eNote 25

Surface and Volume Integrals

Surface and volume integrals are here stated in the same way as the curve and plane integrals in eNote 24, thereby, with the basic general introduction in the Riemann integrals in eNote 23 giving the background for the present eNote. The starting point for the determination of the surface- and space-integrals will be the parametric representations of the surface and the spatial region, respectively. To every parametric representation corresponds a Jacobian function and it is this function that is used for stating and computing the integrals.

Updated: 11.1.2021, D.B.
Updated: 31.1.2023, shsp.

25.1 Surface Integrals

A parameterized surface in 3D space is given by a parametric representation that looks a lot like the parametric representations of plane regions, cf. eNote 24. The difference though is the quite central one, that now the z-coordinate is also a function of the two parameter values u and v:

$$F_{\mathbf{r}} : \quad \mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3 \quad , \quad u \in [a,b] \ , \ v \in [c,d] \quad , \qquad (25\text{-}1)$$

where $x(u,v)$, $y(u,v)$, and $z(u,v)$ are given smooth functions of the two variables u and v.

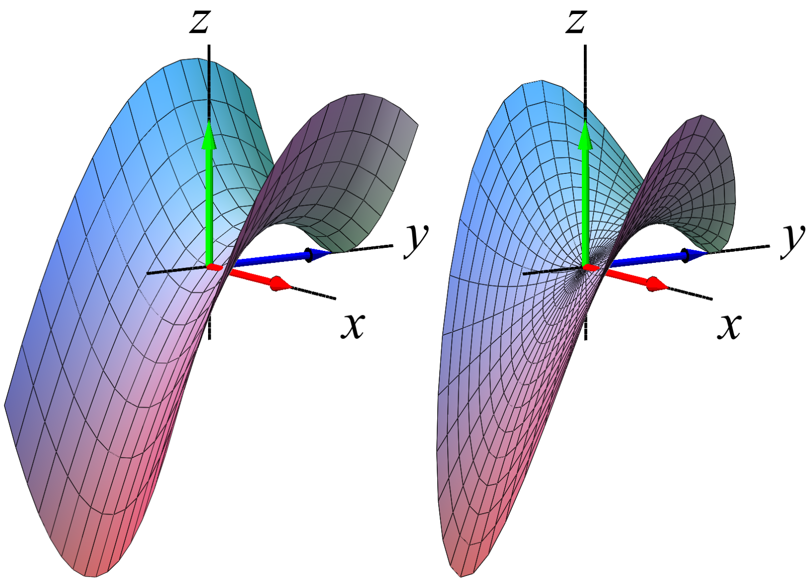

Figure 25.1: The graph surface of the function $h(x,y) = x^2 - y^2 + y$ in part over the square $(x,y) \in [-1,1] \times [-1,1]$ and in part over the circular disc with radius 1 and centre at $(0,0)$ in the (x,y) plane.

|||| Example 25.1 Graph Surfaces of Functions of Two Variables

A function $h(x,y)$ of two variables $(x,y) \in \mathbb{R}^2$ has a graphical surface $\mathcal{F}$ in (x,y,z) space that easily can be 'parameterized' in the stated form:

$$\mathcal{F} \quad : \quad \mathbf{r}(u,v) = (u, v, h(u,v)) \quad , \quad u \in \mathbb{R} \, , \, v \in \mathbb{R} \quad . \tag{25-2}$$

Typically we are only interested in parts of such graphical surfaces, e.g. parts above a rectangle in the (x,y) plane: $x \in [a,b]$, $y \in [c,d]$. This part is parameterized as easily as all of the graphical surface:

$$\widehat{\mathcal{F}} \quad : \quad \widehat{\mathbf{r}}(u,v) = (u, v, h(u,v)) \quad , \quad u \in [a,b] \, , \, v \in [c,d] \quad . \tag{25-3}$$

If we on the contrary are interested in what lies above the circular disc with radius a and centre at $(0,0)$ then we must first parameterize the circular disc in the (x,y) plane with the two parameters u and v and then the graph surface segment can be presented by 'lifting' the points of the circular disc to the correct 'height' with the function $h(x,y)$:

$$\widetilde{\mathcal{F}} \quad : \quad \widetilde{\mathbf{r}}(u,v) = (u \cdot \cos(v), u \cdot \sin(v), h\,(u \cdot \cos(v), u \cdot \sin(v))) \quad , \tag{25-4}$$

where the parameters u and v, now run through the parametrized region for the circular-disc parameterization:

$$u \in [0,a] \quad , \quad v \in [-\pi, \pi] \quad . \tag{25-5}$$

In analogy with the plane integrals (cf. eNote 24) we now define the surface integrals like this:

|||| Definition 25.2 The Surface Integral

Let $f(x, y, z)$ denote a continuous function in $\mathbb{R}^3$. The surface integral of the function $f(x, y, z)$ over the parameterized surface $F_{\mathbf{r}}$ is defined by

$$\int_{F_{\mathbf{r}}} f \, \mathrm{d}\mu = \int_c^d \int_a^b f(\mathbf{r}(u, v)) \, \mathrm{Jacobian}_{\mathbf{r}}(u, v) \, \mathrm{d}u \, \mathrm{d}v \quad , \qquad (25\text{-}6)$$

where the Jacobian function $\mathrm{Jacobian}_{\mathbf{r}}(u, v)$

$$\mathrm{Jacobian}_{\mathbf{r}}(u, v) = |\mathbf{r}'_u(u, v) \times \mathbf{r}'_v(u, v)| \qquad (25\text{-}7)$$

is the area of the parallelogram that at the location $\mathbf{r}(u, v)$ is spanned by the two tangent vectors $\mathbf{r}'_u(u, v)$ and $\mathbf{r}'_v(u, v)$ to the respective coordinate curves through the point $\mathbf{r}(u, v)$ on the surface.

|||| Definition 25.3 Regular Parametric Representation

The parametric representation (25-1) is said to be a *regular parametric representation* if the following applies:

$$\mathrm{Jacobian}_{\mathbf{r}}(u, v) > 0 \quad \text{for all} \quad u \in [a, b] \; , \; v \in [c, d] \quad . \qquad (25\text{-}8)$$

|||| Definition 25.4 One-to-One Parametric Representation

As for parameterized curves the parametric representation in (25-1) is said to be *one-to-one* if different points in the domain are mapped to different points in the range.

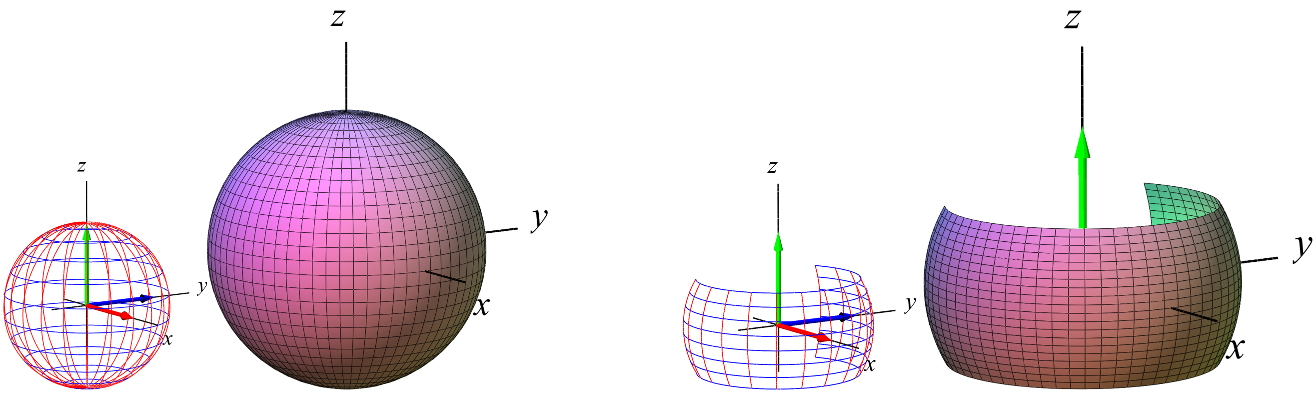

Figure 25.2: The unit spherical surface and a segment of the unit spherical surface. The segment is constructed by reducing the parameter domain to $(u, v) \in [\pi/3, 2\pi/3] \times [-2\pi/3, 2\pi/3]$.

|||| Example 25.5 Spherical Surface Parametrization

All of the spherical surface with radius R and centre at $(0, 0, 0)$ can be parameterized with 'geographical' parameters like this:

$$\mathcal{S}_R \quad : \quad \mathbf{r}(u, v) = (R \cdot \sin(u) \cdot \cos(v), R \cdot \sin(u) \cdot \sin(v), R \cdot \cos(u)) \quad , \tag{25-9}$$

where $u \in [0, \pi]$ and $v \in [-\pi, \pi]$.

