eNote 23

Riemann Integrals

In this eNote we will state and give examples of those techniques, methods, and results that are completely necessary tools when we want to find lengths of curves, areas of plane regions and surfaces, and volumes, centres of mass and moments of inertia of spatial regions etc. In the first place it is about being able to integrate and determine indefinite integrals of given continuous functions, in particular of functions of one variable. Therefore a couple of times we will refer to eNote 3. We shall in this eNote see how infinite sums of infinitely small addends in the limit lead to so-called Riemann integrals that again can be expressed as and computed with the use of suitable indefinite integrals. The methods and the fundamental results for the Riemann integrals are not completely different in the dimensions we consider, but nevertheless we will discuss and analyze concepts, results, and examples explicitly for functions of one, two and three variables for the purpose of being able to use the Riemann integrals most efficiently in the applications that are considered in the eNotes about integration with more variables.

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23.1 Introduction

The point of this and the following eNotes is to motivate, formulate and apply the unique tool that can answer such questions occurring completely naturally in many connections: how long is a curve? How large is a region in the plane? What is the weight of a piece of a surface? What is the volume of that spatial region? What is the total energy-uptake on that solar panel roof for today? How much will that body be deformed when flowing along that vector field?

The tool – the method – that can answer these questions is called *integration*. I.e. we shall be able to integrate given functions $f(x)$ and find indefinite integrals of them. As is well known, an indefinite integral to $f(x)$ is a function $F(x)$ whose derivative is $f(x)$. But there are many indefinite integrals of a given function; if we differentiate $F(x) + c$ where c if a constant, we again get $f(x)$. I.e. if $F(x)$ is an indefinite integral then $F(x) + c$ is also an indefinite integral!

Furthermore it is in no way clear from the outset that such indefinite integrals have anything at all to do with lengths, areas, volumes or weight. And besides, which function $f(x)$ must we use e.g. in order to find the volume of a sphere? And if we, by the way, can find an indefinite integral to $f(x)$, what constant must be added in order to get the right volume? To get an idea about this we must first look at how we at all can try to understand and define the *concept* of volume.

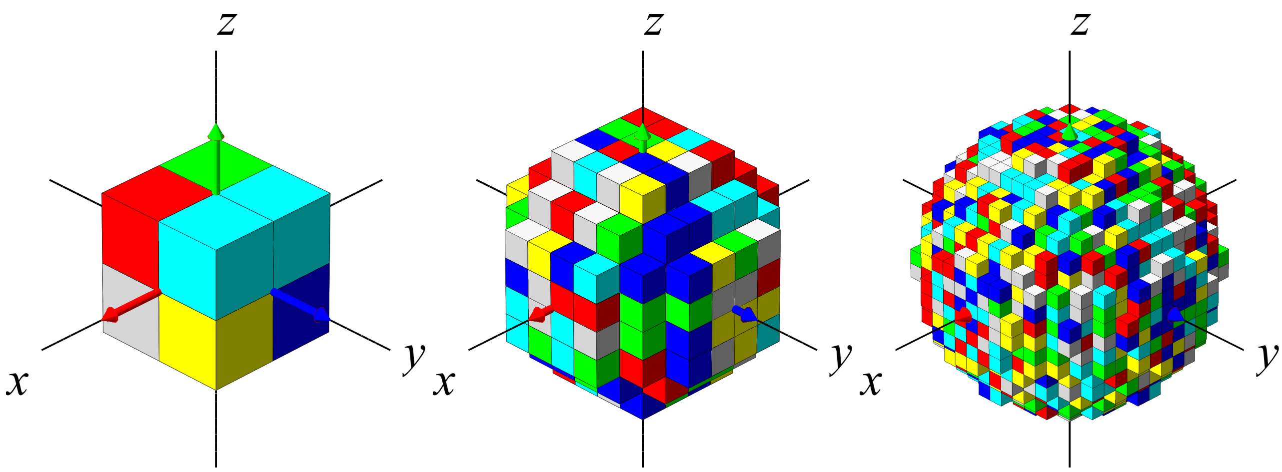

Figure 23.1: Sphere tessellation with cubical blocks. The indicated sphere which we wish to fill with the blocks, has radius 1. To the left there are room for 8 blocks each with a side length of 0.5; in the middle we have used 304 blocks each with a side length of 0.2; to the right we have used 3280 blocks, each with the side length 0.1.

