construct indefinite integrals to a vector field $\mathbf{V}(x,y,z)$ – if the vector field is otherwise a gradient vector field.

- The tangential line integral of $\mathbf{V}(x,y,z)$ along the parametrized curve $K_\mathbf{r}$ can be computed like this:

$$\operatorname{Tan}(\mathbf{V},K_\mathbf{r}) = \int_a^b \mathbf{V}(\mathbf{r}(u))\cdot\mathbf{r}'(u)\,\mathrm{d}u \quad . \tag{27-48}$$

  If the curve $K_\mathbf{r}$ is closed then the curve is denoted $K_\mathbf{r}^\circ$, the tangential line integral is in this case named the circulation of the vector field along the closed curve, and we write

$$\operatorname{Circ}(\mathbf{V},K_\mathbf{r}^\circ) = \operatorname{Tan}(\mathbf{V},K_\mathbf{r}^\circ) \quad . \tag{27-49}$$

- The indefinite integral theorem gives a necessary and sufficient condition for a given vector field being a gradient vector field:

  $\mathbf{V}(x,y,z)$ is a gradient vector field if and only if $\mathbf{Curl}(\mathbf{V})(x,y,z) = (0,0,0)$.

- The circulation theorem expresses a corresponding necessary and sufficient condition for the gradient field property expressed by the circulation of the vector field along closed curves:

  $\mathbf{V}(x,y,z)$ is a gradient vector field if and only if $\operatorname{Circ}(\mathbf{V},K_\mathbf{r}^\circ) = 0$ for all closed curves $K_\mathbf{r}^\circ$.

- The star-function $F^*(x,y,z)$ corresponding to the gradient vector field $\mathbf{V}(x,y,z)$ is an indefinite integral to $\mathbf{V}(x,y,z)$. The function is computed as the tangential line integral from $(0,0,0)$ to the point(x,y,z) like this:

$$\begin{aligned} F^*(x,y,z) &= \operatorname{Tan}(\mathbf{V},K_\mathbf{r}) \\ &= \int_0^1 \mathbf{V}(\mathbf{r}(u))\cdot\mathbf{r}'(u)\,\mathrm{d}u \\ &= (x,y,z)\cdot\int_0^1 \mathbf{V}(u\cdot x, u\cdot y, u\cdot z)\,\mathrm{d}u \quad . \end{aligned} \tag{27-50}$$