The Jacobian function corresponding to this parameterization is determined by the following computations:

$$\begin{aligned} \mathbf{r}'_u &= R \cdot (\cos(u) \cdot \cos(v), \cos(u) \cdot \sin(v), -\sin(v)) \quad , \\ \mathbf{r}'_v &= R \cdot (-\sin(u) \cdot \sin(v), \sin(u) \cdot \cos(v), 0) \quad , \\ \text{Jacobian}_{\mathbf{r}}(u, v) &= |\mathbf{r}'_u \times \mathbf{r}'_v| = R^2 \cdot \sin(u) \quad . \end{aligned} \tag{25-10}$$

This parameterization is neither regular everywhere nor one-to-one in the given parametrized region. Why not? Where and how is the regularity broken? Where and how is the one-to-one property broken? See Section 24.4 in eNote 24.

|||| Example 25.6 Graph Surfaces

Every standard parameterization of a graph surface of a smooth function $h(x, y)$ of two variables is one to one and regular everywhere. If we look at the standard parameterization

$$\mathbf{r}(u, v) = (u, v, h(u, v)) \quad , \quad u \in [a, b] \; , \; v \in [c, d] \tag{25-11}$$

we get the Jacobian function by the following general computation:

$$\begin{aligned} \mathbf{r}'_u(u,v) &= (1,0,h'_u(u,v)) \quad , \\ \mathbf{r}'_v(u,v) &= (0,1,h'_v(u,v)) \quad , \\ \mathbf{r}'_u(u,v)\times\mathbf{r}'_v(u,v) &= (-h'_u(u,v),-h'_v(u,v),1) \quad , \\ \text{Jacobian}_{\mathbf{r}}(u,v) &= \sqrt{1+(h'_u(u,v))^2+(h'_v(u,v))^2} = \sqrt{1+|\boldsymbol{\nabla} h(u,v)|^2} \quad , \end{aligned} \tag{25-12}$$

that clearly is positive for all (u,v) in the parametrized region.

|||| Definition 25.7 The Area of a Surface

The area of the parameterized surface

$$F_{\mathbf{r}}: \quad \mathbf{r}(u,v) = (x(u,v),y(u,v),z(u,v)) \quad , \quad u\in[a,b]\ ,\ v\in[c,d]$$

is defined as the surface integral of the constant function 1 over the surface:

$$\text{Area}(F_{\mathbf{r}}) = \int_{F_{\mathbf{r}}} 1\,\mathrm{d}\mu = \int_c^d\int_a^b \text{Jacobian}_{\mathbf{r}}(u,v)\,\mathrm{d}u\,\mathrm{d}v \quad . \tag{25-13}$$

|||| Example 25.8 The Area of a Graph Surface

The area of the graph surface of the function $h(x,y)$ over the rectangular region $[a,b]\times[c,d]$ in the (x,y) plane is therefore:

$$\text{Area}(\widehat{\mathcal{F}}) = \int_{\widehat{\mathcal{F}}} 1\,\mathrm{d}\mu = \int_c^d\int_a^b \text{Jacobian}_{\mathbf{r}}(u,v)\,\mathrm{d}u\,\mathrm{d}v \quad , \tag{25-14}$$

where

$$\begin{aligned} \widehat{\mathcal{F}} = F_{\mathbf{r}} \quad &: \quad \mathbf{r}(u,v) = (u,v,h(u,v)) \quad , \quad u\in[a,b]\ ,\ v\in[c,d] \quad , \\ \text{Jacobian}_{\mathbf{r}}(u,v) &= \sqrt{1+|\nabla h(u,v)|^2} \end{aligned} \tag{25-15}$$

such that

$$\text{Area}(\widehat{\mathcal{F}}) = \int_c^d\int_a^b \sqrt{1+|\nabla h(u,v)|^2}\,\mathrm{d}u\,\mathrm{d}v \quad . \tag{25-16}$$

|||| Exercise 25.9

Determine the area of that part of the graph surface of the function $h(x, y) = x + y - 1$ that lies above the square $(x, y) \in [0, 1] \times [0, 1]$ in the (x, y) plane. Note that this part of the graph surface is a parallelogram. Use both double integration as in Example 25.8 and the classical area determination by use of base and height.

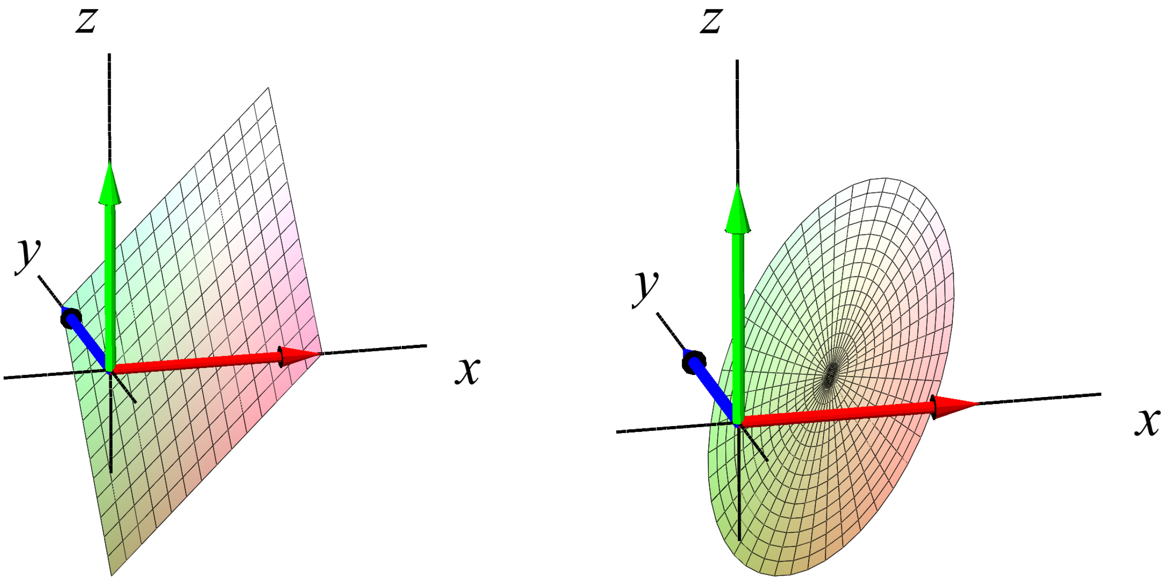

Figure 25.3: Parts of a plane graph surface.

|||| Exercise 25.10

Determine the area of the part of the graph surface of the function $h(x, y) = x + y - 1$ that lies above the circular disc in the (x, y) plane that has the radius $1/2$ and centre at $(1/2, 1/2)$. Note that this part of the graph surface is a plane region bounded by an ellipsis.

|||| Exercise 25.11

Determine the area of that part of the graph surface of the function $h(x, y) = x \cdot y$ that lies above the square $(x, y) \in [-1, 1] \times [-1, 1]$ in the (x, y) plane.

|||| Example 25.12 The Area of a Spherical Surface

A part of a spherical surface with radius R and centre at $(0,0,0)$ is parameterized like this:

$$\widehat{\mathcal{S}} \quad : \quad \widehat{\mathbf{r}}(u,v) = (R\cdot\sin(u)\cdot\cos(v), R\cdot\sin(u)\cdot\cos(v), R\cdot\cos(u)) \quad , \tag{25-17}$$

where $u \in [a,b] \subset [0,\pi]$ and $v \in [c,d] \subset [-\pi,\pi]$.

The area of that part of the spherical surface is then, since $\text{Jacobian}_{\widehat{\mathbf{r}}}(u,v) = R^2\cdot\sin(u)$:

$$\begin{aligned} \text{Area}(\widehat{\mathcal{S}}) &= \int_c^d \int_a^b R^2\cdot\sin(u)\,\mathrm{d}u\,\mathrm{d}v \\ &= R^2\cdot(d-c)\cdot[-\cos(u)]_{u=a}^{u=b} \\ &= R^2\cdot(d-c)\cdot(\cos(a)-\cos(b)) \quad . \end{aligned} \tag{25-18}$$

Therefore we also get in particular that the area of all of the spherical surface with $a = 0$, $b = \pi$, $c = -\pi$, and $d = \pi$:

$$\text{Area}(\mathcal{S}_R) = 4\pi\cdot R^2 \quad . \tag{25-19}$$

25.1.1 Motivation for the Surface Integral

If we – in keeping with the motivation for the line integral – partition *both* the intervals $[a,b]$ and $[c,d]$ in n and m equal parts, respectively, then every u-subinterval has the length $\delta_u = (b-a)/n$ and every v-subinterval has the length $\delta_v = (d-c)/m$. Similarly the coordinates of the division points in the (u,v)-parameter region (which is the rectangle $[a,b]\times[c,d]$ i $\mathbb{R}^2$) become - cf. the section on double integral sums in eNote 23:

$$\begin{aligned} (u_1,v_1) &= (a,c), \\ (u_1,v_j) &= (a, c+(j-1)\delta_v), \\ (u_i,v_1) &= (a+(i-1)\delta_u, c), \\ (u_i,v_j) &= (a+(i-1)\delta_u, c+(j-1)\delta_v), \\ &\;\;.... \\ (b,d) &= (a+n\delta_u, c+m\delta_v) \quad . \end{aligned} \tag{25-20}$$

With each of these given points (u_i, v_j) as development points we can consider Taylor's limit formula for every of the 3 coordinate functions of $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ to the first order with the corresponding epsilon functions:

$$\begin{aligned}\mathbf{r}(u,v) &= \mathbf{r}(u_i,v_j) \\ &\quad + \mathbf{r}'_u(u_i,v_j)\cdot(u-u_i) \\ &\quad + \mathbf{r}'_v(u_i,v_j)\cdot(v-v_j) \\ &\quad + \rho_{ij}\cdot\boldsymbol{\varepsilon}_{ij}(u-u_i, v-v_i) \quad ,\end{aligned} \tag{25-21}$$

where $u \in [\,u_i\,,\, u_i+\delta_u\,]$, $v \in [\,v_j\,,\, v_j+\delta_v\,]$. Here $\rho_{ij} = \sqrt{(u-u_i)^2+(v-v_j)^2}$ denotes the distance between the variable point $(u\,,v)$ and the given development point $(u_i\,,v_j)$ in the parametrized region. It applies here that $\boldsymbol{\varepsilon}_{ij}(u-u_i, v-v_j) \to (0,0,0) = \mathbf{0}$ for $(u-u_i, v-v_j) \to (0,0)$.

Every subrectangle $[u_i, u_i+\delta_u] \times [v_j, v_j+\delta_v]$ is mapped onto the surface *segment* $\mathbf{r}(u,v)$, $u \in [u_i, u_i+\delta_u], v \in [v_j, v_j+\delta_v]$ and this surface segment we can approximate with the linear part of the expression in (25-21) that we get by removing the $\boldsymbol{\varepsilon}_{ij}$-term form the right-hand side in (25-21):

$$\mathbf{r}_{\mathrm{app}_{ij}}(u,v) = \mathbf{r}(u_i,v_j) + \mathbf{r}'_u(u_i,v_j)\cdot(u-u_i) + \mathbf{r}'_v(u_i,v_j)\cdot(v-v_j) \quad , \tag{25-22}$$

where u and v still run through the subintervals $u \in [\,u_i\,,\, u_i+\delta_u\,]$, $v \in [\,v_j\,,\, v_j+\delta_v\,]$.