23.2 The Volume Problem

The volume of a given region in (x, y, z) space, e.g. a massive sphere with radius 1, can be built by standard elements with the simplest box like geometry, e.g. cubical blocks.

But the result of such a construction can only be a crude approximation to the sphere, see Figure 23.1. And therefore the sum of the volumes of the cubical blocks gives only a crude approximation to the volume of the sphere.

However, if we fill the same sphere with cubical blocks each with a volume that is 1000 times smaller (i.e. a side length $1/10$ of the former) it is obvious that the sphere by this can be approximated much better b the use of (more than 1000 times) more cubical blocks; and still it is (in principle) easy to add up the volumes of all the blocks. This also gives a better value for the volume of the sphere.

The example in Figure 23.1 shows the principle: The approximation of a sphere with radius 1 with 8 cubical blocks with side length $1/2$ has the volume $8 \cdot (1/2)^3 = 1$; in the middle we have 304 cubical blocks (all with side length $1/5$) giving an approximated volume of $304 \cdot (1/5)^3 = 2.432$, while the approximation with 3280 blocks (of side length 0.1) in the figure to the right obviously gives a better approximation: $3280 \cdot (1/10)^3 = 3.280$. By way of comparison, already Archimedes knew that the exact volume of the unit sphere is $4\pi/3 \approx 4.1888$.

When first this is clear then we wish of course to 'proceed to the limit' by letting the number of standard blocks grow towards infinity while at the same time the blocks we use are correspondingly reduced in size for each trial. And in this way we shall pack and fill the sphere better and better with more and more, smaller and smaller cubes and in this way obtain ***Archimedes' result*** in the limit.

But how do we add infinitely many infinitely small volumes? And does it really work? The concept of integration and the corresponding determination of indefinite integrals give precise instructions and surprisingly positive answers to both of these questions.

We show in this eNote which formal considerations and methods lie behind the success and in the following eNotes we continue to use the integration techniques for the determination of lengths of curves, areas of pieces of surfaces, volumes of spatial regions, etc.

In the eNote about integration in three variables it is e.g. shown how the volume of the sphere can be computed exactly by the use of 'tessellation' with box-formed blocks (of different sizes and shapes along the lines of the tessellation of the sphere above), see Figure 23.1 and 23.2 and Example 23.26.

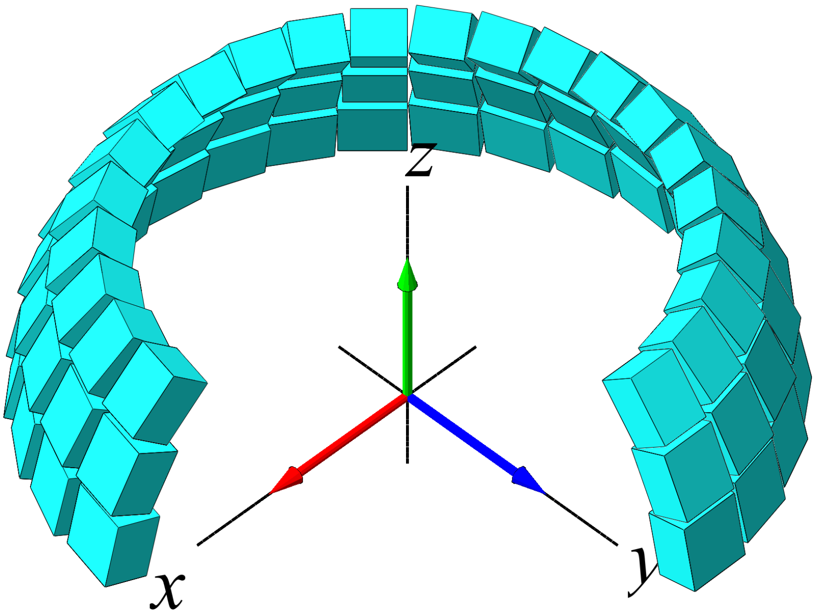

Figure 23.2: Construction in part of a spherical shell.

