eNote 26

Vector Fields

In eNote 10 vectors in the 2D plane and in 3D space were introduced and studied. In eNote 20 we considered the gradients of functions $f(x,y)$ of two variables. A gradient vector field of a function of two variables is – as the name amply hints – an example of a plane vector field. In this eNote we will begin to study vector fields in general, both in the (x,y) plane and in 3-dimensional (x,y,z) space. We will clarify what it means to flow with a given vector field *and compute where you then arrive at in the space or in the plane in this way after a given period of time. In order to find these so-called* flow curves *we need to be able to solve (suitably simple) systems of first-order differential equations. Thus eNote 16 together with the two eNotes above become background material for the present eNote. We will also begin to investigate what happens to larger systems of points or particles when they individually flow with the vector field.*

Updated: 31-01-2023, shsp.

26.1 Vector Fields

A vector field $\mathbf{V}$ in *3D space* is given by 3 smooth functions $V_1(x,y,z)$, $V_2(x,y,z)$ and $V_3(x,y,z)$ that are all functions of the three variables x, y, and z like this:

$$\mathbf{V}(x,y,z) = (V_1(x,y,z)\,, V_2(x,y,z)\,, V_3(x,y,z)\,) \quad \text{for} \quad (x,y,z) \in \mathbb{R}^3 \quad . \tag{26-1}$$

A vector field $\mathbf{V}(x, y, z)$ *is drawn* and usually stated in (x, y, z) space by – at a suitable number of chosen points (x_i, y_i, z_i) – plotting the vector as an arrow with the base point at (x_i, y_i, z_i) and the end point at $(x_i + V_1(x_i, y_i, z_i), y_i + V_2(x_i, y_i, z_i), z_i + V_3(x_i, y_i, z_i))$. There can be good reasons to state the vector field in other ways. E.g. if there is great variation in the length of the vectors in a given vector field, then it can be advantageous to use the *thickness* of the arrows as indication of the vector length.

A vector field in the plane is similarly given by the two coordinate functions $V_1(x, y)$ and $V_2(x, y)$ that are both smooth functions of the two variables x and y:

$$\mathbf{V}(x, y) = (V_1(x, y), V_2(x, y)) \quad \text{for} \quad (x, y) \in \mathbb{R}^2 \quad . \tag{26-2}$$

The gradient vector field of a smooth function $f(x, y)$ of two variables (introduced and studied in eNote 20) is an example of a vector field in the plane, see Figure 26.1.

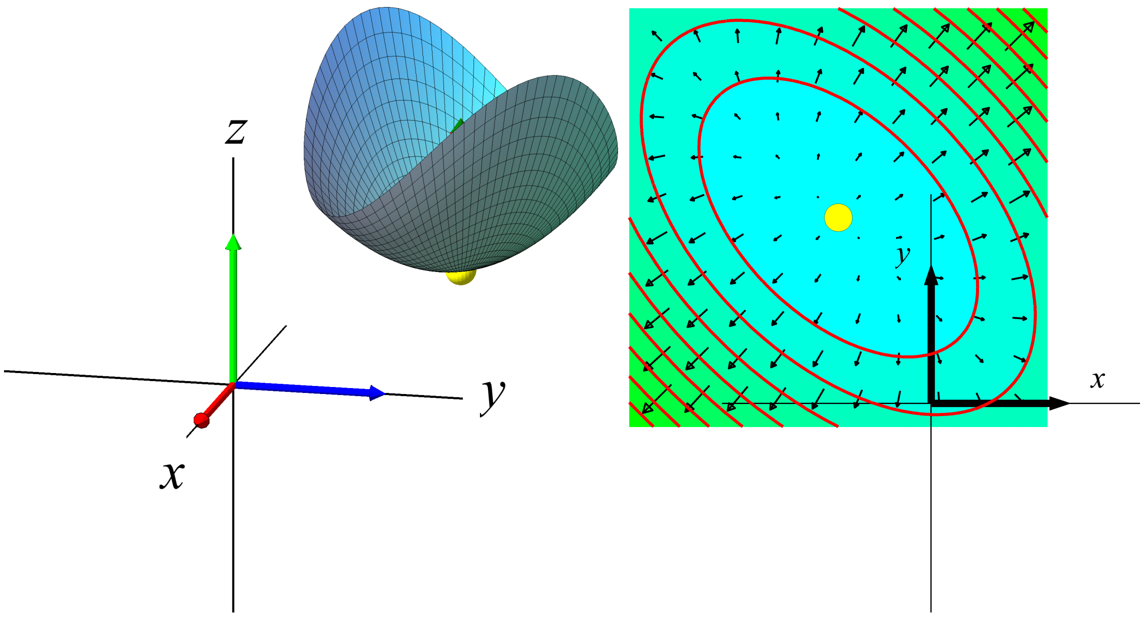

Figure 26.1: A graph surface of a function of two variables and the corresponding gradient vector field in the plane together with some of the level curves. The gradient vector field is everywhere perpendicular to the level curves, see eNote 20.

The gradient vector field of functions $f(x, y, z)$ of three variables is defined similarly to that of two variables:

|||| Definition 26.1 Gradient Field in (x, y, z) Space

Let $f(x,y,z)$ denote a smooth function of three variables in $\mathbb{R}^3$. Then the gradient vector field of $f(x,y,z)$ is defined in the following way by the use of the first three partial derivatives of $f(x,y,z)$:

$$\nabla f(x,y,z) = \Big(f'_x(x,y,z)\,,\, f'_y(x,y,z)\,,\, f'_z(x,y,z) \Big) \quad , \quad (x,y,z) \in \mathbb{R}^3 \quad . \tag{26-3}$$

|||| Example 26.2 A Gradient Field in (x, y, z) Space

We let $f(x,y,z)$ denote the quadratic polynomial

$$f(x,y,z) = 2 \cdot x^2 + 2 \cdot y^2 + 2 \cdot z^2 - 2 \cdot x \cdot z - 2 \cdot x - 4 \cdot y - 2 \cdot z + 3 \quad . \tag{26-4}$$

The gradient vector field of $f(x,y,z)$ is then:

$$\nabla f(x,y,z) = (4 \cdot x - 2 \cdot z - 2\,,\, 4 \cdot y - 4\,,\, -2 \cdot x + 4 \cdot z - 2) \quad . \tag{26-5}$$

See Figure 26.2. We refer to Example 24.5 in eNote 24 about the construction of the ellipsoidal level surface $\mathcal{K}_0(f)$ of $f(x,y,z)$ shown. The level surface and the computed gradient vector field are indicated in Figure 26.2. The gradient vector field is seen to be perpendicular to the level surface.