These linear approximations are parallelograms spanned by the two tangent vectors $\mathbf{r}'_u(u_i,v_j)\cdot\delta_u$ and $\mathbf{r}'_v(u_i,v_j)\cdot\delta_v$. See Figure 25.4 where the approximating parallelograms are shown for a parameterization of a conic surface.

Area

Each of the total of $n\,m$ approximating parallelograms has an area. The area of the (i,j)th parallelogram is the length of the cross product of the two vectors that span the parallelogram in question:

$$\Delta\,\mathrm{Area}_{ij} = |(\mathbf{r}'_u(u_i,v_j)\cdot\delta_u)\times(\mathbf{r}'_v(u_i,v_j)\cdot\delta_v)| = \mathrm{Jacobian}_{\mathbf{r}}(u_i,v_j)\cdot\delta_u\delta_v \quad . \tag{25-23}$$

|||| Exercise 25.13

Prove this statement: The area of a parallelogram is equal to the length of the cross product of the two vectors that span the parallelogram.

The sum of this total of $n \cdot m$ areas is clearly a good approximation to the area of the whole surface segment, such that we have

$$\text{Area}_{\text{app}}(n,m) = \sum_{j=1}^{m}\sum_{i=1}^{n} \Delta\,\text{Area}_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} \text{Jacobian}_{\mathbf{r}}(u_i, v_j) \cdot \delta_u \delta_v \quad . \tag{25-24}$$

Since the sum above is an integral sum for the continuous function $\text{Jacobian}_{\mathbf{r}}(u,v)$ over the parametrized rectangle $[a,b] \times [c,d]$ we get in the limit, where n and m both tend towards infinity (cf. eNote 23):

$$\text{Area}_{\text{app}}(n,m) \rightarrow \text{Area} = \int_c^d \int_a^b \text{Jacobian}_{\mathbf{r}}(u,v)\,\mathrm{d}u\,\mathrm{d}v = \int_{F_{\mathbf{r}}} 1\,\mathrm{d}\mu \quad \text{for} \quad n\,,\, m \rightarrow \infty \quad . \tag{25-25}$$

This is the basic argument for the definition of the area of a parameterized surface as stated above, viz. as the surface integral of the constant function 1.

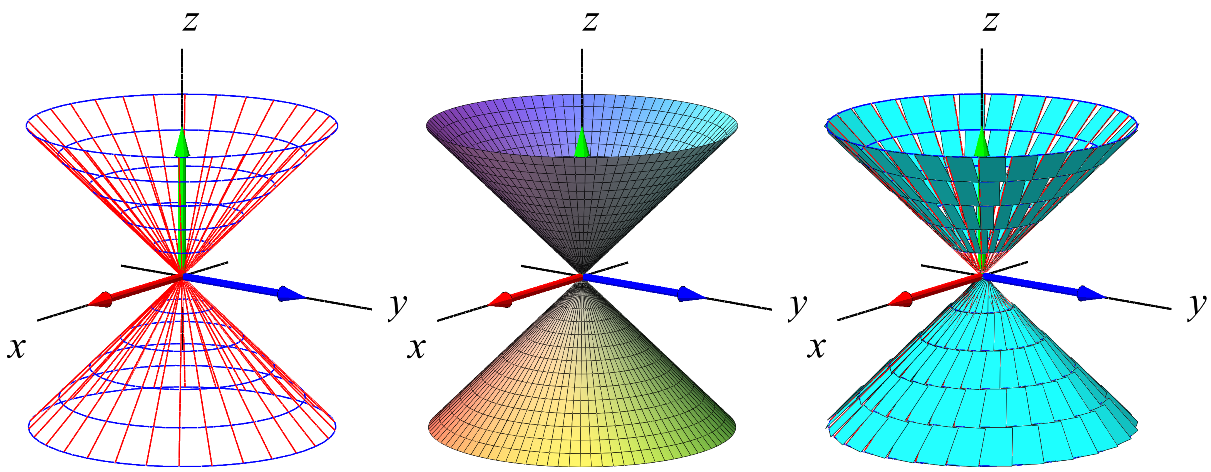

Figure 25.4: The conic surface is given by the parametric representation $\mathbf{r}(u,v) = (u\cos(v), u\sin(v), u)$, $u \in [-1,1]$, $v \in [-\pi,\pi]$. A system of coordinate curves on the surface is shown to the left and the corresponding area-approximating parallelograms are shown to the right.

|||| Exercise 25.14

Show that the given parametric representation in Figure 25.4 is neither a regular nor a one-to-one mapping in the given parametrized region. Consider whether a regular parametric representation of the conic surface can be found.

|||| Exercise 25.15

Why are the approximating parallelograms on the upper half of the conic surface in Figure 25.4 smaller than the corresponding parallelograms (with equal distance to the vertex) on the lower part?

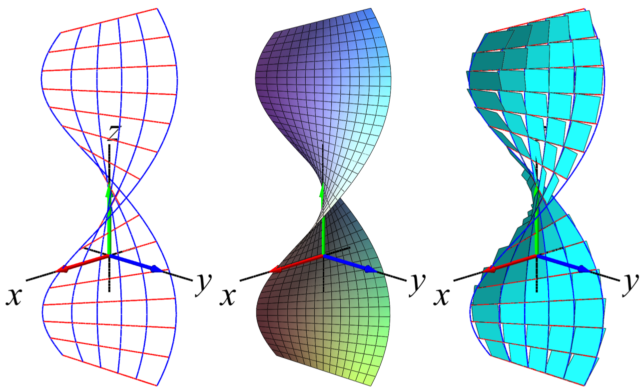

Figure 25.5: This helicoid is given by the parametric representation $\mathbf{r}(u,v) = (\sinh(u)\cos(v), \sinh(u)\sin(v), v)$. The figure also shows an approximation of the surface with parallelograms that are all in fact *squares* of different sizes. See Exercise 25.16.

|||| Exercise 25.16

Show that the approximating parallelograms in Figure 25.5 are all squares.

The Mass of a Surface

If we now assume that each individual parallelogram in (25-22) is allotted a constant mass density given by the value of the function $f(x,y,z)$ at the point of contact between the parallelogram and the surface, then we get the mass of the (i, j)th parallelogram :

$$\begin{aligned}\Delta\,\mathrm{M}_{ij} &= f(x(u_i, v_j), y(u_i, v_j), z(u_i, v_j))\,\mathrm{Jacobian}_{\mathbf{r}}(u_i, v_j)\cdot\delta_u\delta_v \\ &= f(\mathbf{r}(u_i, v_j))\,\mathrm{Jacobian}_{\mathbf{r}}(u_i, v_j)\cdot\delta_u\delta_v \quad .\end{aligned} \tag{25-26}$$

Therefore the total mass of the whole system of parallelograms is the following, which is a good approximation to the mass of the whole surface when this is given the mass density $f(\mathbf{r}(u, v))$ in the point $\mathbf{r}(u, v)$.

$$\mathrm{M}_{\mathrm{app}}(n, m) = \sum_{j=1}^{m}\sum_{i=1}^{n}\Delta\,\mathrm{M}_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} f(\mathbf{r}(u_i, v_j))\,\mathrm{Jacobian}_{\mathbf{r}}(u_i, v_j)\cdot\delta_u\delta_v \quad . \tag{25-27}$$

This is a double integral sum for the continuous function $f(\mathbf{r}(u, v))\,\mathrm{Jacobian}_{\mathbf{r}}(u, v)$ over the parametrized rectangle $[a, b] \times [c, d]$. So we get in the limit, where n and m tend towards infinity:

$$\mathrm{M}_{\mathrm{app}}(n, m) \to \mathrm{M} = \int_c^d \int_a^b f(\mathbf{r}(u, v))\mathrm{Jacobian}_{\mathbf{r}}(u, v)\,\mathrm{d}u\,\mathrm{d}v \quad \text{for} \quad n\,,\, m \to \infty \quad . \tag{25-28}$$

Thus we have motivated the definition of the mass of a parameterized surface with the mass density $f(\mathbf{r}(u, v))$ and hereby also the general definition of the surface integral, definition 25.2.

25.1.2 Surfaces of Revolution

Surfaces of revolution are the special surfaces that appear by rotating a plane curve about a straight line (the axis of rotation) in the plane of the curve. The curve is called a *profile curve* or a *generatrix*. It is assumed that the profile curve does not intersect the axis of rotation. The profile curve is typically chosen in the (x, z) plane and is rotated about the z-axis in an (x, y, z) coordinate system. The profile curve can then be represented by a parametric representation like this:

$$G_{\mathbf{p}}: \quad \mathbf{p}(u) = (g(u), 0, h(u)) \in \mathbb{R}^3 \quad , \quad u \in [a, b] \quad , \tag{25-29}$$

where $g(u) > 0$ and $h(u)$ are given functions of the parameter u. The surface of revolution that appears by rotating $G_{\mathbf{p}}$ a whole turn around the z-axis therefore has the parametric representation:

$$FG_{\mathbf{r}}: \quad \mathbf{r}(u, v) = (g(u)\cos(v), g(u)\sin(v), h(u)) \quad , \quad u \in [a, b] \,,\, v \in [-\pi, \pi] \quad . \tag{25-30}$$