23.3 Approximating Sums and Exact Integrals

On the real u-axis we consider a chosen continuous real function $f(u)$ on the interval $[0,1]$, e.g. $f(u) = 1 + u + u^2$. For a given integer $n > 0$ we now do the following. First we divide the interval $[0,1]$ into n subintervals of the same length which thus is $\delta_u = \frac{1}{n}$. The left endpoints of the subintervals have the $u-$coordinates:

$$u_1 = 0, \quad u_2 = \frac{1}{n}, \quad u_3 = \frac{2}{n}, \quad u_4 = \frac{3}{n}, \cdots, \quad u_{n-1} = \frac{n-2}{n}, \quad u_n = \frac{n-1}{n} \quad .$$

I.e. that the left endpoint of the i'th interval has the $u-$coordinate $u_i = (i-1)\frac{1}{n} = (i-1)\,\delta_u$, where $i = 1, 2, 3, ..., n-1, n$.

 

Figure 23.3: Gottfried Wilhelm von Leibniz (to the left) and Georg Friedrich Bernhard Riemann.

|||| Exercise 23.1

Note that if we increase the number of subintervals n by 1, and now wish a partition of $[0,1]$ in $n+1$ subintervals of equal size then all the previously placed n left endpoints in the interval $[0,1]$ need to be moved (apart from u_1) to give room for the extra subinterval. By how much?

For a given number of subintervals, n, we find the function value of f in each of the left endpoints of the subinterval that is the n values $f(0)$, $f(\frac{1}{n})$, $f(\frac{2}{n})$, $f(\frac{3}{n})$, ..., $f(\frac{n-1}{n})$.

The sum of these values will normally depend to a great extent on the number n of function values, but if we first multiply the function value by the length of the subinterval δ_u we get the following so-called *weighted sum* of the function values, this being just an approximation to the (with sign) area of the region between the $u-$axis and the graph for $f(u)$ over the interval (cf. Figure 23.4):

$$\mathrm{I}(f,n,[0,1]) = \sum_{i=1}^{i=n} f\left((i-1)\frac{1}{n}\right)\frac{1}{n} = \sum_{i=1}^{i=n} f\left((i-1)\delta_u\right)\delta_u = \sum_{i=1}^{i=n} f\left(u_i\right)\delta_u \quad . \tag{23-1}$$

|||| Exercise 23.2

Show that the weighted sum of the functional values of f in equation (23-1) has an upper limit, the maximum value of f and and a lower limit, the minimal value of f in the interval $[0,1]$.

The weighted sum is not only limited for all n. Viz. it also turns out that it has a limit value for n going towards infinity – at least if $f(u)$ is continuous. It is this limit value we call the Riemann integral of $f(u)$ over the interval $[0,1]$ (after B. Riemann, see Figure 23.3). The limit value itself has (after G. Leibniz, see Figure 23.3) the following notation

$$\lim_{n\to\infty} \mathrm{I}(f,n,[0,1]) = \int_0^1 f(u)\,\mathrm{d}u \quad . \tag{23-2}$$