The question is now, do *all* smooth vector fields in the plane and *all* smooth vector fields in 3D space *stem from* a function in the way that each individually is the gradient vector field of some function of two and three variables, respectively? But it is not that simple!

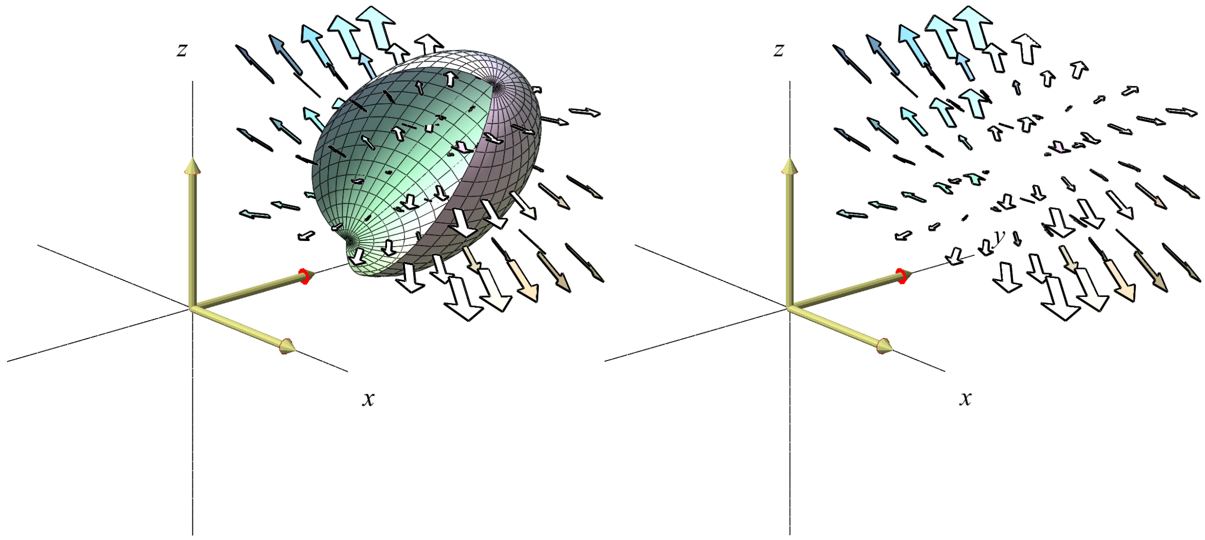

Figure 26.2: The level surface (in an open version) of a function (quadratic polynomial) of three variables and some corresponding gradient vectors from the gradient vector field in (x, y, z) space.

|||| Example 26.3 A Vector Field that Is Not a Gradient Vector Field

Let $\mathbf{V}(x, y)$ denote the very simple vector field in the plane $\mathbf{V}(x, y) = (-y, x)$, where $(x, y) \in \mathbb{R}^2$. Then no function $f(x, y)$ exists that satisfies $\boldsymbol{\nabla} f(x, y) = \mathbf{V}(x, y)$.

Viz. if we (until we reach a contradiction) assume that such a function with this property exists:

$$\begin{aligned} \boldsymbol{\nabla} f(x, y) &= \mathbf{V}(x, y) \quad , \quad \text{such that} \\ \left(f'_x(x, y), f'_y(x, y)\right) &= (-y, x) \quad , \quad \text{then we get that} \end{aligned}$$

$$\begin{aligned} f'_x(x, y) &= -y \\ f'_y(x, y) &= x \quad , \quad \text{and thereby} \end{aligned} \tag{26-6}$$

$$\begin{aligned} f''_{xy}(x, y) &= -1 \\ f''_{yx}(x, y) &= 1 \quad , \end{aligned}$$

and this is not compatible with the fact that for all smooth functions

$$f''_{xy}(x, y) = f''_{yx}(x, y) \quad , \quad \text{for all} \quad (x, y) \in \mathbb{R}^2 \quad . \tag{26-7}$$

This shows that a function whose gradient vector field is the given vector field does not exist.

The gradient vector fields are thus only examples of vector fields – but a very large and very important collection of examples of vector fields.

A vector field $\mathbf{V}(x,y)$ in the (x,y) plane can easily be extended to a vector field $\mathbf{W}(x,y,z)$ in (x,y,z) space by simply displacing all the vectors from the plane in the direction of the z-axis and in addition putting $W_3(x,y,z) = 0$ for all $(x,y,z) \in \mathbb{R}^3$:

The quite general plane vector field $\mathbf{V}(x,y) = (V_1(x,y), V_2(x,y))$ thus has the following spatial extension:

$$\mathbf{W}(x,y,z) = (V_1(x,y), V_2(x,y), 0) \quad \text{i.e.}$$

$$\begin{aligned} W_1(x,y,z) &= V_1(x,y) \quad , \\ W_2(x,y,z) &= V_2(x,y) \quad , \\ W_3(x,y,z) &= 0 \quad . \end{aligned} \tag{26-8}$$

See Figure 26.3 that hints at the extensions of the three different vector fields $\mathbf{V}(x,y) = (1,0)$, $\mathbf{V}(x,y) = (x,y)$, and $\mathbf{V}(x,y) = (-y,x)$, that is, $\mathbf{W}(x,y,z) = (1,0,0)$, $\mathbf{W}(x,y,z) = (x,y,0)$, and $\mathbf{W}(x,y,z) = (-y,x,0)$, respectively.

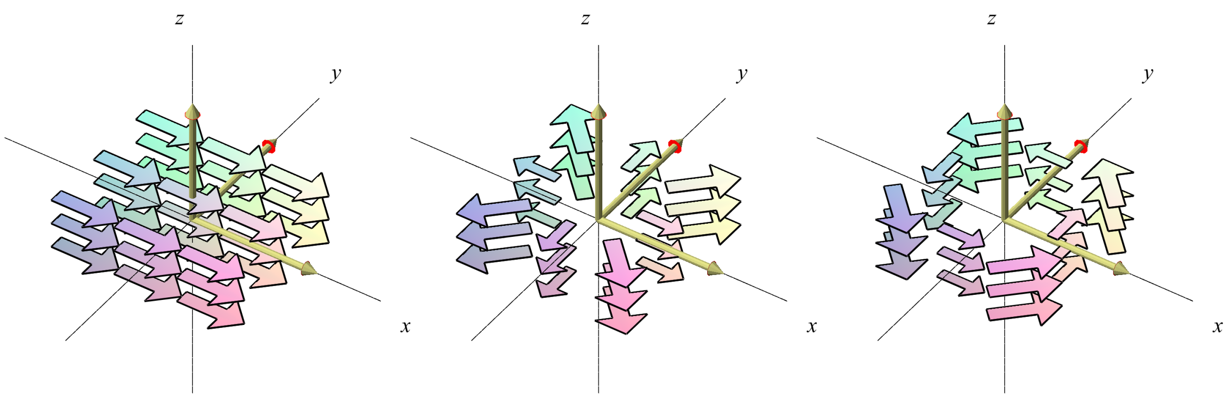

Figure 26.3: Three plane vector fields are here extended to spatial vector fields.

