eNote 24

Line and Plane Integrals

With the methods and results stated in eNote 23 as our starting point we will in this eNote show how Riemann integrals can be used to find lengths of curves and areas of plane regions, inasmuch they are described in suitable parametric form.

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24.1 Line Integrals

Initially we will consider curves that are parameterized in the following way with a parametric description.

A parameterized curve $K_{\mathbf{r}}$ in (x, y, z) space is given by a parametric description like this:

$$K_{\mathbf{r}}: \quad \mathbf{r}(u) = (x(u), y(u), z(u)) \in \mathbb{R}^3 \quad , \quad u \in [a, b] \quad . \tag{24-1}$$

Usually a given curve can be parameterized in infinitely many ways. Figure 24.1 shows three different parameterizations of the straight line segment from $(0, -2, \frac{1}{2})$ to $(0, 2, \frac{1}{2})$. Figure 24.2 similarly shows two different parameterizations of a circle with radius 1 and centre at $(0, 0, 0)$. Figure 24.4 similarly shows 2 different parameterizations of a helic.

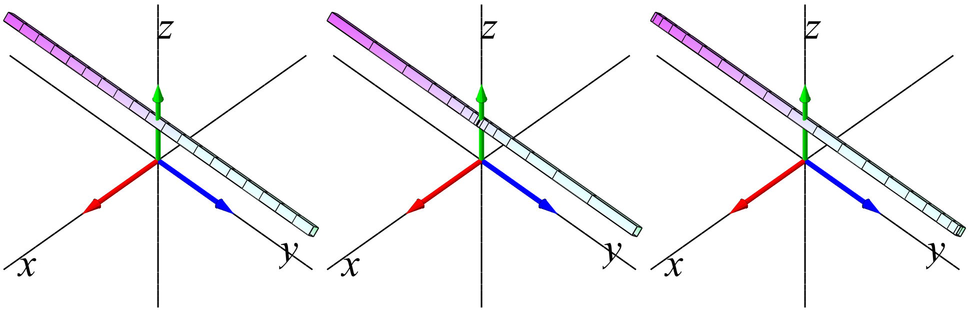

Figure 24.1: The line segment from $(0,-2,\frac{1}{2})$ to $(0,\,2,\,\frac{1}{2})$ is here parameterized in 3 different ways: $\mathbf{r}_1(u) = (0,2u,\,\frac{1}{2})\,,\; u \in [-1,\,1]\,;\; \mathbf{r}_2(u) = (0,\,2\,u^3,\,\frac{1}{2})\,,\; u \in [-1,1]\,,$ and $\mathbf{r}_3(u) = (0,\,2\sin(\frac{\pi}{2}u),\,\frac{1}{2})\,,\; u \in [-1,\,1]\,.$ The marks on the individual line segments stem from the partition of the *common parameter interval* $[-1,\,1]$ consisting of 20 subintervals of equal size. Note that the lengths of the three 'curves' clearly are equal, even though the parameterizations are quite different. See Exercise 24.16.

We assume here and in what follows that the three coordinate functions $x(u)$, $y(u)$ and $z(u)$ in the parametric representations are well-behaved functions of u, i.e. that we assume that all three are smooth functions of u such that they can be differentiated arbitrarily many times. In particular the derivatives $x'(u)$, $y'(u)$ and $z'(u)$ are therefore continuous in the interval $[a,b]$. Then we also have that

$$|\mathbf{r}'(u)| = \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2} \tag{24-2}$$

is a continuous function in the interval $[a,b]$. Therefore this function can be *integrated* over the interval, cf. eNote 23 and we need this in the definition 24.5 below.

|||| Definition 24.1 Regular Parametric Representation

A parametric representation $\mathbf{r}(u)$ of a curve $K_{\mathbf{r}}$ - which in (24-1) - is said to be a *regular parametric representation* if the following condition is fulfilled:

$$\mathbf{r}'(u) \neq \mathbf{0} \quad \text{for all} \quad u \in [a,b] \quad . \tag{24-3}$$

|||| Exercise 24.2

Which of the parametric representations in the Figures 24.1, 24.2, 24.4, and 24.5 are regular?

A parameterized curve is more than just the image (the set of points) $\mathbf{r}([a, b])$, since the parameterization itself can e.g. state that the parts of the set of points shall be met more than once, see Example 24.10 below.

One can think of the interval $[a, b]$ as a straight elastic at rest. The vector map $\mathbf{r}$ deforms the rubber band (into the space) by bending, stretching or compressing it. A local stretch will of course make the band longer locally, while a local compression makes the band shorter locally. Therefore a first natural question is how long is the whole rubber band after we have used the map $\mathbf{r}$. The line integral is introduced i.a. for the purpose of finding the total length of the deformed curve in 3D space.

Similarly we can imagine that the parameterized curve itself is massless, but that on the other hand it after the deformation by $\mathbf{r}$ is colored with paint in such a way that density of paint along the curve (e.g. in grams per centimeter) is given as a function f of the position (x, y, z) in 3D space – so that the density of the paint at the position $\mathbf{r}(u)$ is $f(\mathbf{r}(u))$. The task is then to find the total mass of the deformed and colored parameterized curve. Note that with a little imagination we can allow that the density f to assume negative values.

These imaginations shall only be a help to get a suitable intuitive understanding of the concepts introduced; we shall see, in related eNotes, several other interpretations and uses of the line integral.

|||| Example 24.3 Helix

The helix in Figure 24.4 is presented by two *different parameterizations*:

$$\mathbf{r}_1(u) = \left(\cos(2\pi u), \sin(2\pi u), \tfrac{\pi}{5}u\right), \ u \in [-1, 1] \text{ , og}$$

$$\mathbf{r}_2(u) = \left(\cos(2\pi\, u^3), \sin(2\pi\, u^3), \tfrac{\pi}{5}u^3\right), \ u \in [-1, 1] \,.$$

The marks stem from the partitioning *parameter interval* $[-1, 1]$ that consists of 200 subintervals of equal size. Again the curves are clearly of equal length (see Exercise 24.16).

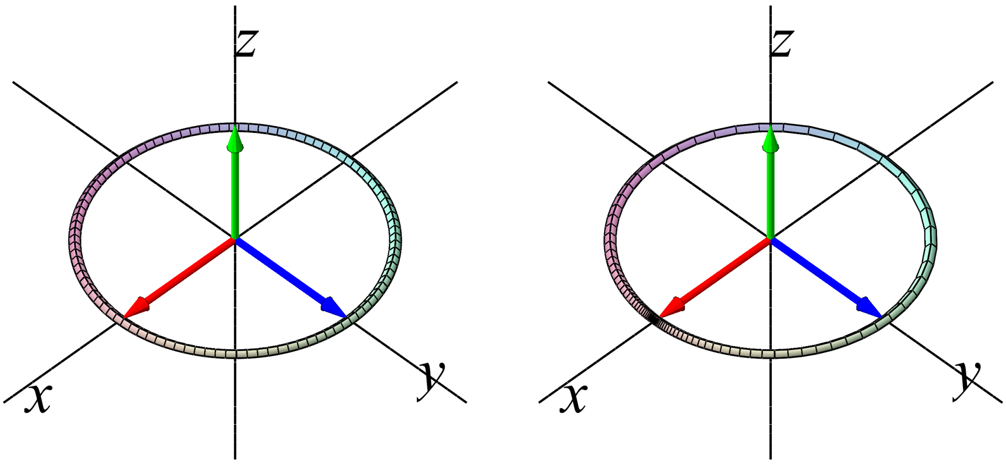

Figure 24.2: A circle in the (x, y) plane is here parameterized in two different ways: $\mathbf{r}_1(u) = (\cos(\pi u), \sin(\pi u), 0)$, $u \in [-1, 1]$, and $\mathbf{r}_2(u) = (\cos(\pi u^3), \sin(\pi u^3), 0)$, $u \in [-1, 1]$. The marks stem from the partitioning of the *parameter interval* $[-1, 1]$ that consists of 100 subintervals of equal size. The circumference of the circle is 2π – independent of the parameterization.

|||| Example 24.4 Knot

The knot in Figure 24.5 has the somewhat complicated parametric representation
$\mathbf{r}(u) = \left(-\frac{1}{3}\cos(u) - \frac{1}{15}\cos(5u) + \frac{1}{2}\sin(2u),\ \frac{1}{3}\sin(u) - \frac{1}{15}\sin(5u) - \frac{1}{2}\cos(2u),\ \frac{1}{3}\cos(3u)\right)$,
with $u \in [-\pi, \pi]$.