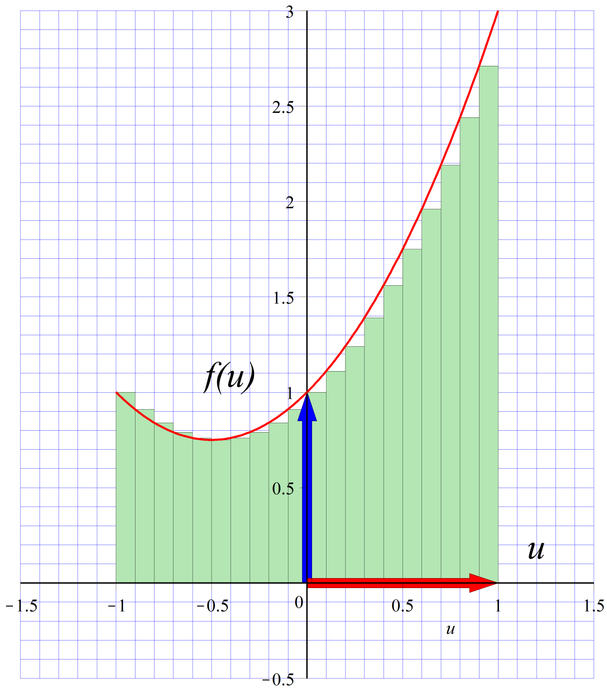

Figure 23.4: The figure shows the area representation of the integral sum $\mathrm{I}(f,n,[-1,1])$ for the function $f(u) = 1+u+u^2$ (see Exercise 23.14) with $n = 20$ subintervals $[a,b] = [-1,1]$. The 20 addends in the sum are represented by rectangular columns with common width $(b-a)/20 = 1/10$ and heights given by the values of the function $f(u) = 1+u+u^2$ in the left endpoints of the intervals.

|||| Example 23.3 Constant Function

Let $f(u) = \alpha$ for all $u \in [0, 1]$, where α is a constant. Then

$$\begin{aligned} \mathrm{I}(f, n, [0,1]) &= \sum_{i=1}^{i=n} f(u_i)\, \delta_u \\ &= \sum_{i=1}^{i=n} \alpha \cdot \frac{1}{n} \\ &= \frac{\alpha}{n} \cdot \sum_{i=1}^{i=n} 1 \\ &= \frac{\alpha}{n} \cdot n \\ &= \alpha \quad , \end{aligned} \tag{23-3}$$

so that

$$\lim_{n \to \infty} \mathrm{I}(f, n, [0,1]) \;=\; \int_0^1 \alpha \, \mathrm{d}u = \alpha \quad . \tag{23-4}$$

|||| Example 23.4 Linear Polynomial

Let $f(u) = \alpha + \beta \cdot u$ for $u \in [0, 1]$, where α and β are constants. I.e. that the graph for $f(u)$ is a straight line segment over the interval $u \in [0, 1]$ on the u-axis. Then

$$\begin{aligned} \mathrm{I}(f, n, [0,1]) &= \sum_{i=1}^{i=n} f(u_i)\, \delta_u \\ &= \sum_{i=1}^{i=n} (\alpha + \beta \cdot u_i) \cdot \frac{1}{n} \\ &= \sum_{i=1}^{i=n} \left(\alpha + \beta \cdot (i-1) \cdot \frac{1}{n}\right) \cdot \frac{1}{n} \\ &= \alpha \cdot \left(\sum_{i=1}^{i=n} \frac{1}{n}\right) + \beta \cdot \left(\sum_{i=1}^{i=n} (i-1) \cdot \frac{1}{n^2}\right) \\ &= \alpha + \beta \cdot \sum_{i=1}^{i=n} \frac{i}{n^2} - \beta \cdot \sum_{i=1}^{i=n} \frac{1}{n^2} \\ &= \alpha + \beta \cdot \frac{1}{n^2} \sum_{i=1}^{i=n} i - \beta \cdot \frac{1}{n} \quad , \end{aligned} \tag{23-5}$$

where we need the following identity:

$$\frac{1}{n^2} \sum_{i=1}^{i=n} i = \frac{1}{n^2} \cdot \frac{(n+1) \cdot n}{2} = \frac{(n+1)}{2 \cdot n} \quad , \tag{23-6}$$

such that

$$I(f, n, [0,1]) = \alpha + \beta \cdot \frac{(n+1)}{2 \cdot n} - \beta \cdot \frac{1}{n} \quad . \tag{23-7}$$

From this it follows that

$$\lim_{n \to \infty} I(f, n, [0,1]) = \alpha + \beta \cdot \frac{1}{2} \quad , \tag{23-8}$$

and then

$$\int_0^1 \alpha + \beta \cdot u \, \mathrm{d}u = \alpha + \frac{1}{2} \cdot \beta \quad . \tag{23-9}$$

Note that the value found is exactly the area of the region between the graph for $f(u) = \alpha + \beta \cdot u$ and the u-axis over the interval $[0,1]$ inasmuch that the graph lies entirely above the u-axis.