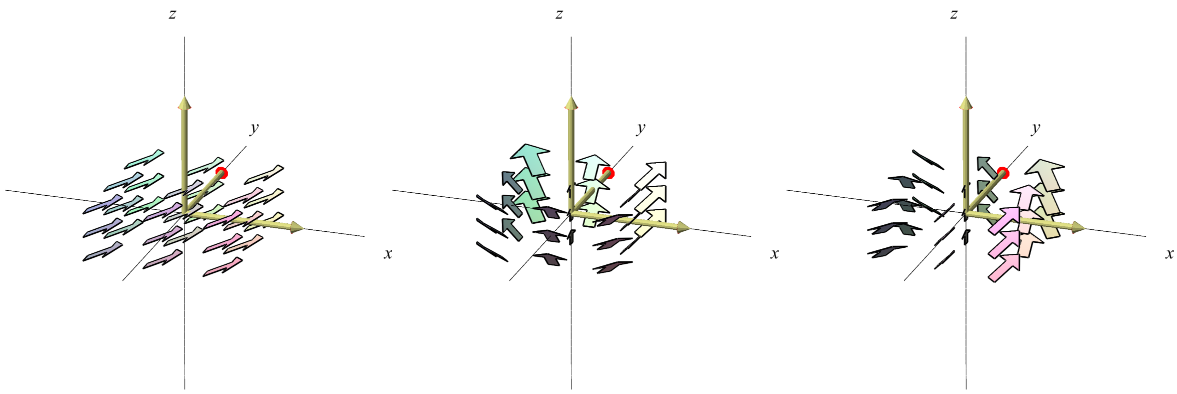

Figure 26.4: The three spatial vector fields from Figure 26.3 are here modified to have $W_3 = 1/2$ in place of $W_3 = 0$.

Some vector fields are particularly simple. In particular this applies to those vector fields where all three coordinate functions are polynomials of at the most the first degree in the spatial variables (x, y, z), i.e.

$$\begin{aligned} \mathbf{V}(x,y,z) = (&a_{11}x + a_{12}y + a_{13}z + b_1 \;, \\ &a_{21}x + a_{22}y + a_{23}z + b_2 \;, \\ &a_{31}x + a_{32}y + a_{33}z + b_3\,) \quad . \end{aligned} \tag{26-9}$$

In this case the vector field can be written in short form by use of the matrix, $\mathbf{A}$, that has the elements a_{ij} and the vector, $\mathbf{b}$, that has the coordinates b_i :

|||| Definition 26.4 Vector Field of the First Degree

A *vector field of the first degree* is a vector field $\mathbf{V}(x,y,z)$ that can be written in the following form by the use of a constant matrix $\mathbf{A}$ and a constant vector $\mathbf{b}$:

$$\mathbf{V}^\top = (\mathbf{V}(x,y,z))^\top = \mathbf{A} \cdot \begin{bmatrix} x & y & z \end{bmatrix}^\top + \mathbf{b}^\top \quad , \tag{26-10}$$

where $^\top$ means transposition of the respective matrices, such that

$$\begin{bmatrix} V_1(x,y,z) \\ V_2(x,y,z) \\ V_3(x,y,z) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad . \tag{26-11}$$

|||| Example 26.5 Constant Vector Field

A constant vector field can e.g. model a constant wind locally close to the (x, y) plane (the face of the earth):

$$\mathbf{V}(x, y, z) = \mathbf{b} \quad , \tag{26-12}$$

where $\mathbf{b}$ is a constant vector, e.g. $\mathbf{b} = (0, 7, 0)$ – if the wind blows with $7km/h$ in the direction of the y-axis.

|||| Example 26.6 Rotating Vector Field

An example of a so-called *rotating vector field* is given by

$$\mathbf{V}(x, y, z) = (-y, x, 0) \quad . \tag{26-13}$$

See Figure 26.3 in the middle.

|||| Definition 26.7

The trace of a square $n \times n$ matrix A with the elements a_{ij} is the sum of the n diagonal elements of the matrix:

$$\textbf{trace}(A) = \sum_{i=1}^{i=n} a_{ii} \quad . \tag{26-14}$$

|||| Exercise 26.8

Find $\mathbf{A}$ and $\mathbf{b}$ (as in Definition 26.4) for the vector field in example 26.6. What is the trace of $\mathbf{A}$ in this case? Can $\mathbf{A}$ be diagonalized (diagonalization is described in eNote 14)?

|||| Example 26.9 Explosion and Implosion Vector Fields

An example of what we could call an *explosion vector field* is given by the following coordinate functions (see why in Figure 26.6):

$$\mathbf{V}(x, y, z) = (x, y, z) \quad . \tag{26-15}$$

Similarly the following is an example of a vector field that we can call an *implosion vector field* (see why in Figure 26.7):

$$\mathbf{V}(x, y, z) = (-x, -y, -z) \quad . \tag{26-16}$$

|||| Exercise 26.10

Find $\mathbf{A}$ and $\mathbf{b}$ for the vector fields in Example 26.9. What is the trace of $\mathbf{A}$ for the two vector fields? Can $\mathbf{A}$ be diagonalized?

We will now argue the (dynamic) names that we have given the vector fields in the above examples 26.6 and 26.9. To do so we will move together with – or flow along – the vector field in a very precise way that we shall now define.

26.2 Flow Curves of a Vector Field

Let us first repeat that if we are given a curve with a parametric representation

$$K_{\mathbf{r}} : \quad \mathbf{r}(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3 \quad , \quad t \in [a, b] \quad , \tag{26-17}$$

then this curve has for every value of the parameter t a tangent vector, viz.

$$\mathbf{r}'(t) = (x'(t),\, y'(t),\, z'(t)) \quad . \tag{26-18}$$

If we consider the parameter $t \in [a, b]$ as a *time parameter* for the motion (of a particle) in (x, y, z) space that is given by $\mathbf{r}(t)$ then $\mathbf{r}'(t)$ is the velocity of the particle at time t.

If we construct sufficiently many curves (a curve through every point in the given space), each curve intersecting neither itself nor any other curve, then we in this way get a vector field in the space.