We define line integration in the following way and motivate the definition in section 24.1.1 below:

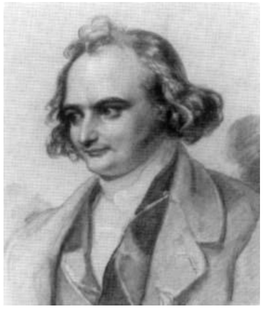

Figure 24.3: Carl Gustav Jakob Jacobi. See Biography.

|||| Definition 24.5 Line Integral

Let $f(x, y, z)$ denote a continuous function on $\mathbb{R}^3$. The line integral of the function f over a parameterized curve $K_{\mathbf{r}}$ is defined by

$$\int_{K_{\mathbf{r}}} f \, \mathrm{d}\mu = \int_a^b f(\mathbf{r}(u)) \, \mathrm{Jacobian}_{\mathbf{r}}(u) \, \mathrm{d}u \quad , \tag{24-4}$$

where the Jacobian function $\mathrm{Jacobian}_{\mathbf{r}}(u)$ is given by:

$$\mathrm{Jacobian}_{\mathbf{r}}(u) = |\mathbf{r}'(u)| \quad . \tag{24-5}$$

The Jacobian function $\mathrm{Jacobian}_{\mathbf{r}}(u)$ thus denotes the length of the tangent vector $\mathbf{r}'(u)$ to the curve at the position $\mathbf{r}(u)$.

Note that the symbol that is on the left-hand side of the equality sign in (24-4) *only* is a *symbol* for the line integral. The integral which we shall compute is on the right-hand side. And this is possible to integrate, because both f, $\mathbf{r}$ and $|\mathbf{r}'|$ are continuous, such that the integrand is continuous.

If we substitute $\mathbf{r}(u) = (x(u), y(u), z(u))$ in the expression for the line integral we get:

$$\int_{K_{\mathbf{r}}} f \, \mathrm{d}\mu = \int_a^b f(x(u), y(u), z(u)) \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2} \, \mathrm{d}u \quad . \tag{24-6}$$

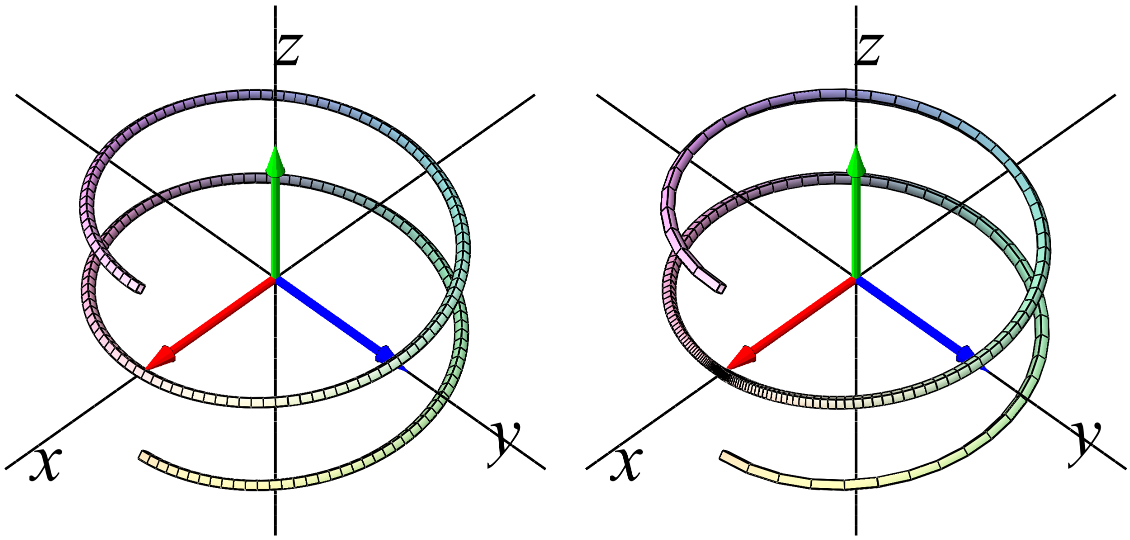

Figure 24.4: A helix in 3D space parameterized in two different ways. See Example 24.3.

|||| **Remark 24.6**

The parametric representation (24-1) of the curve is *regular* if the Jacobian function of the parametric representation is positive: $\text{Jacobian}_{\mathbf{r}}(u) = |\mathbf{r}'(u)| > 0$ for all u in the given interval $[a, b]$.

|||| **Example 24.7 A Weighted Circle**

Given the function $f(x, y, z) = 7x$ and a parameterized circle segment

$$C_{\mathbf{r}}: \mathbf{r}(u) = (x(u), y(u), z(u)) = (\cos(u), \sin(u), 0), \quad u \in [-\frac{\pi}{2}, \pi] \quad .$$

The line integral of f over $C_{\mathbf{r}}$ is

$$\begin{aligned} \int_{C_{\mathbf{r}}} f \, \mathrm{d}\mu &= \int_{-\pi/2}^{\pi} f(x(u), y(u), z(u)) \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2} \, \mathrm{d}u \\ &= \int_{-\pi/2}^{\pi} 7\cos(u) \sqrt{(-\sin(u))^2 + (\cos(u))^2} \, \mathrm{d}u \\ &= \int_{-\pi/2}^{\pi} 7\cos(u) \, \mathrm{d}u = 7 \quad . \end{aligned}$$

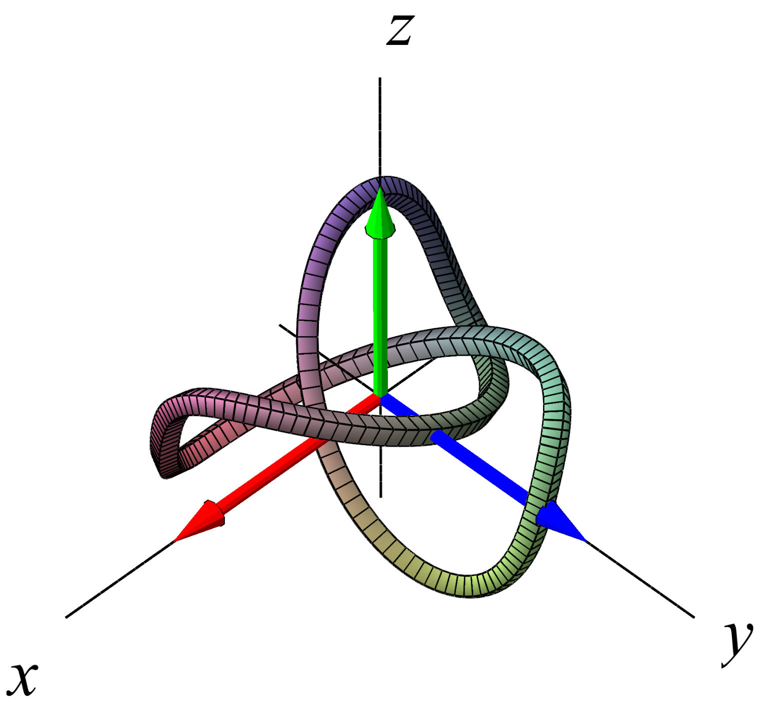

Figure 24.5: A knot. See Example 24.4

As mentioned above and as we will substantiate below – in section 24.1.1 about *Motivation for the Line Integral* – the line integral can be applied for finding lengths of parameterized curves and to find the total mass of parameterized curves with given mass densities. If the mass density is the constant 1 we get the length (i.e. one can find the length of such a curve by determining its weight):