If we use the same strategy as above, but now with a partition of the more general interval $[a, b]$ on the u-axis in n subintervals of the equal size, we have the following theorem:

|||| Theorem 23.5 Integral Sum and Riemann Integral

Let $f(u)$ denote a continuous function on the interval $[a, b]$. For every n the interval is partitioned into n subintervals of equal size each with the length $\delta_u = (b-a)/n$. The left endpoints of these subintervals then have the coordinates $u_i = a + (i-1)\delta_u$ for $i = 1, 2, 3, ..., n-1, n$. Let $I(f, n, [a, b])$ denote the following sum:

$$\begin{aligned} I(f, n, [a,b]) &= \sum_{i=1}^{i=n} f\left(a + (i-1)\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) \\ &= \sum_{i=1}^{i=n} f\left(a + (i-1)\delta_u\right) \delta_u = \sum_{i=1}^{i=n} f\left(u_i\right) \delta_u \quad . \end{aligned} \tag{23-10}$$

Then $I(f, n, [a, b])$ has a limit value for n going towards ∞. The limit value is called the ***Riemann integral*** of $f(u)$ over $[\,a,\, b\,]$ and is denoted $\int_a^b f(u)\,\mathrm{d}u$:

$$I(f, n, [a,b]) = \sum_{i=1}^{i=n} f\left(u_i\right) \delta_u \to \int_a^b f(u)\,\mathrm{d}u \quad \text{for} \quad \mathrm{n} \to \infty \quad . \tag{23-11}$$

|||| Exercise 23.6

Let $f(u) = \alpha + \beta \cdot u$ for given constants α and β and let $[a, b]$ denote an interval on the u-axis. Determine for every n the value of

$$\mathrm{I}(f, n, [a, b]) \quad , \tag{23-12}$$

then find the limit value

$$\lim_{n \to \infty} \mathrm{I}(f, n, [a, b]) \quad , \tag{23-13}$$

and compare this to the area of the region between the (straight) graph for $f(u)$ and the u-axis.

Sums of the type $\mathrm{I}(f, n, [a, b])$ we will in the following call ***integral sums***. It is the existence of limit values of these integral sums that is completely decisive for our enterprise. E.g. note that the limit value $\int_a^b f(u)\,\mathrm{d}u$ now is the best bet on what we shall understand by the area of the plane region between the $u-$axis and the graph for $f(u)$ over the interval $[a, b]$ (inasmuch that $f(u)$ is positive in all of the interval). In exercise 23.14 and in the examples 23.15 and 23.16 we find other examples of how such limit values (and areas) can be computed directly from an analysis on how the sums $\sum_{i=1}^{i=n} f(u_i)\ \delta_u$ behave as $n \to \infty$.

Note that the Riemann integral is constructed and appears precisely as an infinite sum of infinitely small addends $\sum_{i=1}^{i=n} f(u_i)\ \delta_u$ for $n \to \infty$, i.e. just the way we needed it in connection with our deliberation about the volume of the sphere above and in Figure 23.1.

23.4 Determination of Indefinite Integrals

Riemann-integrals can be determined by the use of indefinite integrals – this is precisely the content of the fundamental theorem that we will formulate and use in the next section. We will assume in the following that we already for suitable elementary functions $f(x)$ are able to find the indefinite integrals $F(x)$ to $f(x)$. As is well known this consists in finding all the functions $F(x)$ that fulfill that $F'(x) = f(x)$. These functions we denote as follows using the ***integral sign*** and we say that the ***integrand*** $f(x)$ is *integrated* and yields *the integral* or *the indefinite integral* $F(x)$:

$$F(x) = \int f(x)\,\mathrm{d}x \quad . \tag{23-14}$$

If $F(x)$ is an indefinite integral to $f(x)$ and c is a real constant then $F(x) + c$ is also an indefinite integral to $f(x)$. And *all* the indefinite integrals to $f(x)$ are obtained by finding one indefinite integral and to this add arbitrary constants c.