The obvious *inverse* question is now: Given an initial point $p = (x_0, y_0, z_0)$ and given a vector field $\mathbf{V}(x, y, z)$ in (x, y, z) space, then does a parametrized curve $\mathbf{r}(t)$ through p (with $\mathbf{r}(0) = (x_0, y_0, z_0)$) exist, such that the tangent vector field of the curve all the way *along the curve* precisely is the vector field $\mathbf{V}(x, y, z)$ *along the curve*? If this is the case then we shall call the curve $\mathbf{r}(t)$ an *integral curve* or a *flow curve* of the vector field. These names are partly due to the fact that the curve can be found by integration (solution to a system of differential equations) and partly that motion along the curve is similar to flying or floating with the given vector field, i.e. with a speed and a direction given by the vector field at every point of the motion, since the requirements to the motion $\mathbf{r}(t)$ are expressed by:

|||| Definition 26.11 Flow Curves, Integral Curves

Let $\mathbf{V}(x,y,z)$ denote a smooth vector field in (x,y,z) space. A parametrized curve

$$K_{\mathbf{r}}: \quad \mathbf{r}(t) = (x(t), y(t), z(t)) \quad , \quad t \in [a,b] \quad , \tag{26-19}$$

is called a *flow curve* or an *integral curve* of the vector field $\mathbf{V}(x,y,z)$ if $\mathbf{r}(t)$ fulfills the flow curve equation:

$$\mathbf{V}(\mathbf{r}(t)) = \mathbf{r}'(t) \quad \text{for all} \quad t \in [a,b] \quad , \tag{26-20}$$

which is equivalent to the following system of first-order differential equations

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \left(\mathbf{V}(x(t),y(t),z(t))\right)^{\top} = \begin{bmatrix} V_1(x(t),y(t),z(t)) \\ V_2(x(t),y(t),z(t)) \\ V_3(x(t),y(t),z(t)) \end{bmatrix} \quad . \tag{26-21}$$

If $\mathbf{V}(x,y,z)$ is given and if we have been given an initial point $p = \mathbf{r}(a)$ for a flow curve then the task is of course the typical one, to find the solution to the system of differential equations with this initial condition, i.e. to find the coordinate functions $x(t)$, $y(t)$, and $z(t)$ so that $p = (x(a), y(a), z(a))$.

In other words: If we are given a vector field in 3D space then the task is to start a particle (a small ball) moving along the vector field such that the velocity vector of the ball at every instant is given by the value of the vector field in the point where the ball is positioned at that point in time. And it is of course interesting to be able to decide *where* the ball is after a long period of time. And it is interesting to find out how a multitude of balls (particles that to begin with are close to each other) develop in time – is the multitude of balls more dense or more thin, squeezed together or stretched out?

The following existence and uniqueness theorem applies and will be the foundation for our first examples and the first consideration about the natural questions concerning flow curves and their behaviour.

|||| Theorem 26.12 Existence and Uniqueness

Let $\mathbf{V}(x,y,z)$ be a vector field of the *first degree*, given by a coefficient matrix $\mathbf{A}$ and a vector $\mathbf{b}$ as in Definition 26.4. Let (x_0, y_0, z_0) denote an arbitrary point in (x,y,z) space. Then exactly one curve $\mathbf{r}(t)$ exists that fulfills the two conditions:

$$\begin{aligned} \mathbf{r}(0) &= (x_0, y_0, z_0) \quad \text{and} \\ \mathbf{r}'(t) &= \mathbf{V}(x(t), y(t), z(t)) \quad \text{for} \quad \text{all} \quad t \in [-\infty, \infty] \quad . \end{aligned} \tag{26-22}$$

The last equation (26-22) is equivalent to the following system of differential equations with a constant coefficient matrix $\mathbf{A}$:

$$\begin{aligned} \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} &= (\mathbf{V}(x(t), y(t), z(t)))^\top \\ &= \mathbf{A} \cdot \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}^\top + \mathbf{b}^\top \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad . \end{aligned} \tag{26-23}$$

If we are given a vector field of the first degree, then we can therefore ”start” a point, a particle, at an arbitrary position in the space and let it ”flow” with the vector field such that the particle is situated on a uniquely determined flow curve to every time thereafter.

Two flow curves cannot intersect each other, because if they did there could not be a *unique* flow curve through the point of intersection.

The theorem can be extended to vector fields that are not necessarily of the first degree, but then it is no longer certain that all the time-parameter intervals for the flow curves will be of the double infinity interval $\mathbb{R} =]-\infty, \infty[$. The integral curves of a vector field of the first degree can be found and shown with Maple and are exemplified in the figures 26.5, 26.6, and 26.7.

If the vector field is not of the first degree there is as previously stated no guarantee that flow curves can be determined explicitly (not even using Maple), but in certain cases numerical tools can anyway be applied with success inside "windows" where the solutions exist and are well-defined.

The argument, the proof, for Theorem 26.12 is known from the study of systems of linear coupled differential equations, see eNote 17. Let us shortly repeat the considerations needed in order to find the flow curves of some of the simplest vector fields.

|||| Example 26.13 Flow Curves of a Constant Vector Field

The constant vector field $\mathbf{V}(x, y, z) = (0, 7, 0)$ has flow curves $(x(t), y(t), z(t))$ that fulfill the two conditions: The initial condition $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ and the 3 differential equations for $x(t)$, $y(t)$, and $z(t)$ following from the velocity vector condition

$$\mathbf{r}'(t) = (x'(t), y'(t), z'(t)) = \mathbf{V}(x(t), y(t), z(t)) = (0, 7, 0) \quad . \tag{26-24}$$

The task is to find the three coordinate functions $x(t)$, $y(t)$, and $z(t)$ such that

$$\begin{aligned} x'(t) &= 0 \\ y'(t) &= 7 \\ z'(t) &= 0 \quad . \end{aligned} \tag{26-25}$$

The 3 differential equations in this case are not coupled and they are solved directly with the given initial conditions with the following result: $x(t) = x_0$, $y(t) = y_0 + 7\,t$, and $z(t) = z_0$. I.e. the flow curves are (not surprisingly) all the straight lines parallel to the y-axis, parametrized such that all have the speed 7.

|||| Example 26.14 Flow Curves of a Rotating Vector Field

The example with the rotating vector field $\mathbf{V}(x, y, z) = (-y, x, 1)$ has corresponding flow curves that now have to fulfill the conditions: $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ together with the differential equations

$$\mathbf{r}'(t) = (x'(t), y'(t), z'(t)) = (-y(t), x(t), 1) \quad . \tag{26-26}$$

Therefore the task is here to find the three coordinate functions $x(t)$, $y(t)$, and $z(t)$ such that

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t) \\ z'(t) &= 1 \quad . \end{aligned} \tag{26-27}$$

The differential equations for $x(t)$ and $y(t)$ are coupled linear differential equations with constant coefficients and are solved precisely as in eNote 17. Note that the system matrix has already been found in Exercise 26.8. The result is $x(t) = x_0 \cos(t) - y_0 \sin(t)$, $y(t) = x_0 \sin(t) + y_0 \cos(t)$, and $z(t) = z_0 + t$. These flow curves can be found and inspected with Maple. It is also apparent from this that it is quite reasonable to call the vector field a rotating vector field. See Figure 26.5.