|||| Definition 24.8 The Length of a Curve

The length of the parameterized curve

$$K_{\mathbf{r}} : \quad \mathbf{r}(u) = (x(u), y(u), z(u)) \quad , \quad u \in [a,b]$$

is defined as the line integral

$$\mathrm{L}(K_{\mathbf{r}}) = \int_{K_{\mathbf{r}}} 1 \, \mathrm{d}\mu = \int_a^b |\mathbf{r}'(u)| \, \mathrm{d}u \quad . \tag{24-7}$$

|||| Example 24.9 The Length of a Circle Segment

The parameterized circle segment

$$C_{\mathbf{r}}: \ \mathbf{r}(u) \ = \ (\cos(u), \sin(u), 0)\,, \ \ u \in [-\frac{\pi}{2}, \pi]$$

has the length

$$\begin{aligned} L(C_{\mathbf{r}}) \ &= \int_{C_{\mathbf{r}}} 1\, d\mu \ = \int_{-\pi/2}^{\pi} \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2}\; du \\ &= \int_{-\pi/2}^{\pi} \sqrt{(-\sin(u))^2 + \cos(u))^2}\; du \\ &= \int_{-\pi/2}^{\pi} 1 \; du \ = \ \frac{3\,\pi}{2} \quad . \end{aligned}$$

|||| Example 24.10 More Than One Circular Winding

The parameterized plane Curve

$$\widetilde{C}_{\mathbf{r}}: \ \mathbf{r}(u) \ = \ (\cos(u), \sin(u), 0)\,, \ \ u \in [-\frac{\pi}{2}, 7\pi]$$

has the length $L(\widetilde{C}_{\mathbf{r}}) \ = \ \frac{15\pi}{2}$ corresponding to the fact that the parameterization 'winds' the long interval a number of times around the unit circle!

|||| Example 24.11 The Length of a Plane Spiral

The parameterized plane spiral (see Figure 24.6)

$$K_{\mathbf{r}}: \ \mathbf{r}(u) \ = \ (\, u\cos(u), u\sin(u), 0\,)\ , \ \ u \in [0, \pi/2]$$

has the length

$$\begin{aligned} L(K_{\mathbf{r}}) \ &= \int_{K_{\mathbf{r}}} 1\, d\mu \ = \int_{0}^{\pi/2} \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2}\; du \\ &= \int_{0}^{\pi/2} \sqrt{(\cos(u) - u\sin(u))^2 + (\sin(u) + u\cos(u))^2}\; du \\ &= \int_{0}^{\pi/2} \sqrt{1+u^2}\; du \\ &= \left[(1/2)u\sqrt{1+u^2} + (1/2)\,\mathrm{arcsinh}(u)\right]_0^{\pi/2} \\ &= (\pi/4)\sqrt{1+(\pi/2)^2} + (1/2)\,\mathrm{arcsinh}(\pi/2) \\ &= (\pi/8)\sqrt{4+\pi^2} + (1/2)\ln(2) - (1/2)\ln(-\pi + \sqrt{4+\pi^2}) \\ &= 2.079 \quad . \end{aligned} \tag{24-8}$$

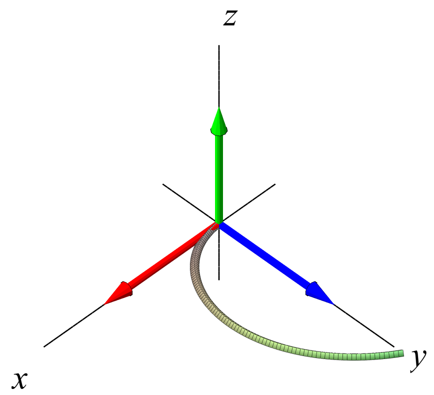

Figure 24.6: Part of a plane spiral. See Example 24.11.

|||| Example 24.12 The Length of the Ellipsis

The length of an ellipsis. The parameterized ellipsis (see Figure 24.7)

$$K_{\mathbf{r}}: \ \mathbf{r}(u) = (\, a\cos(u), b\sin(u), 0\,) \ , \ u \in [-\pi, \pi]$$

has the length

$$\begin{aligned} \mathrm{L}(K_{\mathbf{r}}) = \int_{K_{\mathbf{r}}} 1 \, \mathrm{d}\mu &= \int_{-\pi}^{\pi} \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2} \, \mathrm{d}u \\ &= \int_{-\pi}^{\pi} \sqrt{a^2 \sin^2(u) + b^2 \cos^2(u)} \, \mathrm{d}u \\ &= 4aE\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right) \quad , \end{aligned}$$

where E denotes the so-called complete elliptic integral of the 2nd order. About the function $E(u)$ we only mention here that the function value in $u = 0$ is $E(0) = \pi/2$, such that the result above means that when the ellipsis becomes a circle, i.e. when $a = b$, then we get the correct circumference of the circle with radius a: $L = 2\pi a$.

|||| Example 24.13 The Length of the Helix

The parameterized helix

$$K_{\mathbf{r}}: \ \mathbf{r}(u) = (\cos(u), \sin(u), u)\,, \ \ u \in [-2\pi, 2\pi]$$

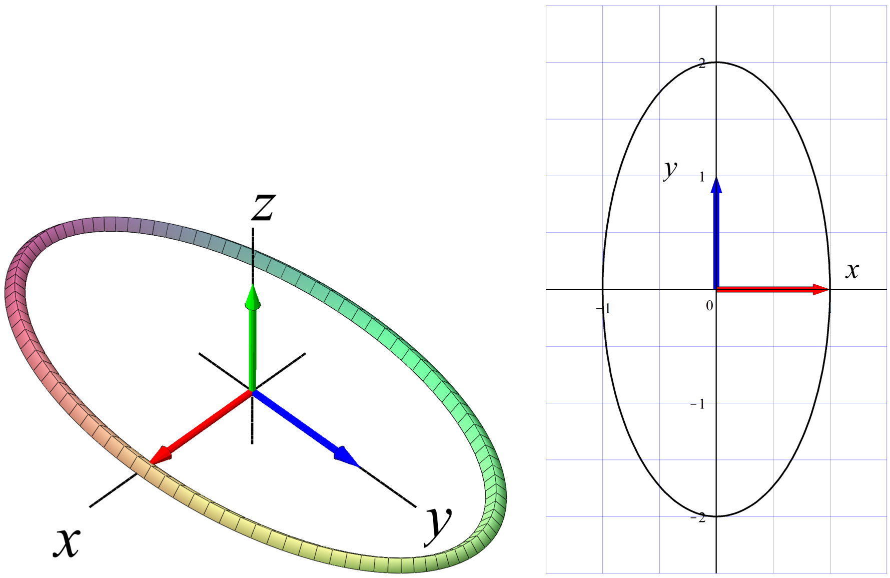

Figure 24.7: An Ellipsis with semi-axes $a = 1$ and $b = 2$. See Example 24.12.

has the length

$$\begin{aligned} \mathrm{L}(K_{\mathbf{r}}) = \int_{K_{\mathbf{r}}} 1 \, d\mu &= \int_{-2\pi}^{2\pi} \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2} \, du \\ &= \int_{-2\pi}^{2\pi} \sqrt{(-\sin(u))^2 + (\cos(u))^2 + 1} \, du \\ &= \int_{-2\pi}^{2\pi} \sqrt{2} \, du = 4\pi \sqrt{2} \quad . \end{aligned}$$

|||| Definition 24.14 One-to-One Parametric Representation

The parametric representation (24-1) of the curve $K_{\mathbf{r}}$ is said to be *one-to-one* if for all $u_1 \in [a, b]$ and for all $u_2 \in [a, b]$ the following applies:

$$u_1 \neq u_2 \quad \text{implies that} \quad \mathbf{r}(u_1) \neq \mathbf{r}(u_2) \quad . \tag{24-9}$$

|||| Exercise 24.15

Which of the parametric representations in the figures 24.1, 24.2, and 24.4 – and in the examples 24.9, 24.10 and 24.13, respectively – are one-to-one?

|||| Exercise 24.16

Show that Definition 24.8 gives the same length for the three parameterizations of the line segment in Figure 24.1, the same length of the two circle segments in Figure 24.2 and the same length of the two helices in Figure 24.4.

|||| Exercise 24.17

Find the length (with 3 decimals) of the knot in Figure 24.5.

|||| Exercise 24.18

Find regular, one-to-one parametric representations of the line segment (Figure 24.1), the circle (Figure 24.2), and the helix (Figure 24.4), such that all parametric representations have the common parameter interval $[0, \pi]$.