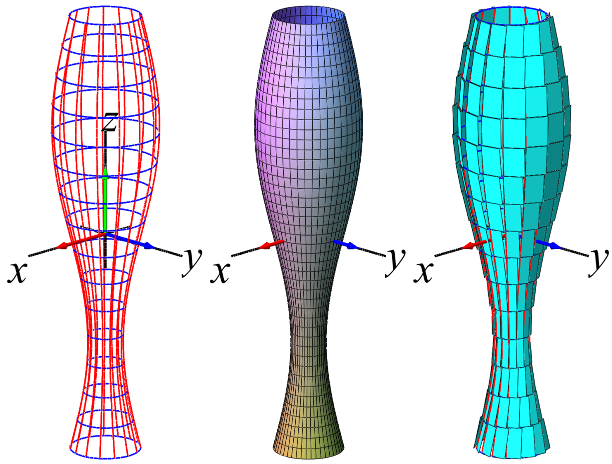

Figure 25.6: The surface of revolution here is given by the parametric representation $\mathbf{r}(u,v) = (g(u)\cos(v), g(u)\sin(v), h(u))$, $u \in [-\pi,\pi]$, $v \in [-\pi,\pi]$, where $g(u) = \frac{1}{2} + \frac{1}{4}\sin(u)$ and $h(u) = u$.

|||| Exercise 25.17

Show that the Jacobian function $\text{Jacobian}_{\mathbf{r}}(u,v)$ for the parametric representation $\mathbf{r}(u,v)$ for the general surface of revolution $FG_{\mathbf{r}}$ in (25-30) is given by

$$\text{Jacobian}_{\mathbf{r}}(u,v) = g(u)\sqrt{(h'(u))^2 + (g'(u))^2} \quad . \tag{25-31}$$

|||| Example 25.18 Torus Area

A given torus is parameterized in the following way:

$$\mathcal{T} \quad : \quad \mathbf{r}(u,v) = (g(u)\cos(v), g(u)\sin(v), h(u)) \ , \ u \in [-\pi,\pi] \ , \ v \in [-\pi,\pi] \quad , \tag{25-32}$$

where $g(u) = 2 + \cos(u)$ and $h(u) = \sin(u)$.

The Jacobian function is

$$\text{Jacobian}_{\mathbf{r}}(u,v) = g(u)\sqrt{(h'(u))^2 + (g'(u))^2} = 2 + \cos(u) \quad , \tag{25-33}$$

so the area of this torus is simply:

$$\text{Area}(\mathcal{T}) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(2+\cos(u))\,\mathrm{d}u\,\mathrm{d}v = 8\pi^2 \quad . \tag{25-34}$$

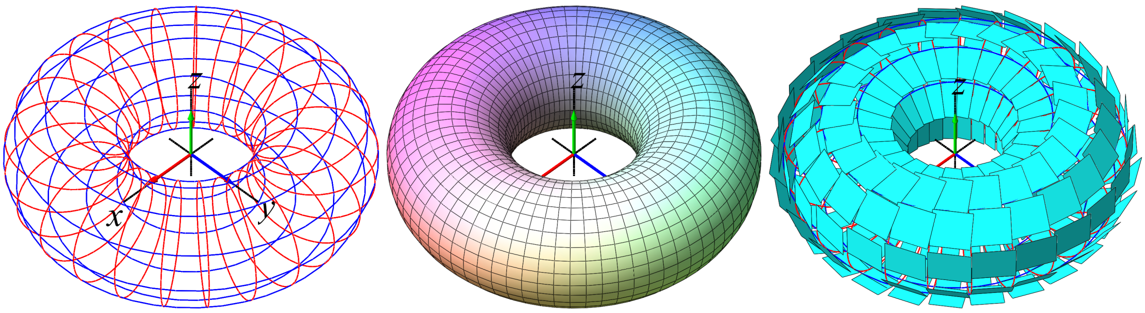

Figure 25.7: This so-called torus is a surface of revolution given by the parametric representation $\mathbf{r}(u,v) = (g(u)\cos(v), g(u)\sin(v), h(u))$, $u \in [-\pi,\pi]$, $v \in [-\pi,\pi]$, where now $g(u) = 2+\cos(u)$ and $h(u) = \sin(u)$.

If we 'load' this torus with a weight function, mass density, given by the function $f(x,y,z) = 2+z$ we get the total weight of the torus:

$$\mathrm{M}(\mathcal{T}) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(2+\sin(u))\cdot(2+\cos(u))\,\mathrm{d}u\,\mathrm{d}v = 16\pi^2 \quad . \tag{25-35}$$

25.2 Triple Integrals

A parameterized spatial region is similar to curves and surfaces given by a parametric representation, now with the following form where the three coordinate functions x, y, and z now are functions of the three parameter variables u, v, and w:

$$\begin{aligned}\Omega_{\mathbf{r}}: \quad \mathbf{r}(u,v,w) &= (x(u,v,w), y(u,v,w), z(u,v,w)) \in \mathbb{R}^3 \quad , \\ & u \in [a,b] \ , \ v \in [c,d], \ \ w \in [h,l] \quad .\end{aligned} \tag{25-36}$$

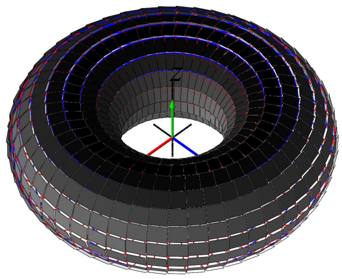

Figure 25.8: This torus is the same as in Figure 25.7, but is here given by the weight function $f(x,y,z) = 2+z$. The total mass of the weighted torus is $M = 16\pi^2$.

|||| Definition 25.19 Triple Integral, Volume Integral

Let $f(x,y,z)$ denote a continuous function in $\mathbb{R}^3$. The triple integral or volume integral of the function $f(x,y,z)$ over the parameterized spatial region $\Omega_{\mathbf{r}}$ is defined by

$$\int_{\Omega_{\mathbf{r}}} f \, \mathrm{d}\mu = \int_h^l \int_c^d \int_a^b f(\mathbf{r}(u,v,w)) \,\mathrm{Jacobian}_{\mathbf{r}}(u,v,w) \, \mathrm{d}u \, \mathrm{d}v \, dw \quad , \tag{25-37}$$

where the Jacobian function $\mathrm{Jacobian}_{\mathbf{r}}(u,v,w)$ is now given by

$$\mathrm{Jacobian}_{\mathbf{r}}(u,v,w) = |\,(\mathbf{r}'_u(u,v,w) \times \mathbf{r}'_v(u,v,w)) \cdot \mathbf{r}'_w(u,v,w)\,| \quad . \tag{25-38}$$

That is, $\mathrm{Jacobian}_{\mathbf{r}}(u,v,w)$ is the volume (here computed as a spatial product) of the parallelepiped that at the position $\mathbf{r}(u,v,w)$ is spanned by the three coordinate curve tangent vectors $\mathbf{r}'_u(u,v,w)$, $\mathbf{r}'_v(u,v,w)$ and $\mathbf{r}'_w(u,v,w)$.

|||| Exercise 25.20

Show that the Jacobian function $\text{Jacobian}_{\mathbf{r}}(u,v,w)$ can also be found as the numerical value of the determinant of the matrix that as columns has the coordinates of the three vectors $\mathbf{r}'_u(u,v,w)$, $\mathbf{r}'_v(u,v,w)$ and $\mathbf{r}'_w(u,v,w)$.

|||| Definition 25.21 Regular Parametric Representation

The parametric representation in (25-36) is called a *regular parametric representation* if $\text{Jacobian}_{\mathbf{r}}(u,v,w) > 0$ for all $u \in [a,b]$, $v \in [c,d]$, $w \in [h,l]$.

|||| Definition 25.22 One-to-One Parametric Representation

As for curves and surfaces we will call the parametric representation in (25-36) one-to-one if different points in the domain are mapped to different points in the range.

|||| Definition 25.23 Volume

The volume of the spatial region

$$\Omega_{\mathbf{r}} : \quad \mathbf{r}(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w)) \quad , \tag{25-39}$$

where

$$u \in [a,b] \quad , \quad v \in [c,d] \quad , \quad \text{and} \quad w \in [h,l] \quad , \tag{25-40}$$

is defined as the triple integral of the constant function 1:

$$\text{Vol}(\Omega_{\mathbf{r}}) = \int_{\Omega_{\mathbf{r}}} 1 \, d\mu = \int_h^l \int_c^d \int_a^b \text{Jacobian}_{\mathbf{r}}(u,v,w) \, du \, dv \, dw \quad . \tag{25-41}$$

|||| Exercise 25.24

Show that the parametric representation in Figure 25.9 is regular and one-to-one.

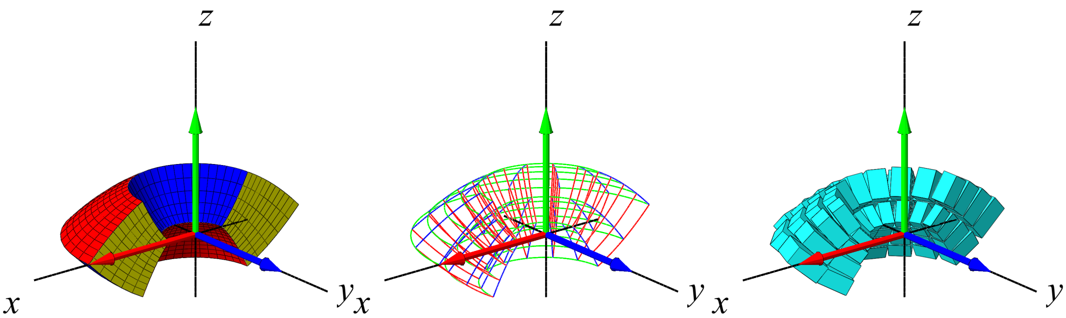

Figure 25.9: Image of the spatial region given by the parametric representation $\mathbf{r}(u,v,w) = (u\,v\cos(w), u\,v\sin(w), \frac{1}{2}(u^2 - v^2))$, $u \in [\frac{1}{2}, 1]$, $v \in [\frac{1}{2}, 1]$, $w \in [\pi, 2\pi]$.