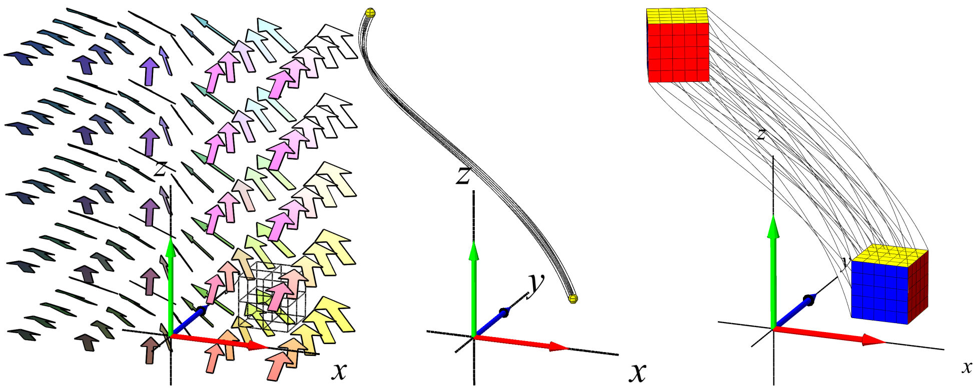

Figure 26.5: The rotating vector field from Example 26.14, one "flow curve" for an individual particle and the system of flow curves, passing through a cube (the nethermost cube flows along the flow curves with the vector field until the time π).

|||| Exercise 26.15

Let $\mathbf{V}(x, y, z) = (-y, x, 0)$ and use Maple to find flow curves and the motion of points in the same cube as in Figure 26.5 when the time interval of the flow is $T = [0, 2\pi]$. Next compare with 'the effect' of the vector fields $\mathbf{W}(x, y, z) = (-y, -x, 0)$ $\mathbf{W}(x, y, z) = (-y, 2x, 0)$ on the points of the cube for the same time interval. Explain the difference between the three 'effects' of the three different vector fields on the cube.

|||| Example 26.16 Flow Curves of an Explosion and Implosion Vector Field

The explosion vector field

$$\mathbf{V}(x,y,z) = (x,y,z) \tag{26-28}$$

has flow curves that satisfy an initial condition

$$(x(0), y(0), z(0)) = (x_0, y_0, z_0) \tag{26-29}$$

and the differential equations

$$\mathbf{r}'(t) = (x'(t), y'(t), z'(t)) = (x(t), y(t), z(t)) \quad . \tag{26-30}$$

We find the three coordinate functions $x(t)$, $y(t)$, and $z(t)$ such that

$$\begin{aligned} x'(t) &= x(t) \\ y'(t) &= y(t) \\ z'(t) &= z(t) \quad . \end{aligned} \tag{26-31}$$

The differential equations for $x(t)$, $y(t)$, and $z(t)$ are here uncoupled linear differential equations that can easily be solved, one at a time. The result is

$$x(t) = x_0 \exp(t) \quad , \qquad y(t) = y_0 \exp(t) \quad , \quad \text{and} \quad z(t) = z_0 \exp(t) \quad . \tag{26-32}$$

Note that if $(x(0), y(0), z(0)) = (0,0,0)$ then $(x(t), y(t), z(t)) = (0,0,0)$ for all $t \in [-\infty, \infty]$. Therefore the flow curve 'through' the point $(0,0,0)$ is not a proper curve but consists only of the point itself. Note also that all other flow curves run arbitrarily close to the point $(0,0,0)$ for $t \to -\infty$, since $\exp(t) \to 0$ for $t \to -\infty$, but they do not run through the point. Therefore if we follow the flow curves in Figure 26.6 back in time from $t = 0$ through negative values we will see an exponentially decreasing *implosion* of the cube. If we on the contrary follow the flow curves forward in time from $t = 0$ through larger and larger positive values for t we will see an exponentially increasing explosion of the cube. The flow curves can again be found and inspected using Maple. See Figure 26.6.

The implosion vector field is given by

$$\mathbf{V}(x,y,z) = (-x, -y, -z) \tag{26-33}$$

with "time-reversed" solution (as related to the explosion vector field)

$$x(t) = x_0 \exp(-t) \quad , \qquad y(t) = y_0 \exp(-t) \quad , \quad \text{og} \quad z(t) = z_0 \exp(-t) \quad . \tag{26-34}$$

See Figure 26.7.

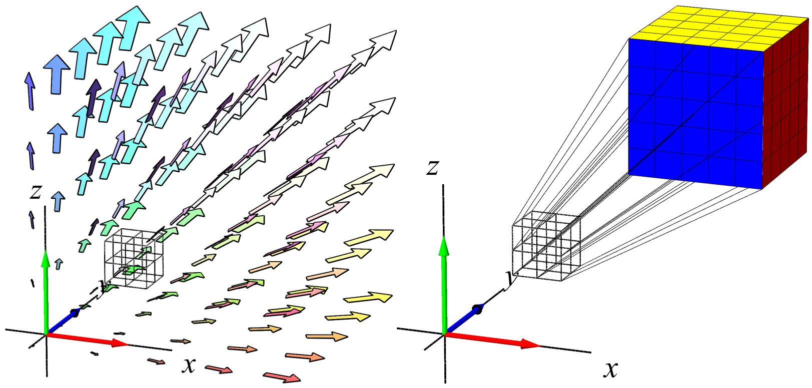

Figure 26.6: The explosion vector field from example 26.16 together with the integral curves passing through a cube. The cube is shown as a solid to the time $t = 1$ and as "open" to the time $t = 0$. Compare with Figure 26.7.

|||| **Exercise 26.17**

Let $\mathbf{V}$ denote the vector field $\mathbf{V}(x, y, z) = (-x, -2y, -3z)$. Find and show a suitable number of the flow curves of the vector field through the ball that has its centre at $(1, 0, 0)$ and a radius of $\frac{1}{4}$.

26.3 The Divergence of a Vector Field

For the purpose of geometric analysis of vector fields and their flow curve properties we will here introduce two tools, two concepts, for the local description of general smooth vector fields. The description is local because both concepts are expressed by the partial derivatives of the coordinate functions of the vector field.