24.1.1 Motivation for the Line Integral

If we equidistantly partition the interval $[a, b]$ in n parts, then every subinterval has the length $\delta_u = (b-a)/n$ and the coordinates of the division points in $[a, b]$ become:

$$
\begin{aligned}
u_1 &= a, \\
u_2 &= u_1 + \delta_u = a + \delta_u, \\
u_3 &= u_2 + \delta_u = a + 2\delta_u, \\
u_4 &= u_3 + \delta_u = a + 3\delta_u, \\
&\ldots. \\
b &= u_n + \delta_u = a + n\delta_u \quad .
\end{aligned}
\tag{24-10}
$$

Using these chosen values of u_i as development points we can use Taylor's limit formula for each of the 3 coordinate functions $x(u)$, $y(u)$, and $z(u)$ of $\mathbf{r}(u) = (x(u), y(u), z(u))$ to first order and with the corresponding epsilon functions as follows, see eNote 21:

$$\begin{aligned} x(u) &= x(u_i) + x'(u_i)\,(u - u_i) + \varepsilon_x(u - u_i) \cdot |u - u_i| \\ y(u) &= y(u_i) + y'(u_i)\,(u - u_i) + \varepsilon_y(u - u_i) \cdot |u - u_i| \\ z(u) &= z(u_i) + z'(u_i)\,(u - u_i) + \varepsilon_z(u - u_i) \cdot |u - u_i| \quad . \end{aligned} \tag{24-11}$$

These 3 formulas we can gather and express with vector notation like this:

$$\mathbf{r}(u) = \mathbf{r}(u_i) + \mathbf{r}'(u_i) \cdot (u - u_i) + \boldsymbol{\varepsilon}_i(u - u_i) \cdot \rho_i \quad , \tag{24-12}$$

where we use the short way of writing $\rho_i = |u - u_i| = \sqrt{(u - u_i)^2}$ for the distance between the variable value u and the given value u_i in the parameter interval. Furthermore we have that the vector $\boldsymbol{\varepsilon}_i(u - u_i) = (\,\varepsilon_x(u - u_i),\, \varepsilon_y(u - u_i),\, \varepsilon_z(u - u_i)\,) \to (0, 0, 0) = \mathbf{0}$ for $u \to u_i$.

Every subinterval $[u_i, u_i + \delta_u]$ on the u-axis is mapped onto the curve *segment* $\mathbf{r}(u)$, $u \in [u_i, u_i + \delta_u]$, and this curve segment we can approximate with the linear part of the expression in (24-12) that is found by removing the $\boldsymbol{\varepsilon}_i$-term from the right-hand side in (24-12):

$$\mathbf{r}_{\mathrm{app}_i}(u) = \mathbf{r}(u_i) + \mathbf{r}'(u_i) \cdot (u - u_i) \ , \quad u \in [u_i, u_i + \delta_u] \quad . \tag{24-13}$$

See the figures 24.8 and 24.9 where the approximating line segments are shown for a parameterized circle, for two different parameterizations and for different values of n. The i'th line segment by definition is in contact with the curve in one of its end points. This we call the contact point for the line segment.

The Length of a Curve

Each of the in total n approximating line segments has a length, see Figure 24.8. The length of the i'th line segment is, according to (24-13),

$$\Delta\,\mathrm{L}_i = |\mathbf{r}_{\mathrm{app}_i}(u_i + \delta_u) - \mathbf{r}_{\mathrm{app}_i}(u_i)| = |\mathbf{r}'(u_i)| \cdot \delta_u \quad . \tag{24-14}$$

The sum of these n lengths is (for large values of n) clearly a good approximation to the length of the curve, so that we can write

$$\mathrm{L}_{\mathrm{app}}(n) = \sum_{i=1}^{n} \Delta\,\mathrm{L}_i = \sum_{i=1}^{n} |\mathbf{r}'(u_i)| \cdot \delta_u \quad , \tag{24-15}$$

Since the above sum is an integral sum (see eNote 23) for the continuous function $|\mathbf{r}'(u)|$ over the interval $[a,b]$, we get in the limit, where n goes towards infinity:

$$\mathrm{L}_{\mathrm{app}}(n) \;\rightarrow\; \mathrm{L} = \int_a^b |\mathbf{r}'(u)|\,\mathrm{d}u \quad \text{for} \quad n \rightarrow \infty \quad . \tag{24-16}$$

By this we have motivated the definition of the length of a curve as stated above, viz. as the line integral of the constant function 1 over the parameterized curve.

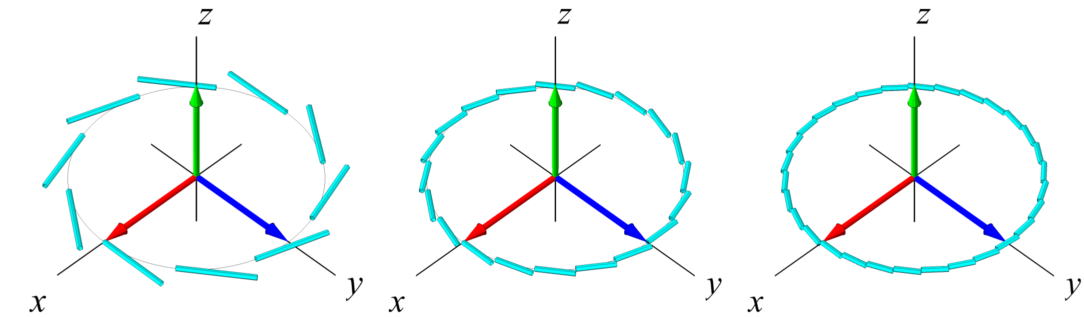

Figure 24.8: The curve $\mathbf{r}(u) = (\cos(2\pi u), \sin(2\pi u), 0)$, $u \in [-1,1]$, with 10, 20, and 30 approximating line segments, respectively. It is reasonable to define the length of the curve as the total length of the approximating line segments in the limit where the number of line segments tends toward infinity.

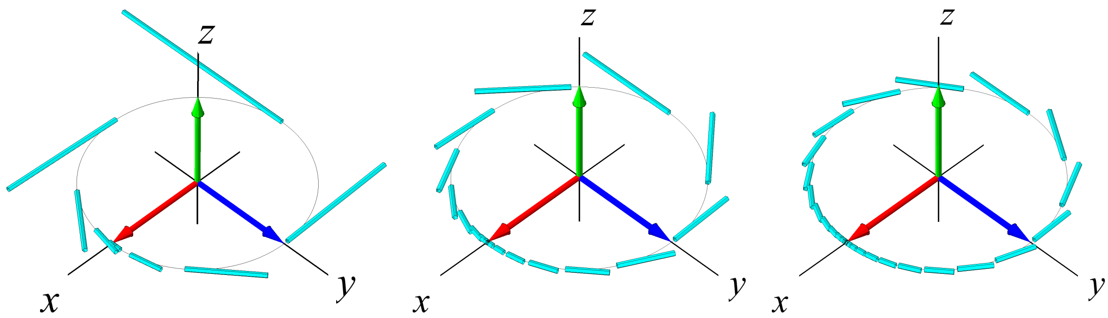

Figure 24.9: The curve $\mathbf{r}(u) = \left(\cos(2\pi u^3), \sin(2\pi u^3), 0\right)$, $u \in [-1,1]$, with 30, 60 and 100 approximating line segments. It is reasonable to define the length of the curve as the total length of the approximating line segments in the limit where the number of approximating line segments tends towards infinity.