25.2.1 Motivation for the Triple Integral

The intervals $[a,b]$, $[c,d]$ and $[h,l]$ are partitioned into n, m and q equal parts, respectively. Then every u-subinterval has the length $\delta_u = (b-a)/n$, every v-subinterval has the length $\delta_v = (d-c)/m$ and every w-interval has the length $\delta_w = (l-h)/q$. Correspondingly the coordinates of every division point in the (u,v,w)-parameter region (which here is the right-angled 'box-region' $[a,b] \times [c,d] \times [h,k]$ in $\mathbb{R}^3$.

$$
\begin{aligned}
(u_1, v_1, w_1) &= (a, c, h), \\
&\;\;.... \\
(u_i, v_j, w_k) &= (a + (i-1)\delta_u, c + (j-1)\delta_v, h + (k-1)\delta_w), \\
&\;\;.... \\
(b, d, l) &= (a + n\delta_u, c + m\delta_v, h + q\delta_w) \quad .
\end{aligned}
\tag{25-42}
$$

With any of these given points (u_i, v_j, w_k) as development points we can again use Taylor's limit formula for each of the 3 coordinate functions of

$$
\mathbf{r}(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))
$$

to the first order and with corresponding epsilon functions:

$$
\begin{aligned}
\mathbf{r}(u,v,w) = \; & \mathbf{r}(u_i, v_j, w_k) \\
& + \mathbf{r}'_u(u_i, v_j, w_k) \cdot (u - u_i) \\
& + \mathbf{r}'_v(u_i, v_j, w_k) \cdot (v - v_j) \\
& + \mathbf{r}'_w(u_i, v_j, w_k) \cdot (w - w_k) \\
& + \rho_{ijk} \cdot \boldsymbol{\varepsilon}_{ijk}(u - u_i, v - v_j, w - w_k) \quad ,
\end{aligned}
\tag{25-43}
$$

where $u \in [u_i, u_i+\delta_u]$, $v \in [v_j, v_j+\delta_v]$, $w \in [w_j, w_j+\delta_w]$. The distance between the variable point (u, v, w) and the given point (u_i, v_j, w_k) in the parametrized region is denoted by ρ_{ijk} and as before we have $\boldsymbol{\varepsilon}_{ijk}(u-u_i, v-v_j, w-w_k) \to \mathbf{0}$ for $(u-u_i, v-v_j, w-w_k) \to (0,0,0)$.

Every parameter subregion or sub-box $[u_i, u_i+\delta_u] \times [v_j, v_j+\delta_v] \times [w_k, w_k+\delta_w]$ is mapped onto the spatial image region $\mathbf{r}(u,v,w)$, $u \in [u_i, u_i+\delta_u], v \in [v_j, v_j+\delta_v], w \in [w_k, w_k+\delta_w]$ in the mapping range and this region we can approximate with the linear part of the expression in (25-43) that we get by removing the $\boldsymbol{\varepsilon}_{ijk}$-term from the right-hand side in (25-43):

$$\begin{aligned} \mathbf{r}_{\text{app}\,ijk}(u,v,w) \;=\; & \mathbf{r}(u_i, v_j, w_k) \\ & + \mathbf{r}'_u(u_i, v_j, w_k) \cdot (u-u_i) \\ & + \mathbf{r}'_v(u_i, v_j, w_k) \cdot (v-v_j) \\ & + \mathbf{r}'_w(u_i, v_j, w_k) \cdot (w-w_k) \quad , \end{aligned} \tag{25-44}$$

where we still have that $u \in [u_i, u_i+\delta_u]$, $v \in [v_j, v_j+\delta_v]$, $w \in [w_j, w_j+\delta_w]$.

These linear spatial approximations are parallelepipeds spanned by the three tangent vectors $\mathbf{r}'_u(u_i, v_j, w_k) \cdot \delta_u$, $\mathbf{r}'_v(u_i, v_j, w_k) \cdot \delta_v$ and $\mathbf{r}'_w(u_i, v_j, w_k) \cdot \delta_w$.

Volume

Each one of the $n\,m\,q$ approximating parallelepipeds has a volume. The volume of the (i,j,k)th parallelepiped is the numerical value of the spatial product of the three vectors spanning the parallelepiped in question:

$$\begin{aligned} \Delta\,\text{Vol}_{ijk} &= \left| \left((\mathbf{r}'_u(u_i, v_j, w_k) \cdot \delta_u) \times (\mathbf{r}'_v(u_i, v_j, w_k) \cdot \delta_v) \right) \bullet (\mathbf{r}'_w(u_i, v_j, w_k) \cdot \delta_w) \right| \\ &= \text{Jacobian}_{\mathbf{r}}(u_i, v_j, w_k) \cdot \delta_u \cdot \delta_v \cdot \delta_w \quad . \end{aligned} \tag{25-45}$$

|||| Exercise 25.25

Prove this statement: The volume of a parallelepiped is the numerical value of the spatial product of its three spanning vectors.

The sum of the total of $n\,m\,q$ volumes is a good approximation to the volume of the whole spatial region, so that we have

$$\begin{aligned}\mathrm{Vol}_{\mathrm{app}}(n,m,q) &= \sum_{k=1}^{q}\sum_{j=1}^{m}\sum_{i=1}^{n}\Delta\,\mathrm{Vol}_{ijk} \\ &= \sum_{k=1}^{q}\sum_{j=1}^{m}\sum_{i=1}^{n}\mathrm{Jacobian}_{\mathbf{r}}(u_i,v_j,w_k)\cdot\delta_u\delta_v\delta_w \quad .\end{aligned} \tag{25-46}$$

Since the above sum is a triple integral sum for the continuous function of the three variables, $\mathrm{Jacobian}_{\mathbf{r}}(u,v,w)$, over the parameter box $[a,b]\times[c,d]\times[h,l]$ we get in the limit, where n, m and q all tend towards infinity:

$$\mathrm{Vol}_{\mathrm{app}}(n,m,q) \rightarrow \mathrm{Vol} = \int_h^l\int_c^d\int_a^b \mathrm{Jacobian}_{\mathbf{r}}(u,v,w)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \quad \text{for} \quad n\,,\,m\,,\,q\rightarrow\infty \quad . \tag{25-47}$$

This is the justification for the definition of the volume of a parameterized region in (x,y,z) space as given above, viz. as the triple integral of the constant function 1.

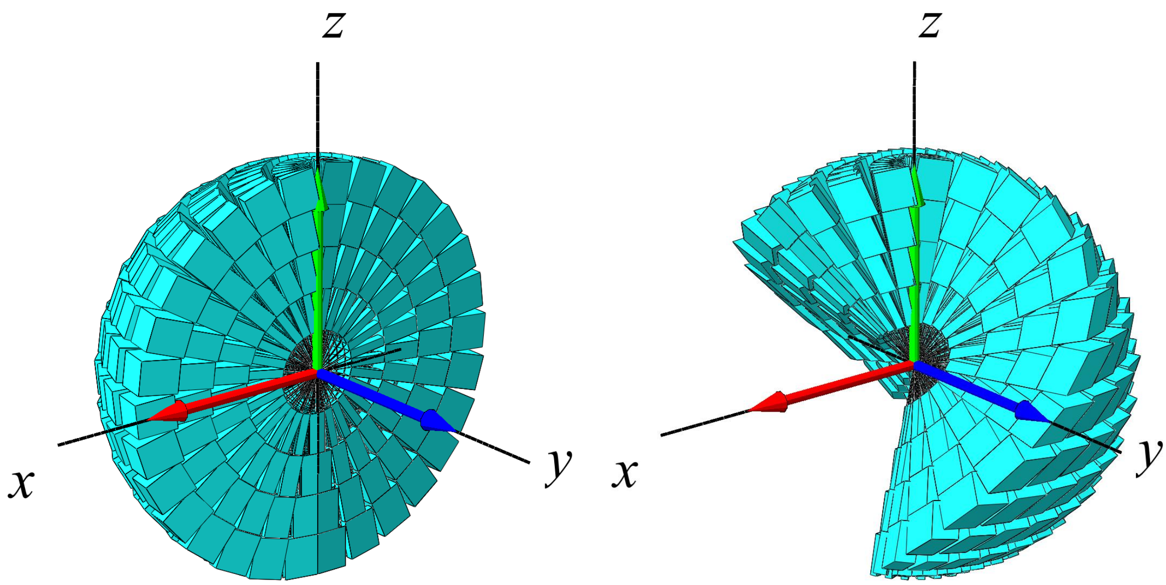

Figure 25.10: Two different partial spherical tessellations with parallelepipeds. See Exercise 25.26.

|||| Exercise 25.26

The spherical parametric representations in Figure 25.10 are as follows, respectively:

$$\begin{aligned}\mathbf{r}_1(u,v,w) &= [\,w\sin(u)\cos(v), w\sin(u)\sin(v), w\cos(u)\,]\\ \mathbf{r}_2(u,v,w) &= [\,w\sin(v)\cos(u+v), w\sin(v)\sin(u+v), w\cos(v)\,]\ ,\end{aligned}$$

where the parameter intervals in both cases are given by:

$$u\in[-\pi,0]\ ,\ v\in[0,\pi]\ ,\ w\in[0,1]\quad .$$

Determine the Jacobian functions of both of the two parametric representations and show that both of the shown spatial regions have the volume $2\pi/3$, that is, exactly half of all of the volume of the unit sphere.