Here are some examples of indefinite integrals for some well-known functions $f(x)$ (we only state one indefinite integral for each of the given functions):

$$\begin{aligned}
f(x) &= a \quad , \quad F(x) = \int f(x)\,\mathrm{d}x = ax \\
f(r) &= 4\pi r^2 \quad , \quad F(r) = \int f(r)\,\mathrm{d}r = \frac{4}{3}\pi r^3 \\
f(t) &= 1/(1+t^2) \quad , \quad F(t) = \int f(t)\,\mathrm{d}t = \arctan(t) \\
f(u) &= 1+u+u^2 \quad , \quad F(u) = \int f(u)\,\mathrm{d}u = u + \frac{1}{2}u^2 + \frac{1}{3}u^3 \\
f(x) &= \sin(x^2) \quad , \quad F(x) = \int f(x)\,\mathrm{d}x = \sqrt{\frac{\pi}{2}} \cdot \mathrm{FresnelS}\left(x \cdot \sqrt{\frac{2}{\pi}}\right) \\
f(x) &= \mathrm{e}^{-x^2} \quad , \quad F(x) = \int f(x)\,\mathrm{d}x = \frac{1}{2}\sqrt{\pi}\,\mathrm{erf}(x) \\
f(x) &= \mathrm{e}^{x} \quad , \quad F(x) = \int f(x)\,\mathrm{d}x = \mathrm{e}^{x} \quad .
\end{aligned} \tag{23-15}$$

Since we in the eNotes about integration with more variables to a great extent are in need of being able to find indefinite integrals – also for a bit more complicated integrand-functions – we mention the following two theorems that can be of help in rewriting of given indefinite integral problems to the determination of simpler indefinite integrals.

|||| Theorem 23.7 Partial Integration

Let $f(x)$ denote a continuous function with an indefinite integral $F(x)$ and let $g(x)$ be a differentiable function with continuous derivative $g'(x)$. Then all the indefinite integrals to the product $f(x) \cdot g(x)$ can be determined by:

$$\int f(x) \cdot g(x)\, \mathrm{d}x = F(x) \cdot g(x) - \int F(x) \cdot g'(x)\, \mathrm{d}x. \tag{23-16}$$

|||| Proof

We only need to show that the two sides of Equation (23-16) have the same derivatives for all x! Two functions are indefinite integrals of the same integrand-function if their difference is a constant. The derivatives are equal since:

$$\begin{aligned} \frac{d}{dx}\left(\int f(x) \cdot g(x)\, \mathrm{d}x\right) &= f(x) \cdot g(x) \\ \frac{d}{dx}\left(F(x) \cdot g(x) - \int F(x) \cdot g'(x)\, \mathrm{d}x\right) &= F'(x) \cdot g(x) + F(x) \cdot g'(x) - F(x) \cdot g'(x) \\ &= f(x) \cdot g(x) \quad . \end{aligned} \tag{23-17}$$

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|||| Example 23.8 Partial Integration

We will determine the indefinite integral to $h(x) = x \cdot \sin(x)$. First we write $h(x) = f(x) \cdot g(x)$ with $f(x) = \sin(x)$ and $g(x) = x$. Then, in accordance with the rule on partial integration, we have:

$$\int x \cdot \sin(x)\, \mathrm{d}x = x \cdot (-\cos(x)) - \int 1 \cdot (-\cos(x))\, \mathrm{d}x = -x \cdot \cos(x) + \sin(x) \quad . \tag{23-18}$$

|||| Exercise 23.9

Determine all indefinite integrals to the function $f(x) = \cos(x) \cdot \sin(x)$.

|||| Theorem 23.10 Integration by Substitution

Let $f(x)$ be a continuous function and let $g(u)$ denote a monotonous, differentiable function with continuous $g'(u)$. Then we can determine an indefinite integral to $f(x)$ by use of the composite function $f(g(u))$ like this:

$$\int f(x)\,\mathrm{d}x = \left(\int f(g(u))\cdot g'(u)\,\mathrm{d}u\right)_{u=g^{\circ -1}(x)} \quad , \tag{23-19}$$

where $g^{\circ -1}(x)$ denotes the inverse function to $g(u)$.