Mass, the Weight of a Curve with Mass Density

If we assume that every individual line segment in (24-13) is allotted a constant mass density given by the value of the function $f(x, y, z)$ in contact point of the line segment with the curve, then we get the mass of the i'th line segment:

$$\Delta\,\mathrm{M}_i \;=\; f(x(u_i), y(u_i), z(u_i))\,|\mathbf{r}'(u_i)|\cdot\delta_u \;=\; f(\mathbf{r}(u_i))\,|\mathbf{r}'(u_i)|\cdot\delta_u \quad .$$

The total mass of the whole system of line segments is therefore the following, which is a good approximation to the mass of the whole curve when the curve is given the mass density $f(\mathbf{r}(u))$ at the position $\mathbf{r}(u)$:

$$\mathrm{M}_{\mathrm{app}}(n) \;=\; \sum_{i=1}^{n}\Delta\,\mathrm{M}_i \;=\; \sum_{i=1}^{n} f(\mathbf{r}(u_i))\,|\mathbf{r}'(u_i)|\cdot\delta_u \quad . \tag{24-17}$$

Again this is an integral sum, but now for the continuous function $f(\mathbf{r}(u))\,|\mathbf{r}'(u)|$ over the interval $[a, b]$. Thus we get in the limit, where n tends toward infinity:

$$\mathrm{M}_{\mathrm{app}}(n) \;\to\; \mathrm{M} \;=\; \int_a^b f(\mathbf{r}(u))|\mathbf{r}'(u)|\,\mathrm{d}u \quad \text{for} \quad n\to\infty \quad . \tag{24-18}$$

We have thus motivated the definition of the mass of a curve with the mass density $f(\mathbf{r}(u))$ (inasmuch this function is positive in $[a, b]$) and hereby the general definition of the line integral, Definition 24.5.

24.2 Plane Integrals

A parameterized region in the plane is given by a parametric representation

$$P_{\mathbf{r}}: \quad \mathbf{r}(u, v) \;=\; (x(u, v), y(u, v)) \in \mathbb{R}^2 \quad , \quad u \in [a, b] \;,\; v \in [c, d] \quad , \tag{24-19}$$

where $x(u, v)$ and $y(u, v)$ are given (typically smooth) functions of the two parameter variables u and v.

Plane integrals are written, named, and computed quite analogously to line integrals:

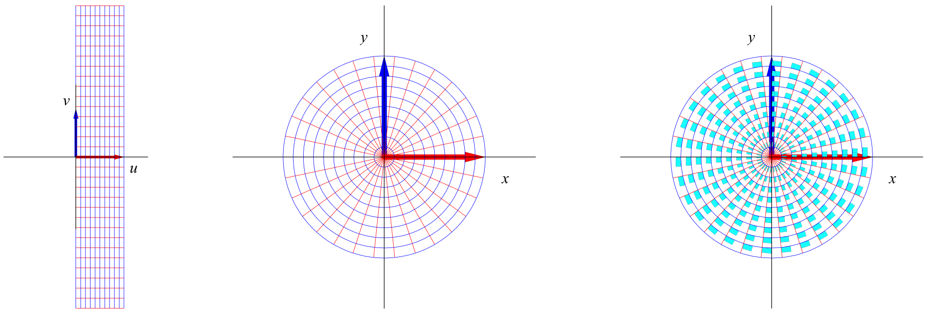

Figure 24.10: This region in the plane is given by the following parametric representation that represents *polar coordinates* in the plane: $\mathbf{r}(u,v) = (u\cos(v), u\sin(v))$, $u \in [0,1]$, $v \in [-\pi,\pi]$. The parametrized rectangle is seen to the left. This is deformed and mapped (by the use of **r**) on the plane region in the middle. To the right the placement and the size (apart from a factor 4) of the approximating parallelograms (here, rectangles) corresponding to the given net.

|||| Definition 24.19 Plane Integrals

Let $f(x,y)$ denote a continuous function on $\mathbb{R}^2$. The plane integral of the function f over the parameterized region $P_{\mathbf{r}}$ is defined by

$$\int_{P_{\mathbf{r}}} f \,\mathrm{d}\mu = \int_c^d \int_a^b f(\mathbf{r}(u,v))\,\mathrm{Jacobian}_{\mathbf{r}}(u,v)\,\mathrm{d}u\,\mathrm{d}v \quad , \tag{24-20}$$

where the Jacobian function $\mathrm{Jacobian}_{\mathbf{r}}(u,v)$,

$$\mathrm{Jacobian}_{\mathbf{r}}(u,v) = |\mathbf{r}'_u(u,v)| \cdot |\mathbf{r}'_v(u,v)| \cdot \sin(\theta(u,v)) \quad , \tag{24-21}$$

is the area of the parallelogram *in the plane*, that at the position $\mathbf{r}(u,v)$ is spanned by the two tangent vectors $\mathbf{r}'_u(u,v)$ and $\mathbf{r}'_v(u,v)$ to the respective coordinate curves through the point $\mathbf{r}(u,v)$ in the plane (the function $\theta(u,v) \in [0,\pi]$ denotes the angle between these tangent vectors).

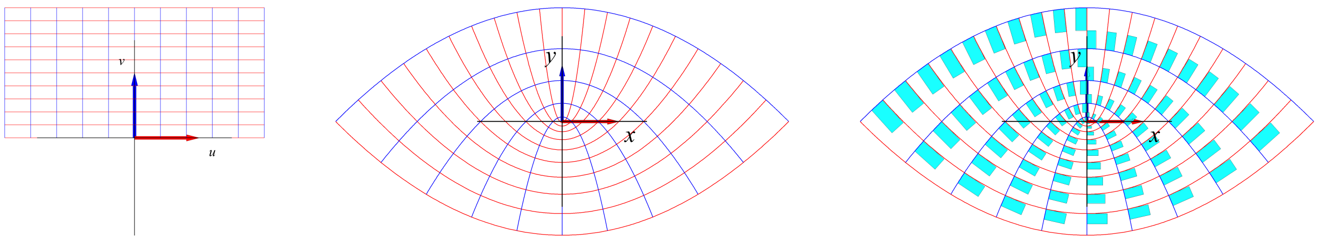

Figure 24.11: A plane region with parabolic coordinates. This region in the plane is given by the parametric representation $\mathbf{r}(u,v) = (u\,v, \frac{1}{2}(u^2 - v^2))$, $u \in [-2,2]$, $v \in [0,2]$. The Figure to the right again hints at a system of area-approximating parallelograms apart from a factor 4.

|||| Definition 24.20 Regular Parametric Representation of a Region in the Plane

The parametric representation (24-19) is said to be a *regular parametric representation* of the plane region if the following applies:

$$\text{Jacobian}_{\mathbf{r}}(u,v) > 0 \quad \text{for all} \quad u \in [a,b] \ , \ v \in [c,d] \quad . \tag{24-22}$$

|||| Definition 24.21 One-to-One Parametric Representations

As for parameterized curves the parametric representation in (24-19) is said to be one-to-one if different points in the domain are mapped to different points in the range in the plane.