Mass

If we now assume that every individual parallelepiped given by (25-44) is allotted a constant mass density that is given by the value of the function $f(x,y,z)$ at the position $\mathbf{r}(u_i,v_j,w_k)$, then the mass of the (i,j,k)th parallelepiped becomes:

$$\begin{aligned}\Delta\,\mathrm{M}_{ijk} &= f(x(u_i,v_j,w_k),y(u_i,v_j,w_k),z(u_i,v_j,w_k))\,\mathrm{Jacobian}_{\mathbf{r}}(u_i,v_j,w_k)\cdot\delta_u\delta_v\delta_w\\ &= f(\mathbf{r}(u_i,v_j,w_k))\,\mathrm{Jacobian}_{\mathbf{r}}(u_i,v_j,w_k)\cdot\delta_u\delta_v\delta_w\quad .\end{aligned}\tag{25-48}$$

The total mass of the whole system of approximating parallelepipeds is therefore the following, which necessarily is a good approximation to the mass of the whole spatial region:

$$\begin{aligned}\mathrm{M}_{\mathrm{app}}(n,m,q) &= \sum_{k=1}^{q}\sum_{j=1}^{m}\sum_{i=1}^{n}\Delta\,\mathrm{M}_{ijk}\\ &= \sum_{k=1}^{q}\sum_{j=1}^{m}\sum_{i=1}^{n} f(\mathbf{r}(u_i,v_j,w_k))\,\mathrm{Jacobian}_{\mathbf{r}}(u_i,v_j,w_k)\cdot\delta_u\delta_v\delta_w\quad .\end{aligned}\tag{25-49}$$

This is a triple integral sum for the continuous function $f(\mathbf{r}(u,v,w))\,\mathrm{Jacobian}_{\mathbf{r}}(u,v,w)$ over the parameter box $[a,b]\times[c,d]\times[h,l]$. We get in the limit, where n, m and q tend towards infinity:

$$\mathrm{M_{app}}(n,m,q) \;\to\; \mathrm{M} \;=\; \int_h^l \int_c^d \int_a^b f(\mathbf{r}(u,v,w))\mathrm{Jacobian}_{\mathbf{r}}(u,v,w)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \quad \text{for} \quad n,m,q \to \infty \quad . \tag{25-50}$$

Hereby we have motivated the definition of the mass of a parameterized region with the mass density $f(\mathbf{r}(u,v,w))$ and hereby also the general definition 25.19 of triple integral.

|||| Exercise 25.27

In Figure 25.11 the following parameterization of a spatial region:

$$\mathbf{r}(u,v,w) \;=\; (u\,\sin(v)\cos(w), u\,\sin(v)\sin(w), u\,\cos(v)) \quad , \tag{25-51}$$

where $u \in [1/2,1]$, $v \in [\pi/3, 2\pi/3]$, and $w \in [-\pi,\pi]$. By mapping the box-shaped parametrized region we expect a total of six side surfaces for the image set, i.e. for the spatial region that appears by the parametric representation. The Figure only shows three of the 6 side surfaces. Where are the others and how do they look?

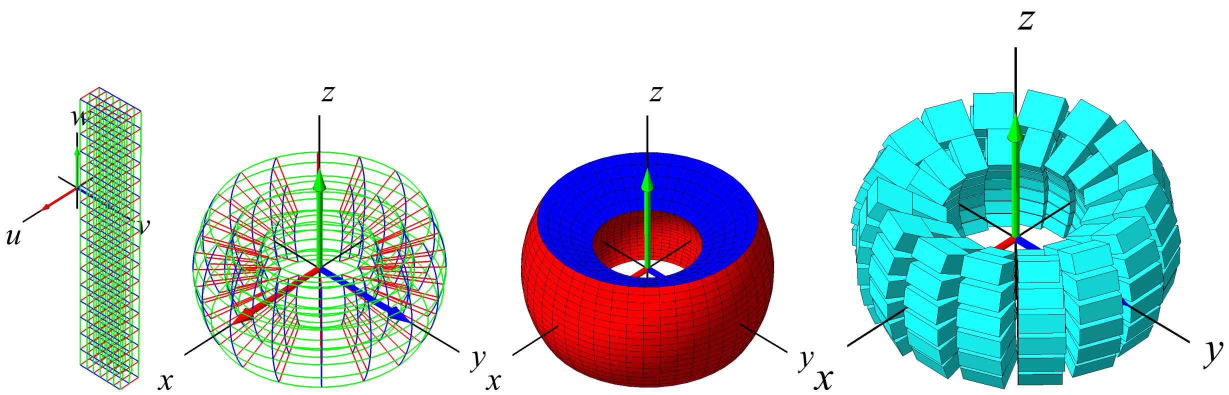

Figure 25.11: This *spatial* region is given by the parametric representation $\mathbf{r}(u,v,w) = (u\,\sin(v)\cos(w), u\,\sin(v)\sin(w), u\,\cos(v))$, $u \in [1/2,1]$, $v \in [\pi/3, 2\pi/3]$, $w \in [-\pi,\pi]$. The parameter box, the coordinate curves, the image of the spatial region by the parametric representation and a system of volume-approximating parallellepipids are shown.

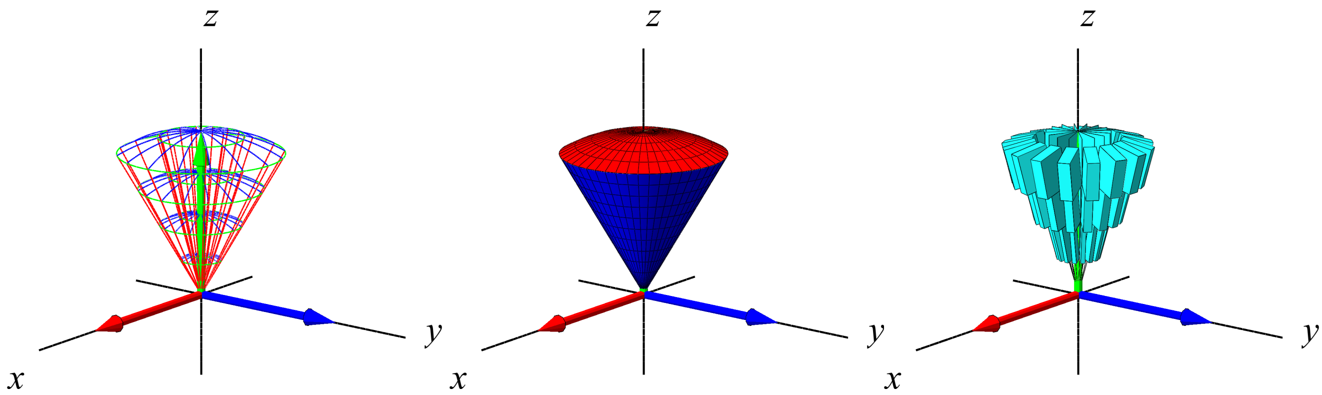

Figure 25.12: This *spatial* region is a 'solid-angle'-segment of the spherical surface represented by $\mathbf{r}(u,v,w) = (u\,\sin(v)\cos(w), u\,\sin(v)\sin(w), u\,\cos(v))$, $u \in [0,1]$, $v \in [0,\pi/6]$, $w \in [-\pi,\pi]$.

|||| Example 25.28 The Volume of a Solid Ellipsoid

A solid ellipsoid with semi-axes a, b, and c has a parametric representation:

$$\mathbf{r}(u,v,w) = (a\,u\,\sin(v)\cos(w), b\,u\,\sin(v)\sin(w), c\,u\,\cos(v)) \quad , \tag{25-52}$$

where $u \in [0,1]$, $v \in [0,\pi]$ and $w \in [-\pi,\pi]$.

The Jacobian function of this parameterization is $\text{Jacobian}(u,v,w) = abc\,u^2\,\sin(v)$, and from this the volume follows by triple integration:

$$\begin{aligned} \text{Vol}(\Omega_\mathbf{r}) &= \int_{\Omega_\mathbf{r}} 1\,\mathrm{d}\mu \\ &= \int_{-\pi}^{\pi}\int_0^{\pi}\int_0^1 \text{Jacobian}_\mathbf{r}(u,v,w)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \\ &= abc\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^1 u^2\,\sin(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \\ &= \frac{4}{3}\pi\,abc \quad . \end{aligned} \tag{25-53}$$

For a solid sphere B_a with radius a we therefore get the volume

$$\text{Vol}(B_a) = \frac{4}{3}\pi \cdot a^3 \quad . \tag{25-54}$$

|||| Example 25.29 Graph Surface Bounded Spatial Region

A positive function $h(x,y)$ of two variables (x,y) has a graph surface that over a given region P in the (x,y) plane bounds a spatial region. It is easy to parameterize the spatial region when first the plane region is parameterized.

If the plane region is given by a rectangle $(x,y) \in [a,b] \times [c,d]$ we have the parametric representation of the spatial region:

$$\Omega_{\mathbf{r}} \quad : \quad \mathbf{r}(u,v,w) = (u,v,w \cdot h(u,v)) \quad , \quad (u,v) \in [a,b] \times [c,d] \quad , \quad w \in [0,1] \quad . \tag{25-55}$$

The Jacobian function is particularly simple:

$$\begin{aligned} \text{Jacobian}_{\mathbf{r}}(u,v,w) &= |((1,0,w \cdot h'_u(u,v)) \times (0,1,w \cdot h'_v(u,v))) \bullet (0,0,h(u,v))| \\ &= |((1,0,*) \times (0,1,**)) \bullet (0,0,h(u,v))| \\ &= h(u,v) \quad . \end{aligned} \tag{25-56}$$

The volume of the spatial region that is bounded by the graph surface together with the plane region in the (x,y) plane is therefore:

$$\text{Vol}(\Omega_{\mathbf{r}}) = \int_0^1 \int_c^d \int_a^b h(u,v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = \int_c^d \int_a^b h(u,v)\,\mathrm{d}u\,\mathrm{d}v \quad . \tag{25-57}$$

If the plane region P in the (x,y) plane is not a rectangle as above, we must first parameterize P. Therefore we now assume that the plane region P is an image of a rectangular parametric region by a vector function $\widehat{\mathbf{r}}$:

$$P \quad : \quad \widehat{\mathbf{r}}(u,v) = (\xi(u,v),\eta(u,v)) \quad , u \in [a,b] \quad , \quad v \in [c,d] \quad , \tag{25-58}$$

where $\xi(u,v)$ and $\eta(u,v)$ are given functions of u and v. Then the spatial region between P and the graph surface $h(x,y)$ is given by the parametric representation:

$$\Omega_{\mathbf{r}} \quad : \quad \mathbf{r}(u,v,w) = (\xi(u,v),\eta(u,v),w \cdot h(\xi(u,v),\eta(u,v))) \quad , \tag{25-59}$$

where $(u,v) \in [a,b] \times [c,d]$ and $w \in [0,1]$ with the corresponding Jacobian function:

$$\begin{aligned} &\text{Jacobian}_{\mathbf{r}}(u,v,w) = \\ &= |((\xi'_u(u,v),\eta'_u(u,v),*) \times (\xi'_v(u,v),\eta'_v(u,v),**)) \bullet (0,0,h(\xi(u,v),\eta(u,v)))| \\ &= h(\xi(u,v),\eta(u,v)) \cdot \text{Jacobian}_{\widehat{\mathbf{r}}}(u,v) \quad . \end{aligned} \tag{25-60}$$