|||| Proof

Again we shall show that the two sides in (23-19) have the same derivative for all x:

$$\frac{d}{dx}\left(\int f(x)\,\mathrm{d}x\right) = f(x) \quad \text{and}$$

$$\begin{aligned} &\frac{d}{dx}\left(\int f(g(u))\cdot g'(u)\,\mathrm{d}u\right)_{u=g^{\circ -1}(x)} \\ &\qquad = \frac{d}{du}\left(\int f(g(u))\cdot g'(u)\,\mathrm{d}u\right)_{u=g^{\circ -1}(x)} \cdot \frac{d}{dx}\left(g^{\circ -1}(x)\right) \\ &\qquad = \left(f(g(u))\cdot g'(u)\right)_{u=g^{\circ -1}(x)} \cdot \frac{1}{g'(u)} \\ &\qquad = f(g(u)) \\ &\qquad = f(x) \quad , \end{aligned} \tag{23-20}$$

where we along the way have used the rule of differentiation of composite functions (the chain rule) and the rule about differentiation of inverse functions, see eNote 3.

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|||| Example 23.11 Substitution

We will determine an indefinite integral to the function

$$f(x) = \frac{\sqrt{x}}{1+x} \quad , \quad x \in \,]0,\infty[\tag{23-21}$$

so:

$$\int f(x)\,\mathrm{d}x = \int \frac{\sqrt{x}}{1+x}\,\mathrm{d}x. \tag{23-22}$$

If we substitute by the function $g(u) = u^2$ for $u \in]0,\infty[$ we get the square root 'removed' and thus we have the ingredients to use in the rule of substitution:

$$\begin{aligned} f(g(u)) &= \frac{u}{1+u^2} \\ g'(u) &= 2\cdot u \quad . \end{aligned} \tag{23-23}$$

We substitute and get:

$$\begin{aligned} \int f(x)\,\mathrm{d}x &= \int \frac{\sqrt{x}}{1+x}\,\mathrm{d}x \\ &= \left(\int \frac{u}{1+u^2}\cdot 2\cdot u\,\mathrm{d}u\right)_{u=g^{\circ -1}(x)} \\ &= \left(\int 2\cdot \frac{u^2+1-1}{1+u^2}\,\mathrm{d}u\right)_{u=\sqrt{x}} \\ &= 2\cdot\left(\int \left(1-\frac{1}{1+u^2}\right)\,\mathrm{d}u\right)_{u=\sqrt{x}} \\ &= 2\cdot(u-\arctan(u))_{u=\sqrt{x}} \\ &= 2\cdot\left(\sqrt{x}-\arctan(\sqrt{x})\right) \quad , \quad x\in\,]0,\infty[\quad . \end{aligned} \tag{23-24}$$

|||| Exercise 23.12

Determine all indefinite integrals to the function $f(x) = x\cdot \mathrm{e}^{x^2}$.

23.5 The Fundamental Theorem

The following fundamental theorem establishes the hinted-at relation between the determination of the indefinite integral and the Riemann integrals, and it is this theorem we will use to a great extent in the eNotes about integration in more variables.

|||| Theorem 23.13 The Fundamental Theorem about Calculus

Let $f(u)$ denote a continuous function on the interval $[a, b]$. Assume that $F(u)$ is an (arbitrary) indefinite integral for $f(u)$. Then the following applies:

$$\lim_{n\to\infty} \mathrm{I}(f, n, [a, b]) = \int_a^b f(u)\,\mathrm{d}u = [\,F(u)\,]_{u=a}^{u=b} = F(b) - F(a) \quad . \tag{23-25}$$