|||| Exercise 24.22

Show that $\text{Jacobian}_{\mathbf{r}}(u,v)$ (in (24-21)) can also be found as the numerical value of the determinant of the (2×2) matrix that as columns has the coordinates of the two vectors $\mathbf{r}'_u(u,v)$ and $\mathbf{r}'_v(u,v)$.

|||| Example 24.23 Standard Graph-Bounded Region

Let $f(x)$ denote a positive function on an x-interval $[a, b]$. Then we can parameterize the region between the x-axis and the graph of the function $f(x)$ in the following simple way:

$$P_{\mathbf{r}} \quad : \quad \mathbf{r}(u, v) = (u, v \cdot f(u)) \quad , \quad u \in [a, b] \quad , \quad v \in [0, 1] \quad . \tag{24-23}$$

See Figure 24.12 (where the function $f(x) = 1 + x + x^2$ is used as an illustration). For the determination of the area of the region between the graph of $f(x)$ and x-axis we now have generally:

$$\text{Jacobian}_{\mathbf{r}}(u, v) = f(u) \quad , \tag{24-24}$$

because

$$\begin{aligned} \mathbf{r}'_u(u, v) &= (1, v \cdot f'(u)) \quad \text{and} \\ \mathbf{r}'_v(u, v) &= (0, f(u)) \quad , \end{aligned} \tag{24-25}$$

such that the determinant of the matrix that has the columns $\mathbf{r}'_u(u, v)$ and $\mathbf{r}'_v(u, v)$, exactly in this case is the function $f(u)$ itself and thereby the Jacobian function is also given by $f(u)$ according to Exercise 24.22. From this we get the wanted area reconstructed as:

$$\begin{aligned} \text{Area}(P_{\mathbf{r}}) = \int_{P_{\mathbf{r}}} 1 \, d\mu &= \int_0^1 \left(\int_a^b f(u) \, du \right) dv \\ &= \int_a^b f(u) \, du \quad . \end{aligned} \tag{24-26}$$

Consider what happens to the integral in (24-26) if we allow $f(x)$ to be negative on given subintervals of $[a, b]$.

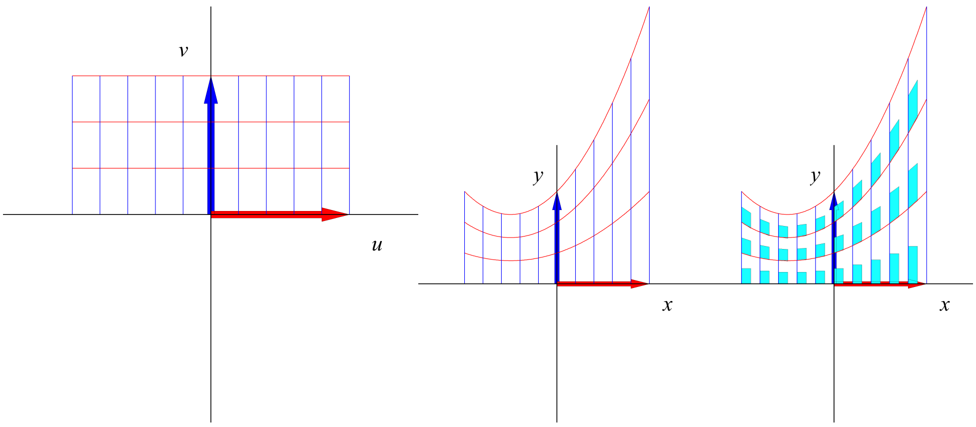

Figure 24.12: Parameterization of the region between the x-axis and the graph of the function $f(x) = 1 + x + x^2$. See Example 24.23.

|||| Example 24.24 Elliptic Region

The parameterized elliptic region in the plane (see Figure 24.13)

$$P_{\mathbf{r}}: \ \mathbf{r}(u,v) \ = \ (\, av\cos(u), \, bv\sin(u)\,) \ , \ u \in [-\pi,\pi] \ , \ v \in [0,1] \, .$$

has the area

$$\text{Area}(P_{\mathbf{r}}) \ = \ \int_{P_{\mathbf{r}}} 1 \, \mathrm{d}\mu \ = \ \int_0^1 \int_{-\pi}^{\pi} abv \ \mathrm{d}u \ \mathrm{d}v \ = \ ab\pi \quad ,$$

since $\text{Jacobian}_{\mathbf{r}}(u,v) \ = \ abv$. Compare with the computation of *the length of the ellipsis* in Example 24.12.

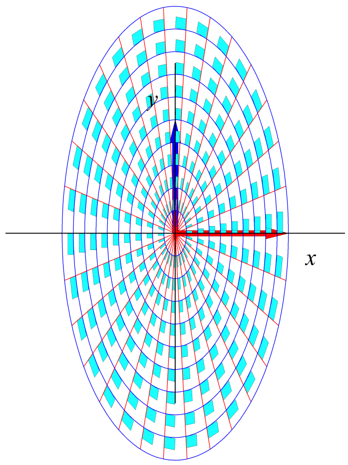

Figure 24.13: An elliptic region in the plane with semi-axes $a = 1$ and $b = 2$. See Example 24.24. Note that the approximating parallelograms (here shown scaled) are not rectangles.

|||| Exercise 24.25 Level-Bounded Elliptic Region

An elliptic region in the plane is bounded by the level curve $\mathcal{K}_0(f)$ of the quadratic polynomial

$$f(x,y) = 2 \cdot x^2 + 2 \cdot y^2 + 2 \cdot x \cdot y - 8 \cdot x - 10 \cdot y + 13 \quad . \qquad \text{(24-27)}$$

Determine the length of the level curve and determine the area of the bounded elliptic region in the (x,y) plane. See Figure 24.14.

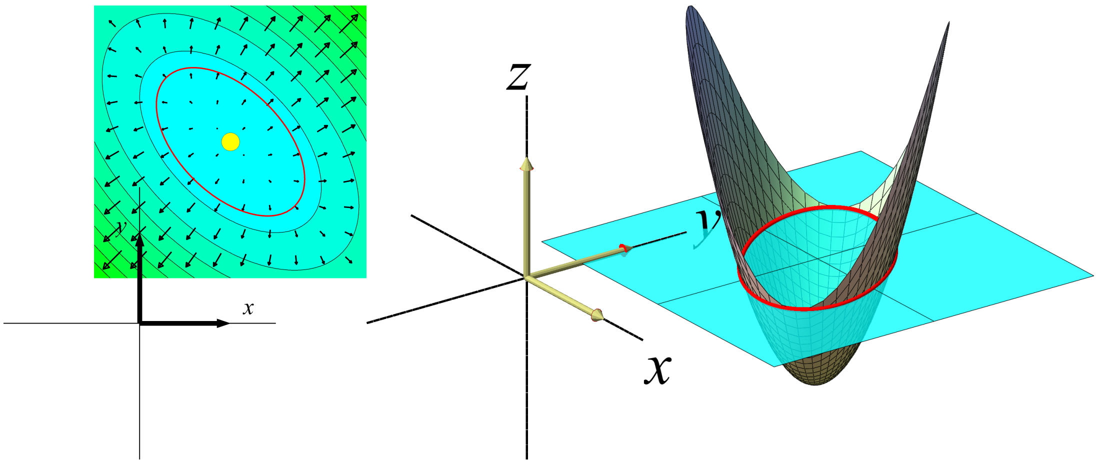

Figure 24.14: An elliptic region in the plane is bounded by an ellipsis that is the level curve $\mathcal{K}_0(f)$ of the quadratic polynomial $f(x,y) = 2 \cdot x^2 + 2 \cdot y^2 + 2 \cdot x \cdot y - 8 \cdot x - 10 \cdot y + 13$. See Exercise 24.25.

|||| Example 24.26 Spiral Region

The parameterized spiral-bounded region in the plane (see Figure 24.15)

$$P_{\mathbf{r}} : \ \mathbf{r}(u,v) = (\, vu\cos(u), vu\sin(u)\,) \ , \ u \in [0, \pi/2] \ , \ v \in [0,1] \, .$$

has the area

$$\text{Area}(P_{\mathbf{r}}) = \int_{P_{\mathbf{r}}} 1 \, \mathrm{d}\mu = \int_0^1 \int_0^{\pi/2} v^2 u \, \mathrm{d}u \, \mathrm{d}v = \pi^3/48 \quad ,$$

since $\text{Jacobian}_{\mathbf{r}}(u,v) = v^2 u$. Compare with the computation of the *length of the spiral* in Example 24.11.