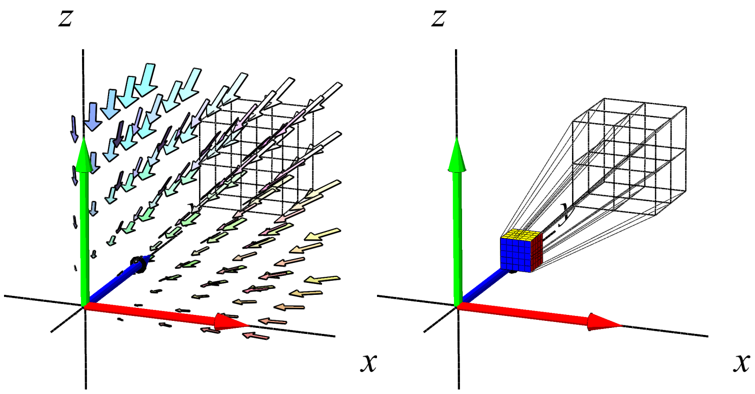

Figure 26.7: The implosion vector field from Example 26.16 together with the integral curves passing through a cube. The cube is shown as a solid to the time $t = 1$ and as "open" to the time $t = 0$. Compare with Figure 26.6.

|||| Definition 26.18

Let $\mathbf{V}(x,y,z) = (V_1(x,y,z),\, V_2(x,y,z),\, V_3(x,y,z))$ be a vector field in (x,y,z) space. The *divergence* of $\mathbf{V}$ at the point (x_0, y_0, z_0) is defined like this:

$$\mathrm{Div}(\mathbf{V})(x_0,y_0,z_0) = \frac{\partial V_1}{\partial x}(x_0,y_0,z_0) + \frac{\partial V_2}{\partial y}(x_0,y_0,z_0) + \frac{\partial V_3}{\partial z}(x_0,y_0,z_0) \quad . \tag{26-35}$$

If $\mathbf{V}(x,y) = (V_1(x,y),\, V_2(x,y))$ is a *plane vector field* we define quite similarly:

$$\mathrm{Div}(\mathbf{V})(x_0,y_0) = \frac{\partial V_1}{\partial x}(x_0,y_0) + \frac{\partial V_2}{\partial y}(x_0,y_0) \quad . \tag{26-36}$$

The divergence of a smooth vector field in $\mathbf{R}^3$ is a smooth *function* in $\mathbf{R}^3$.

Note that the divergence of a plane vector field is the same as the divergence of the spatial extension of the field.

|||| Example 26.19 Simple Divergences

Every constant vector field $\mathbf{V}(x,y,z) = \mathbf{b}$ has the divergence $\mathrm{Div}(\mathbf{V}) = 0$.

The explosion vector field $\mathbf{V}(x,y,z) = (x,y,z)$ has the constant divergence $\mathrm{Div}(\mathbf{V}) = 3$.

The implosion vector field $\mathbf{V}(x,y,z) = (-x,-y,-z)$ has the divergence $\mathrm{Div}(\mathbf{V}) = -3$.

The rotating vector field $\mathbf{V}(x,y,z) = (-y,x,0)$ also has a constant divergence $\mathrm{Div}(\mathbf{V}) = 0$.

|||| Exercise 26.20

Let $\mathbf{V}(x,y,z) = (x+\sin(y),\, z+\cos(y),\, x+y-z)$. Determine $\mathrm{Div}(\mathbf{V})$ at every point in the (x,y,z) space.

|||| Exercise 26.21

Let $\mathbf{V}(x,y,z)$ be a vector field of the first degree with the matrix representation as in Equation (26-10). Show that the divergence of $\mathbf{V}(x,y,z)$ is constant and equal to the trace of $\mathbf{A}$.

|||| Exercise 26.22

Let $\mathbf{V}(x,y,z) = \boldsymbol{\nabla} h(x,y,z)$ be the gradient vector field for a given function $h(x,y,z)$. Show that the divergence of $\mathbf{V}(x,y,z)$ is

$$\mathrm{Div}(\boldsymbol{\nabla} h(x,y,z)) = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \quad . \tag{26-37}$$

In the applications of vector analysis the divergence of gradient vector fields of given functions, $\mathrm{Div}(\boldsymbol{\nabla} h(x,y,z))$ is very often used and therefore is given its own name:

|||| Definition 26.23

Let $h(x,y,z)$ denote a smooth function in $\mathbb{R}^3$. Then we write:

$$\begin{aligned}\Delta h(x,y,z) &= \text{Div}(\boldsymbol{\nabla} h(x,y,z)) \\ &= \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \quad .\end{aligned} \tag{26-38}$$

The function $\Delta h(x,y,z)$ is called the *Laplacian* of the function $h(x,y,z)$.

|||| Example 26.24 The Laplacian

The Laplacian of some elementary functions of three variables:

Function	$\boldsymbol{\nabla} f(x,y,z)$	$\Delta f(x,y,z)$
$f(x,y,z) = a\cdot x + b\cdot y + c\cdot z$	(a,b,c)	0
$f(x,y,z) = x^2 + y^2 + z^2$	$(2x,2y,2z)$	6
$f(x,y,z) = y\cdot\sin(x)$	$(y\cdot\cos(x),\sin(x),\,0)$	$-y\cdot\sin(x)$
$f(x,y,z) = e^x\cdot\cos(z)$	$(e^x\cdot\cos(z),\,0\,,-e^x\cdot\sin(z))$	0

(26-39)

The Laplacian of a smooth function $f(x,y,z)$ is the trace of the 3×3-Hessian matrix of f (see eNote 22):

$$\Delta f(x,y,z) = \textbf{trace}(\boldsymbol{H} f(x,y,z)) \quad . \tag{26-40}$$

26.4 The Curl of a Vector Field

The other quite central concept, a tool for the analysis of vector fields, is the following:

|||| Definition 26.25 The Curl of a Vector Field

Let $\mathbf{V}(x, y, z) = (V_1(x, y, z), V_2(x, y, z), V_3(x, y, z))$ be a vector field in (x, y, z) space. The *curl* of $\mathbf{V}$ at the point (x_0, y_0, z_0) is defined as the following *vector*:

$$
\begin{aligned}
\mathbf{Rot}(\mathbf{V})(x_0, y_0, z_0) = \Big(& \frac{\partial V_3}{\partial y}(x_0, y_0, z_0) - \frac{\partial V_2}{\partial z}(x_0, y_0, z_0)\,, \\
& \frac{\partial V_1}{\partial z}(x_0, y_0, z_0) - \frac{\partial V_3}{\partial x}(x_0, y_0, z_0)\,, \\
& \frac{\partial V_2}{\partial x}(x_0, y_0, z_0) - \frac{\partial V_1}{\partial y}(x_0, y_0, z_0) \Big) \quad .
\end{aligned}
\tag{26-41}
$$

The curl of a smooth vector field in $\mathbf{R}^3$ is in itself a smooth vector field in $\mathbf{R}^3$.

|||| Example 26.26 The Curl of Simple Vector Fields

The explosion vector field $\mathbf{V}(x, y, z) = (x, y, z)$ has constant curl $\mathbf{Curl}(\mathbf{V}) = \mathbf{0}$.