The volume of the spatial region that is bounded by the graph surface together with the plane region in the (x,y) plane is therefore in this more general situation:

$$\begin{aligned} \text{Vol}(\Omega_{\mathbf{r}}) &= \int_0^1 \int_c^d \int_a^b h(\xi(u,v),\eta(u,v)) \cdot \text{Jacobian}_{\widehat{\mathbf{r}}}(u,v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \\ &= \int_c^d \int_a^b h(\xi(u,v),\eta(u,v)) \cdot \text{Jacobian}_{\widehat{\mathbf{r}}}(u,v)\,\mathrm{d}u\,\mathrm{d}v \quad . \end{aligned} \tag{25-61}$$

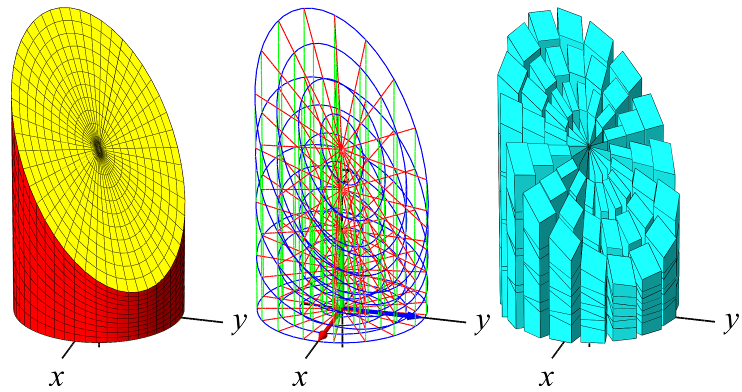

Figure 25.13: The Planetarium is bounded by the graph surface of $f(x, y) = 2 - x - y$ above a circular floor plan in the (x, y) plane with radius 1 and centre at $(0, 0)$. The volume is 2π.

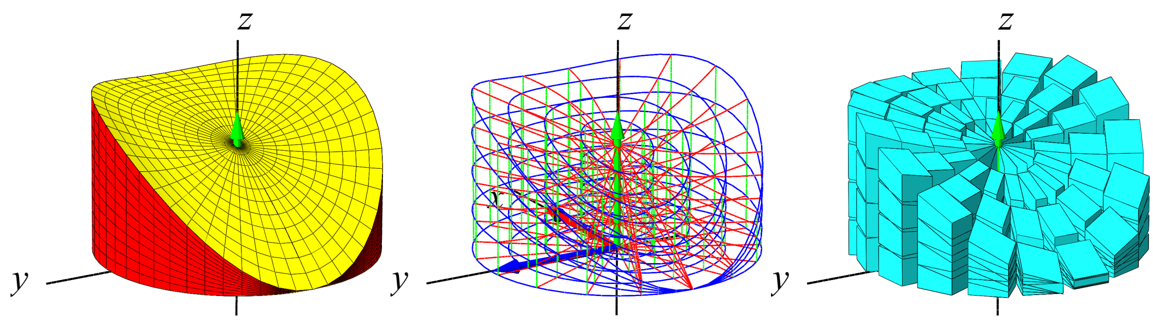

Figure 25.14: A Planetarium-like solid with a little more complicated top surface; the function used is here $f(x, y) = y^2 - x^2 + 2x + 3$. The volume is ca. 2.36.

|||| Example 25.30 The Tycho Brahe Planetarium

The volume of a cylinder between two plane sections (see Figure 25.13) is determined by the parametric representation:

$$\Omega_{\mathbf{r}} \quad : \quad \mathbf{r}(u, v, w) = (\xi(u, v), \eta(u, v), w \cdot h(\xi(u, v), \eta(u, v))) \quad , \tag{25-62}$$

where the height function is $h(x, y) = 2 - x - y$, the parameters are $(u, v) \in [0, 1] \times [-\pi, \pi]$, $w \in [0, 1]$, and the circular ground area is given by

$$P \quad : \quad \widehat{\mathbf{r}}(u, v) = (u \cdot \cos(v), u \cdot \sin(v)) \quad , u \in [0, 1] \quad , \quad v \in [-\pi, \pi] \tag{25-63}$$

with the Jacobian function

$$\text{Jacobian}_{\widehat{\mathbf{r}}}(u, v) = u \quad , \tag{25-64}$$

such that

$$\begin{aligned}
\mathrm{Vol}(\Omega_{\mathbf{r}}) &= \int_{-\pi}^{\pi}\int_{0}^{1} h(\xi(u,v),\eta(u,v))\cdot \mathrm{Jacobian}_{\widehat{\mathbf{r}}}(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{-\pi}^{\pi}\int_{0}^{1} (2-u\cdot\cos(v)-u\cdot\sin(v))\cdot u\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{-\pi}^{\pi}\int_{0}^{1} (2u-u^2\cdot\cos(v)-u^2\cdot\sin(v))\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{-\pi}^{\pi} \left(\left[u^2\right]_{u=0}^{u=1} - \left[\frac{1}{3}u^3\right]_{u=0}^{u=1}\cdot\cos(v) - \left[\frac{1}{3}u^3\right]_{u=0}^{u=1}\cdot\sin(v)\right)\mathrm{d}v \\
&= \int_{-\pi}^{\pi}\left(1-\frac{1}{3}\cdot\cos(v)-\frac{1}{3}\cdot\sin(v)\right)\,\mathrm{d}v \\
&= 2\cdot\pi \quad .
\end{aligned} \tag{25-65}$$

Note that the volume, due to symmetry, also can be found more easily as half that of a cylinder with identical base but ending in a horizontal section at height 4.

25.3 Solids of Revolution

Solids of revolution are the special spatial regions created by rotating a plane region (e.g. defined in the (x,z) plane) about an axis of rotation in the same plane (the z-axis) that is assumed to lie outside the region. Cf. the definition of surfaces of revolution in section 25.1.2.

The plane region - the profile region - is represented by a parametric representation like this:

$$P_{\mathbf{r}}: \quad \mathbf{r}(u,v) \;=\; (g(u,v),0,h(u,v)) \in \mathbb{R}^3 \quad , \; u\in[a,b] \; , \; v\in[c,d] \quad , \tag{25-66}$$

where $g(u,v) > 0$ and $h(u,v)$ are given functions of the parameters u and v. The spatial region, the solid that appears by rotating the profile region a whole turn around the z-axis, therefore has the parametric representation:

$$\begin{aligned}
\Omega P_{\mathbf{r}}: \quad & \mathbf{r}(u,v,w) \;=\; (g(u,v)\cos(w),\, g(u,v)\sin(w),\, h(u,v)) \in \mathbb{R}^3 \quad , \\
& u\in[a,b] \; , \; v\in[c,d] \; , \; w\in[-\pi,\pi] \quad .
\end{aligned} \tag{25-67}$$

Figure 25.9 shows half of a solid of rotation. Figure 25.11 shows the surface of a solid of rotation defined by the use of spherical coordinates. Cylindrical coordinates in 3D space give similarly well-known solids of revolution such as e.g. the one that is shown in Figure 25.15.

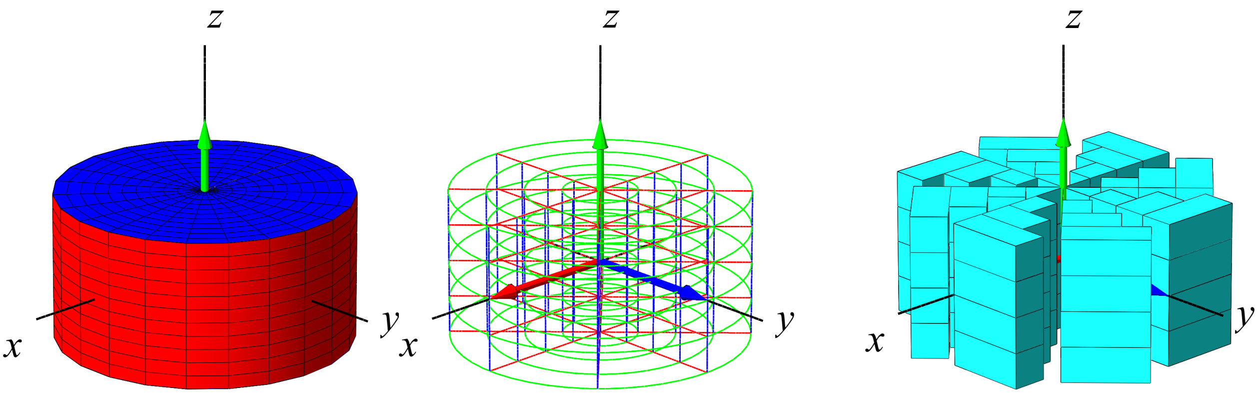

Figure 25.15: A spatial region parameterized through cylindrical coordinates is given by the parametric representation $\mathbf{r}(u,v,w) = (g(u,v)\cos(w), g(u,v)\sin(w), h(u,v))$, $u \in [0, \frac{1}{2}]$, $v \in [-\frac{1}{2}, \frac{1}{2}]$, $w \in [-\pi, \pi]$, where $g(u,v) = u$ and $h(u,v) = v$.

|||| Exercise 25.31

Show that the Jacobian function $\text{Jacobian}_{\mathbf{r}}(u,v,w)$ for the parametric representation $\mathbf{r}(u,v,w)$ for the generalized solid of revolution $\Omega P_{\mathbf{r}}$ in (25-67) is given by

$$\text{Jacobian}_{\mathbf{r}}(u,v,w) = g(u,v)\,|g'_u(u,v)\,h'_v(u,v) - h'_u(u,v)\,g'_v(u,v)| \quad . \tag{25-68}$$