24.2.1 Motivation for the Plane Integral

If we, in analogy to the statement of the line integral, partition *both* the parameter intervals $[a,b]$ and $[c,d]$ in n and m equal parts, then every u-subinterval has the length $\delta_u = (b-a)/n$ and every v-subinterval has the length $\delta_v = (d-c)/m$. Correspondingly the coordinates of the division points in the (u,v)-parameter region (that is the rectangle $[a,b] \times [c,d]$ in $\mathbb{R}^2$) - cf. eNote 20:

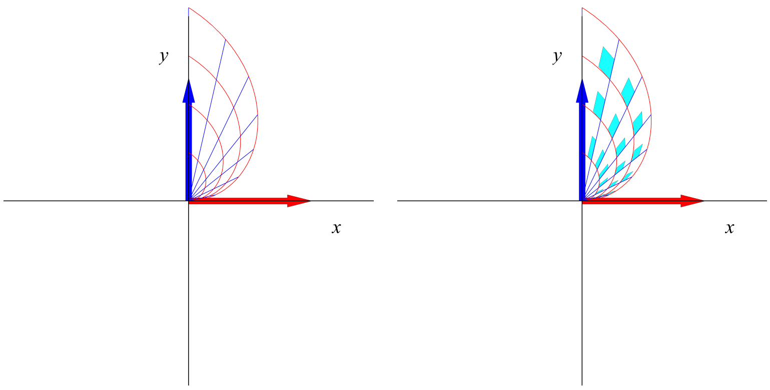

Figure 24.15: A spiral-bounded region in the plane. See Example 24.26.

$$\begin{aligned}
(u_1, v_1) &= (a, c), \\
(u_1, v_j) &= (a, c + (j-1)\delta_v), \\
(u_i, v_1) &= (a + (i-1)\delta_u, c), \\
(u_i, v_j) &= (a + (i-1)\delta_u, c + (j-1)\delta_v), \\
&\dots. \\
(b, d) &= (a + n\delta_u, c + m\delta_v) \quad .
\end{aligned} \tag{24-28}$$

With all these given points (u_i, v_j) as development points we can again use Taylor's limit formula, now for each of the 2 coordinate functions $x(u, v)$ and $y(u, v)$ of $\mathbf{r}(u, v) = (x(u, v), y(u, v))$ to the first order with corresponding epsilon functions:

$$\begin{aligned}
\mathbf{r}(u, v) = \;& \mathbf{r}(u_i, v_j) \\
& + \mathbf{r}'_u(u_i, v_j) \cdot (u - u_i) \\
& + \mathbf{r}'_v(u_i, v_j) \cdot (v - v_j) \\
& + \rho_{ij} \cdot \boldsymbol{\varepsilon}_{ij}(u - u_i, v - v_i) \quad ,
\end{aligned} \tag{24-29}$$

where $u \in [u_i, u_i + \delta_u]$, $v \in [v_j, v_j + \delta_v]$. Here $\rho_{ij} = \sqrt{(u - u_i)^2 + (v - v_j)^2}$ denotes the distance between the variable point (u, v) and the given development point (u_i, v_j) in the parametrized region. Here it applies that the vector function $\boldsymbol{\varepsilon}_{ij}(u - u_i, v - v_j) \to (0, 0) = \mathbf{0}$ for $(u - u_i, v - v_j) \to (0, 0)$.

Every sub-rectangle $[u_i, u_i + \delta_i] \times [v_j, v_j + \delta_j]$ is mapped onto the plane subregion, which we can describe by $\mathbf{r}(u, v)$ evaluated in the parameter-subrectangle $u \in [u_i, u_i + \delta_u], v \in$

$[v_j, v_j + \delta_v]$ and this subregion we can approximate with the expression (24-29) that again exactly can be found by removing the ε_{ij}-term from the right-hand side from (24-29):

$$\mathbf{r}_{\mathrm{app}_{ij}}(u,v) = \mathbf{r}(u_i,v_j) + \mathbf{r}'_u(u_i,v_j)\cdot(u-u_i) + \mathbf{r}'_v(u_i,v_j)\cdot(v-v_j) \quad , \tag{24-30}$$

with u and v still running through the sub-intervals $u \in [u_i\,,\, u_i+\delta_u]\,,\; v \in [v_j\,,\, v_j+\delta_v]\,.$

These linear approximations are parallelograms that are spanned by the two tangent vectors $\mathbf{r}'_u(u_i,v_j)\cdot\delta_u$ and $\mathbf{r}'_v(u_i,v_j)\cdot\delta_v$.

Area of a Plane Region

Each of the (in total) $n\cdot m$ approximating parallelograms has an area. The area of the (i,j)th parallelogram is given by

$$\begin{aligned}\Delta\,\mathrm{Area}_{ij} &= |\mathbf{r}'_u(u_i,v_j)|\cdot|\mathbf{r}'_v(u_i,v_j)|\,\sin(\theta(u_i,v_j))\cdot\delta_u\cdot\delta_v \\ &= \mathrm{Jacobian}_{\mathbf{r}}(u_i,v_j)\cdot\delta_u\cdot\delta_v \quad .\end{aligned} \tag{24-31}$$

|||| Exercise 24.27

Prove this statement: The area of a parallelogram is the product of the lengths of the two spanning vectors and the sine of the interjacent angle. See eNote 10.

The sum of all these (in total) $n\,m$ areas is clearly a good approximation to the area of the whole surface segment, so that we have

$$\mathrm{Area}_{\mathrm{app}}(n,m) = \sum_{j=1}^{m}\sum_{i=1}^{n}\Delta\,\mathrm{Area}_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n}\mathrm{Jacobian}_{\mathbf{r}}(u_i,v_j)\cdot\delta_u\delta_v \quad . \tag{24-32}$$

since the sum above is a double integral sum for the continuous function $\mathrm{Jacobian}_{\mathbf{r}}(u,v)$ over the parametrized rectangle $[a,b]\times[c,d]$ we get in the limit, where n and m both tend to infinity (see eNote 23):

$$\mathrm{Area}_{\mathrm{app}}(n,m) \to \mathrm{Area} = \int_c^d\int_a^b \mathrm{Jacobian}_{\mathbf{r}}(u,v)\,\mathrm{d}u\,\mathrm{d}v \quad \text{for} \quad n\,,\,m\to\infty \quad . \tag{24-33}$$

This is the reason for the definition of the area of a parameterized region in the plane as stated above, viz. as the surface integral of the constant function 1.