The implosion vector field $\mathbf{V}(x, y, z) = (-x, -y, -z)$ has (not surprisingly) also constant curl $\mathbf{Curl}(\mathbf{V}) = \mathbf{0}$.

The rotating vector field (that rotates *counter-clockwise*) $\mathbf{V}(x, y, z) = (-y, x, 0)$ has constant curl that of course is different from $\mathbf{0}$: $\mathbf{Curl}(\mathbf{V}) = (0, 0, 2)$.

The rotating vector field (that rotates *clockwise*) $\mathbf{V}(x, y, z) = (y, -x, 0)$ also has constant curl that of course is the opposite of the counter-clockwise rotation: $\mathbf{Curl}(\mathbf{V}) = (0, 0, -2)$.

|||| Exercise 26.27

Let $\mathbf{V}(x, y, z) = (x + \sin(y), z + \cos(y), x + y - z)$. Determine $\mathbf{Curl}(\mathbf{V})$ at every point in the space.

|||| Exercise 26.28

Let $\mathbf{V}(x,y,z)$ be a vector field of the first degree with the matrix representation as in equation (26-10). Show that the curl of $\mathbf{V}(x,y,z)$ is a constant vector and express the vector by the elements in $\mathbf{A}$.

26.5 A Bridge between Divergence and Curl

We mention here some relations between divergence, curl and gradient vector fields:

|||| Theorem 26.29 Divergence Versus Curl

Let $h(x,y,z)$ denote a smooth function in (x,y,z) space. Then

$$\mathbf{Curl}(\boldsymbol{\nabla} h) = \mathbf{0} \quad . \tag{26-42}$$

Let $\mathbf{V}(x,y,z)$ and $\mathbf{W}(x,y,z)$ denote two vector fields in $\mathbb{R}^3$. Then the following identity applies

$$\mathrm{Div}(\mathbf{V} \times \mathbf{W}) = \mathbf{Curl}(\mathbf{V}) \cdot \mathbf{W} - \mathbf{V} \cdot \mathbf{Curl}(\mathbf{W}) \quad . \tag{26-43}$$

Therefore we have in particular: If $\mathbf{W}$ is a gradient vector field of a function $h(x,y,z)$ in $\mathbb{R}^3$, i.e. in short form $\mathbf{W} = \boldsymbol{\nabla} h$, then

$$\mathrm{Div}(\mathbf{V} \times \boldsymbol{\nabla} h) = \mathbf{Curl}(\mathbf{V}) \cdot \boldsymbol{\nabla} h \quad . \tag{26-44}$$

|||| Exercise 26.30

Show by direct computation that the two equations (26-42) and (26-43) both are satisfied.

From the quite simple considerations and examples that we have been through in this eNote it is reasonable to expect that the divergence is a measure of how much a given collection of particles are spread or squished when they flow with the vector field. This is exactly the content of Gauss' theorem which is so important for the application of these concepts that it is given its own eNote26.

Similarly we must expect that the total rcurl of a collection of particles flowing with the vector field can be expressed by use of the rotation vector field for the given vector field. This is exactly the content of Stokes' Theorem, which therefore also – for the same reason – has its own eNote 27.

26.6 Flows of Curves and Surfaces

As already hinted with the figures 26.5, 26.6 and 26.7 we can let any geometrically well-defined set, surface, or curve flow with a given vector field – in such a way that every point on the object follows, within the vector field, the unique flow curve passing through the point. The idea is to understand the geometry of the vector field by observing how it moves and deforms geometric objects. See also figures 26.8 and 26.9.

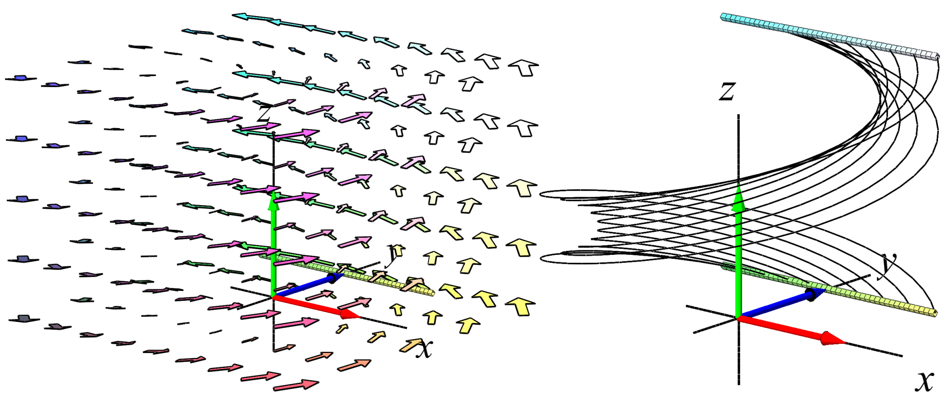

Figure 26.8: A line segment flows with the flow curves of the vector field $\mathbf{V}(x,y,z) = (-y,x,0.3)$.

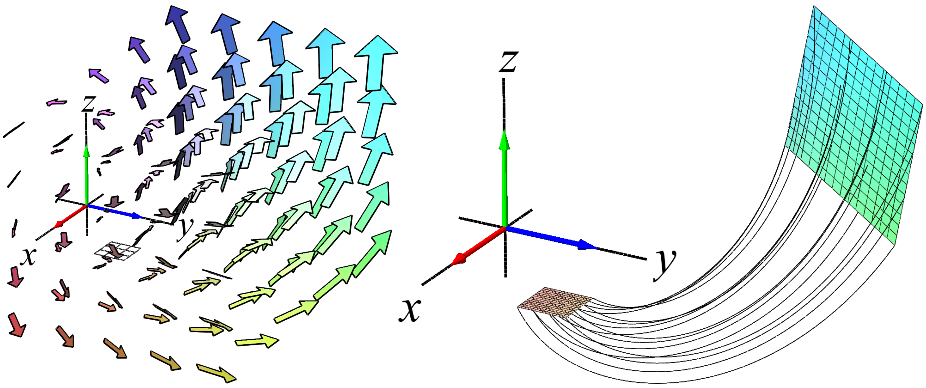

Figure 26.9: A square flows with the flow curves of the vector field $\mathbf{V}(x, y, z) = (-y + (x/9), -z + (y/9), -x + (z/9))$.