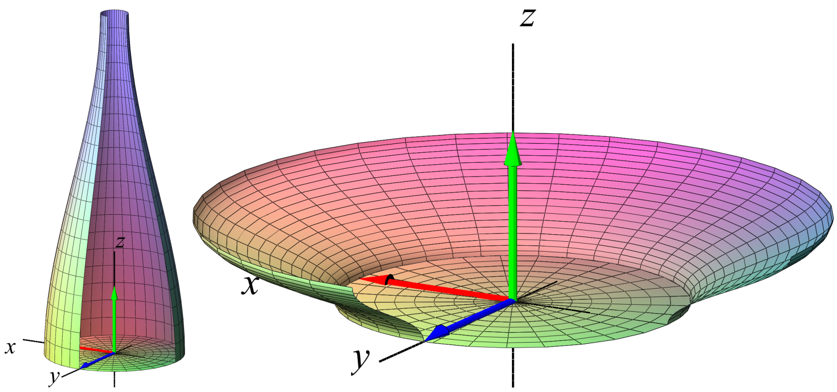

Figure 25.16: Parts of a bottle of revolution and a dish of revolution, respectively, see Exercise 25.32.

|||| Exercise 25.32

A bottle (or dish?) of revolution is given (apart from the base) by the parametric representation of its profile curve in the $(x, z)-$plane like this:

$$G_{\mathbf{r}} : \quad \mathbf{r}(u) = (g(u), 0, h(u)) \in \mathbb{R}^3 \quad , \quad u \in [0, 1] \quad , \tag{25-69}$$

where

$$\begin{aligned} g(u) &= 2(R_1 - R_2)u^3 + 3(R_2 - R_1)u^2 + R_1 \\ h(u) &= H\,u \quad , \end{aligned} \tag{25-70}$$

for suitable choices of *positive* constants R_1, R_2 and H.

1. Plot different versions of these surfaces of revolution, see e.g. Figure 25.16.
2. Show that the surface of revolution is perpendicular to the (x, y) plane for every choice of the positive constants R_1, R_2, and H.
3. How large a volume (of e.g. water) can the surface of revolution 'contain' for given values of the positive constants R_1, R_2, and H?
4. What is the area of the surface of the surface of revolution + base for given values of the positive constants R_1, R_2, and H?
5. Which choice(s) of constants give(s) the largest volume in proportion to the total surface area?

25.4 More Architecturally Motivated Spatial Regions

|||| Exercise 25.33

Inspired by Malmö's Turning Torso it is an interesting exercise to find the volume and the surface area of different 'twisted' buildings:

1. Find a parametric representation of the spatial area shown in Figure 25.17 to the left. Choose the dimensions of your building (base, height) of your building yourself. Hint: The ascending curves (which in a similar, but non-twisted building would have been straight vertical lines)are helices, see Example 24.3 in eNote 24 .
2. What is the volume of your building?

3. What is the area of the surfaces of the side walls of the building?

4. With given height and base: Show that the volume is independent of the number of turns (the number of times the section figure is rotated from the bottom to the top – for the Torso the turn number is $1/5$). How does the surface area depend on the turn number? With which turn number do we get the largest volume in proportion to the total surface area (of the side walls)?

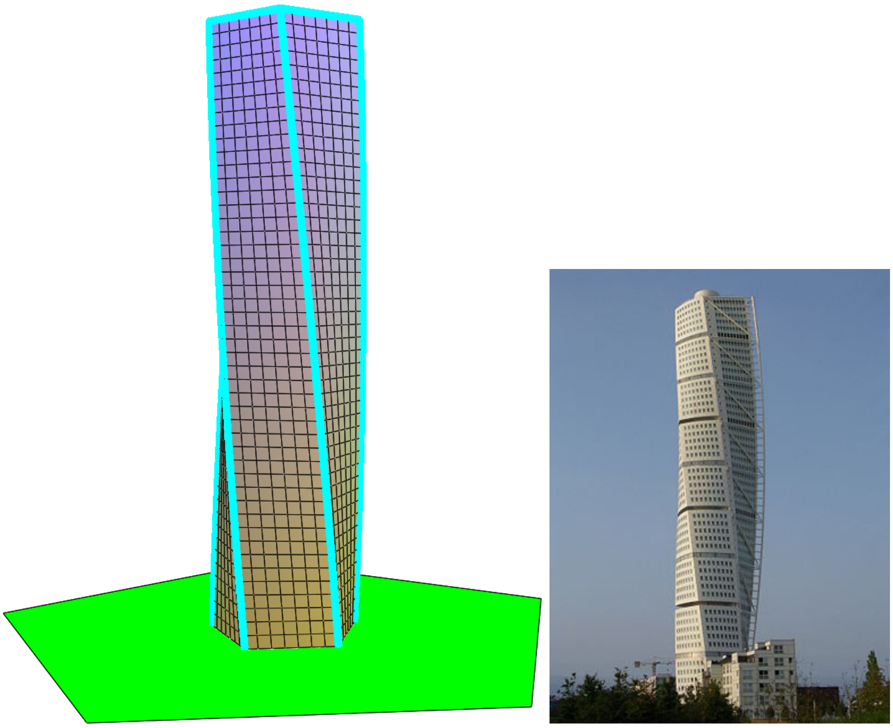

Figure 25.17: Five-edged Turning Torso model and 'the real thing' in Malmö.

|||| Exercise 25.34

Find the parametric representation of the two (highest) towers that are shown in Figure 25.18. Choose the dimensions yourself. Hint: The tower in the left image has an elliptic section. Find volume and surface area of each of the buildings.

Figure 25.18: Chinese–Canadian project.

25.5 Summary

We have in this eNote described those concepts and methods that make it possible to compute (weighted) areas and volumes of surfaces and spatial regions, respectively, – inasmuch they are given by parametric representations from rectangular and box-shaped parametric regions. A series of examples and exercises show how very different surfaces and regions can be parameterized by fairly simple vector functions $\mathbf{r}(u,v)$ and $\mathbf{r}(u,v,w)$.

When first a relevant parameterization is stated the 'remaining task' is only a question about computing the corresponding Jacobian function $\mathrm{Jacobian}_{\mathbf{r}}(u,v)$ or $\mathrm{Jacobian}_{\mathbf{r}}(u,v,w)$, multiply this by a possible weight function $f(x,y,z)$ limited to the image set in the space of the parameterization, and finally compute the double or triple integral of this product over the parametrized rectangle or parameter box:

- For a surface $F_{\mathbf{r}}$ with the parametric representation

$$F_{\mathbf{r}}: \quad \mathbf{r}(u,v) \;=\; (x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3 \quad , \quad u \in [a,b] \; , \; v \in [c,d] \qquad (25\text{-}71)$$

  the integral of the (weight-)function $f(x,y,z)$ over the surface is given by:

$$\int_{F_{\mathbf{r}}} f \,\mathrm{d}\mu \;=\; \int_c^d \int_a^b f(\mathbf{r}(u,v))\,\mathrm{Jacobian}_{\mathbf{r}}(u,v)\,\mathrm{d}u\,\mathrm{d}v \quad , \qquad (25\text{-}72)$$

  where the Jacobian function $\mathrm{Jacobian}_{\mathbf{r}}(u,v)$

$$\mathrm{Jacobian}_{\mathbf{r}}(u,v) \;=\; |\mathbf{r}'_u(u,v) \times \mathbf{r}'_v(u,v)| \qquad (25\text{-}73)$$

  is the area of the parallelogram that at this position $\mathbf{r}(u,v)$ is spanned by the two tangent vectors $\mathbf{r}'_u(u,v)$ and $\mathbf{r}'_v(u,v)$ to the respective coordinate curves through the point $\mathbf{r}(u,v)$ on the surface.

- In particular the area of $F_{\mathbf{r}}$ is determined by:

$$\mathrm{Area}(F_{\mathbf{r}}) = \int_{F_{\mathbf{r}}} 1\,\mathrm{d}\mu = \int_c^d \int_a^b \mathrm{Jacobian}_{\mathbf{r}}(u,v)\,\mathrm{d}u\,\mathrm{d}v \quad . \qquad (25\text{-}74)$$

- For a spatial region $\Omega_{\mathbf{r}}$ with the parametric representation

$$\Omega_{\mathbf{r}}: \quad \mathbf{r}(u,v,w) \;=\; (x(u,v,w), y(u,v,w), z(u,v,w)) \quad , \qquad (25\text{-}75)$$

where $u \in [a,b]$, $v \in [c,d]$, and $w \in [h,l]$ the integral of the (weight-)function $f(x,y,z)$ over the region given by:

$$\int_{\Omega_{\mathbf{r}}} f \,\mathrm{d}\mu = \int_h^l \int_c^d \int_a^b f(\mathbf{r}(u,v,w)) \,\mathrm{Jacobian}_{\mathbf{r}}(u,v,w) \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}w \quad , \tag{25-76}$$

where the Jacobian function $\mathrm{Jacobian}_{\mathbf{r}}(u,v,w)$

$$\mathrm{Jacobian}_{\mathbf{r}}(u,v,w) = \left| \left(\mathbf{r}'_u(u,v,w) \times \mathbf{r}'_v(u,v,w) \right) \cdot \mathbf{r}'_w(u,v,w) \right| \tag{25-77}$$

is the volume of the parallelepiped that at this position $\mathbf{r}(u,v,w)$ is spanned by the three tangent vectors $\mathbf{r}'_u(u,v,w)$, $\mathbf{r}'_v(u,v,w)$, and $\mathbf{r}'_w(u,v,w)$ to the respective coordinate curves through the point $\mathbf{r}(u,v,w)$ in the spatial region.

- In particular the volume of $\Omega_{\mathbf{r}}$ is determined by:

$$\mathrm{Vol}(\Omega_{\mathbf{r}}) = \int_{\Omega_{\mathbf{r}}} 1 \,\mathrm{d}\mu = \int_h^l \int_c^d \int_a^b \mathrm{Jacobian}_{\mathbf{r}}(u,v,w) \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}w \quad . \tag{25-78}$$