Mass, Weight of a Plane Region

If we now assume that every individual parallelogram in (24-30) is allotted a constant mass density given by the value of the function $f(x,y)$ at the contact point of the parallelogram with the surface, then we get the mass of the (i,j)th parallelogram :

$$\begin{aligned} \Delta \mathrm{M}_{ij} &= f(x(u_i,v_j),y(u_i,v_j))\,\mathrm{Jacobian}_{\mathbf{r}}(u_i,v_j)\cdot\delta_u\delta_v \\ &= f(\mathbf{r}(u_i,v_j))\,\mathrm{Jacobian}_{\mathbf{r}}(u_i,v_j)\cdot\delta_u\delta_v \quad . \end{aligned} \tag{24-34}$$

The total mass of the whole system of parallelograms is therefore the following, which is a good approximation to the mass of all of the plane region when this is given the masse density $f(\mathbf{r}(u,v))$ in the point $\mathbf{r}(u,v)$.

$$\mathrm{M}_{\mathrm{app}}(n,m) = \sum_{j=1}^{m}\sum_{i=1}^{n}\Delta \mathrm{M}_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} f(\mathbf{r}(u_i,v_j))\,\mathrm{Jacobian}_{\mathbf{r}}(u_i,v_j)\cdot\delta_u\delta_v \quad . \tag{24-35}$$

This is a double integral sum for the continuous function $f(\mathbf{r}(u,v))\,\mathrm{Jacobian}_{\mathbf{r}}(u,v)$ over the parametrized rectangle $[a,b]\times[c,d]$. Thus we get in the limit, where n and m tend towards infinity:

$$\mathrm{M}_{\mathrm{app}}(n,m) \to \mathrm{M} = \int_c^d \int_a^b f(\mathbf{r}(u,v))\mathrm{Jacobian}_{\mathbf{r}}(u,v)\,\mathrm{d}u\,\mathrm{d}v \quad \text{for} \quad n\,,\,m\to\infty \quad . \tag{24-36}$$

Hereby we have motivated the definition of the mass of a parameterized region in the plane with the mass density $f(\mathbf{r}(u,v))$ and hereby also the general definition of plane integral, Definition 24.19.

|||| Example 24.28 Total Weight of a Ring-Shaped Region

Let $f(x,y) = 1+x$ be a weight function on the plane region in the (x,y) plane that is bounded by the x-axis and the two upper semi-circular arcs of the circles with the radii 1 and $1/2$, respectively, both centred at $(0,0)$, see Figure 24.16. A parameterization of the area is e.g.:

$$P_{\mathbf{r}} \quad : \quad \mathbf{r}(u,v) = (u\cdot\cos(v), u\cdot\sin(v)) \quad , \quad \text{where} \quad u\in[1/2,1] \quad , \quad v\in[0,\pi] \quad . \tag{24-37}$$

When the region locally has the weight given by the weight function $f(x,y)$ the total weight of the region becomes:

$$M(P_{\mathbf{r}}) = \int_{P_{\mathbf{r}}} f\,\mathrm{d}\mu = \int_0^{\pi}\int_{1/2}^{1} f(\mathbf{r}(u,v))\cdot\mathrm{Jacobian}_{\mathbf{r}}(u,v)\,\mathrm{d}u\,\mathrm{d}v \quad , \tag{24-38}$$

where

$$f(\mathbf{r}(u,v)) = 1 + x(u,v) = 1 + u \cdot \cos(v) \quad \text{and} \qquad \text{(24-39)}$$
$$\text{Jacobian}_{\mathbf{r}}(u,v) = u \quad ,$$

such that

$$\begin{aligned} M(P_{\mathbf{r}}) &= \int_0^{\pi} \int_{1/2}^{1} (1 + u \cdot \cos(v)) \cdot u \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_0^{\pi} \left[\frac{1}{2}u^2 + \frac{1}{3}u^3 \cdot \cos(v) \right]_{u=1/2}^{u=1} \mathrm{d}v \\ &= \int_0^{\pi} \left(\frac{3}{8} + \frac{7}{24} \cdot \cos(v) \right) \mathrm{d}v \\ &= \left[\frac{3}{8}v + \frac{7}{24} \cdot \sin(v) \right]_{v=0}^{v=\pi} \\ &= \frac{3}{8} \cdot \pi \quad . \end{aligned} \qquad \text{(24-40)}$$

The total mass of the region with the given weight distribution is therefore $M(P_{\mathbf{r}}) = 3\pi/8$. By way of comparison the area of the region is also $\text{Area}(P_{\mathbf{r}}) = 3\pi/8$ – but this is a coincidence.

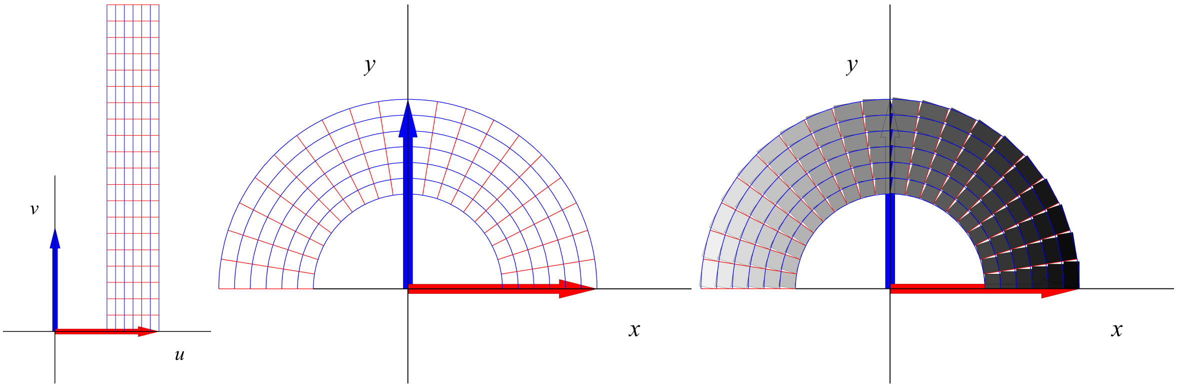

Figure 24.16: Half a circular ring-shaped region in the plane with a weight distribution indicated by shading. See Example 24.28.

24.3 Summary

We have in this eNote stated the concepts and methods that give us precise expressions for lengths of curves, areas of plane regions and more general curve- and plane-integrals of (weight-)functions on curves and plane regions that are parameterized from an interval or a rectangular parametrized region, respectively. Curve- and plane-integrals are stated by use of the respective Jacobian functions that describe how much the parameter interval or the parametrized region, respectively, is deformed locally when mapped into the wanted curve or into the wanted plane region with the chosen vector maps $\mathbf{r}(u)$ and $\mathbf{r}(u,v)$.

- For a space curve $K_\mathbf{r}$ with parametric representation $\mathbf{r}(u) = (x(u), y(u), z(u))$ we have the following motivated line integral of the function $f(x,y,z)$ over the space curve:
$$\int_{K_\mathbf{r}} f \, \mathrm{d}\mu = \int_a^b f(\mathbf{r}(u)) \, \mathrm{Jacobian}_\mathbf{r}(u) \, \mathrm{d}u \quad , \tag{24-41}$$
where the Jacobian function $\mathrm{Jacobian}_\mathbf{r}(u)$ is given by the length of the tangent vector of the parameterization:
$$\mathrm{Jacobian}_\mathbf{r}(u) = |\mathbf{r}'(u)| \quad . \tag{24-42}$$

- For a plane region $P_\mathbf{r}$ with the parametric representation $\mathbf{r}(u,v) = (x(u,v), y(u,v))$ we have similarly motivated the following definition of the plane integral of the function $f(x,y)$ over the region:
$$\int_{P_\mathbf{r}} f \, \mathrm{d}\mu = \int_c^d \int_a^b f(\mathbf{r}(u,v)) \, \mathrm{Jacobian}_\mathbf{r}(u,v) \, \mathrm{d}u \, \mathrm{d}v \quad , \tag{24-43}$$
where the Jacobian function $\mathrm{Jacobian}_\mathbf{r}(u,v)$ now is given by the area of the parallelogram spanned by the tangent vectors of the coordinate curves:
$$\mathrm{Jacobian}_\mathbf{r}(u,v) = |\mathbf{r}'_u(u,v)| \cdot |\mathbf{r}'_v(u,v)| \cdot \sin(\theta(u,v)) \quad , \tag{24-44}$$
where $\theta(u,v) \in [0, \pi]$ denotes the angle between the two tangent vectors $\mathbf{r}'_u(u,v)$ and $\mathbf{r}'_v(u,v)$.