26.6.1 The Flow of Level Curves and Level Surfaces

The gradient vector fields of the functions of two and three variables also deform curve and surfaces via their respective flow curves. One could now be led to believe that level curves and level surfaces probably flow over into other level curves and level surfaces by the gradient vector flow. Yet it is not that simple – but almost.

By closer consideration one will realize that it cannot be the case that the gradient vector field in general should make level sets flow into level sets. If e.g. two neighboring level curves in Figure 26.1 are close to each other then the gradient vector is correspondingly large and vice versa if two neighboring level curves lie further apart then the gradient vector is correspondingly smaller. I.e. where the gradient vectors are large we observe slower flow and where they are small we see faster flow for the level curves to flow into each other.

|||| Theorem 26.31 Level Set Flow

Let $f(x,y,z)$ denote a smooth function of three variables with a proper gradient vector field $\nabla f(x,y,z) \neq \mathbf{0}$. Let $\mathbf{V}(x,y,z)$ be the square normed gradient vector field:

$$\mathbf{V}(x,y,z) = \frac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|^2} \quad . \tag{26-45}$$

If we let every point p on the level surface $\mathcal{K}_c(f)$ flow for the time t_0 with the flow curve of $\mathbf{V}(x,y,z)$ that starts at p, then the entire level surface will flow into the level surface $\mathcal{K}_{c+t_0}(f)$.

A similar result applies to gradient vector fields of smooth functions $f(x,y)$ of two variables and their corresponding level curves in the plane.

|||| Proof

We only have to show that if we start (at time $t = 0$) at a point p where $f(p) = c$ then we end up with the flow curve $\mathbf{r}(t)$ of the vector field $\mathbf{V}(x,y,z)$ after the time t_0 at a point $\mathbf{r}(t_0)$ where $f(x,y,z)$ has the value $f(\mathbf{r}(t_0)) = c + t_0$.

We use the chain rule for function value incrementation along a flow curve, see eNote 19:

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \quad , \tag{26-46}$$

and since $\mathbf{r}(t)$ is a flow curve of $\mathbf{V}(x,y,z)$ we know that $\mathbf{r}'(t) = \mathbf{V}(\mathbf{r}(t))$, which substituted into (26-46) gives:

$$\begin{aligned} \frac{d}{dt} f(\mathbf{r}(t)) &= \nabla f(\mathbf{r}(t)) \cdot \mathbf{V}(\mathbf{r}(t)) \\ &= \nabla f(\mathbf{r}(t)) \cdot \frac{\nabla f(\mathbf{r}(t))}{|\nabla f(\mathbf{r}(t))|^2} \\ &= \frac{\nabla f(\mathbf{r}(t)) \cdot \nabla f(\mathbf{r}(t))}{|\nabla f(\mathbf{r}(t))|^2} \\ &= 1 \quad . \end{aligned} \tag{26-47}$$

From this we get the result we wanted directly:

$$\begin{aligned} f(\mathbf{r}(t_0)) &= c + \int_0^{t_0} \frac{d}{dt} f(\mathbf{r}(t))\, \mathrm{d}t \\ &= c + \int_0^{t_0} 1 \, \mathrm{d}t \\ &= c + t_0 \quad . \end{aligned} \tag{26-48}$$

26.7 Summary

We have in this eNote established the first concepts and methods for the analysis of vector fields in the plane and 3D space.

- Some but not all vector fields are gradient vector fields of functions $f(x,y,z)$ (here of three variables):

$$\boldsymbol{\nabla} f(x,y,z) = \Big(f'_x(x,y,z)\,,\, f'_y(x,y,z)\,,\, f'_z(x,y,z)\Big) \quad , \quad (x,y,z) \in \mathbb{R}^3 \quad . \tag{26-49}$$

- Every vector field $\mathbf{V}(x,y,z)$ of the first degree can be written and stated by use of a system matrix $\mathbf{A}$ and a constant vector $\mathbf{b}$:

$$\begin{bmatrix} V_1(x,y,z) \\ V_2(x,y,z) \\ V_3(x,y,z) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad . \tag{26-50}$$

- To understand the geometry of a given vector field it is important to be able to determine the flow curves of the vector field – i.e. the parametrized curves $\mathbf{r}(t) = (x(t), y(t), z(t))$, $t \in [a,b]$, which at all curve-points have the given vector field as the tangent vector. If the vector field is given by $\mathbf{V}(x,y,z) = (V_1(x,y,z), V_2(x,y,z), V_3(x,y,z))$ then the flow curve equation is:

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \left(\mathbf{V}(x(t), y(t), z(t))\right)^\top = \begin{bmatrix} V_1(x(t), y(t), z(t)) \\ V_2(x(t), y(t), z(t)) \\ V_3(x(t), y(t), z(t)) \end{bmatrix} \quad . \tag{26-51}$$

- The divergence of a vector field $\mathbf{V}(x,y,z)$ we have defined as the function that at an arbitrary point (x_0, y_0, z_0) has the value:

$$\mathrm{Div}(\mathbf{V})(x_0,y_0,z_0) \;=\; \frac{\partial V_1}{\partial x}(x_0,y_0,z_0) + \frac{\partial V_2}{\partial y}(x_0,y_0,z_0) + \frac{\partial V_3}{\partial z}(x_0,y_0,z_0) \quad , \tag{26-52}$$

  and we have indicated through very simple examples that the divergence is a local measure for how much the vector field spreads or squishes a given set of particles flowing with the vector field, that is, follows the flow curves of the vector field.

- The curl of a vector field we have defined as the following vector field

$$\begin{aligned}\mathbf{Curl}(\mathbf{V})(x_0,y_0,z_0) \;=\; (\; & \frac{\partial V_3}{\partial y}(x_0,y_0,z_0) - \frac{\partial V_2}{\partial z}(x_0,y_0,z_0)\,, \\ & \frac{\partial V_1}{\partial z}(x_0,y_0,z_0) - \frac{\partial V_3}{\partial x}(x_0,y_0,z_0)\,, \\ & \frac{\partial V_2}{\partial x}(x_0,y_0,z_0) - \frac{\partial V_1}{\partial y}(x_0,y_0,z_0)\;) \quad ,\end{aligned} \tag{26-53}$$

and we have indicated through very simple examples that the curl is a local measure for how much the vector field rotates a given set of particles that flow with the vector field, i.e. follows the flow curves of the vector field.