We will not here prove the fundamental theorem – only note that one can expect that the indefinite integral to $f(u)$ can give the limit value of the integral sum as in Theorem 23.13: If we for given n let

$$\widehat{F}(x_0) = \mathrm{I}(f, n, [a, x_0]) \tag{23-26}$$

and if we put $x_0 = u_{n+1}$ and $x = x_0 + \delta_u$, then

$$\begin{aligned} \widehat{F}(x) &= \mathrm{I}(f, n+1, [a, x]) \\ &= \mathrm{I}(f, n, [a, x_0]) + f(u_{n+1}) \cdot \delta_u \\ &= \widehat{F}(x_0) + f(x_0) \cdot (x - x_0) \quad , \end{aligned} \tag{23-27}$$

If we initially allow ourselves to neglect the n-dependencies and the necessary involved limit values for $n \to \infty$, then it 'follows' from (23-27) that $\widehat{F}(x)$ is differentiable in x_0 with the 'derivative' $f(x_0)$, see the definition on differentiability of a function of one variable in eNote 3. In this meaning we must expect that $\widehat{F}(x)$ is actually an 'indefinite integral' to $f(x)$. This crude consideration naturally is only an indication in the direction of a proper proof of the fundamental theorem.

Hereafter the Riemann integrals can be computed in two ways, partly as the limit value of integral sums and partly as a difference between the values of a indefinite integral at the end points of the interval. Normally it is the last method that is the smartest method to use if the relevant indefinite integrals can indeed be found or determined.

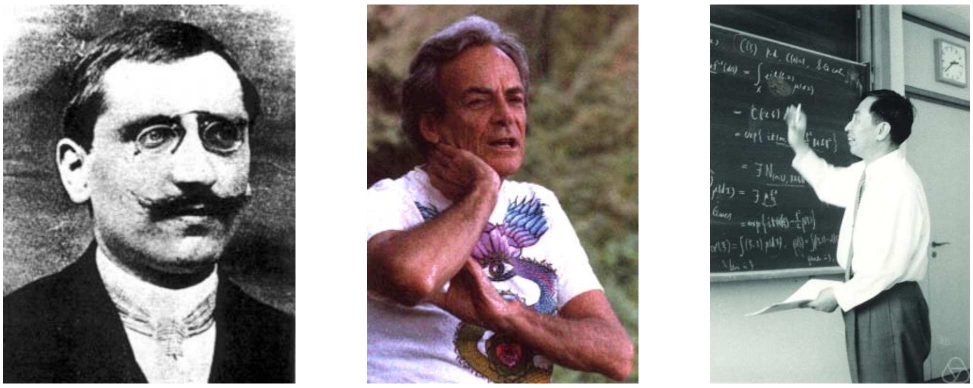

Figure 23.5: Henri Léon Lebesgue, Richard P. Feynman, and Kiyosi Ito.

Riemann integration has later been developed extensively for the benefit and use in numerous applications. Lebesgue's integral and measure theory from 1901 makes it e.g. possible to extend the concept of length, area and volume to also give a consistent meaning to e.g. fractal geometric objects. Ito-calculus, Santalo's integral geometri, and Feynman's path integrals are among the newest developments with exciting applications in such widely different disciplines as financial mathematics and quantum field theory.

Here we illustrate the fundamental theorem and the two ways of computation in the form of an exercise and a pair of examples:

|||| Exercise 23.14

Let $f(u) = 1 + u + u^2,\ u \in [-1, 1]$. Then

$$\begin{aligned} \mathrm{I}(f, n, [-1,1]) &= \sum_{i=1}^{i=n} \left(1 + \left(-1 + (i-1)\frac{2}{n} \right) + \left(-1 + (i-1)\frac{2}{n} \right)^2 \right) \frac{2}{n} \\ &= \sum_{i=1}^{i=n} \left(\frac{8 + 4\,n + 2\,n^2 - 16\,i - 4\,i\,n + 8\,i^2}{n^3} \right) \quad . \end{aligned} \tag{23-28}$$

Use the table on the partial sums below in order to calculate the sum in (23-28) as a function of n and then for the purpose of proving

$$\lim_{n\to\infty} \mathrm{I}(f, n, [-1,1]) = \int_{-1}^{1} \left(1 + u + u^2 \right) \mathrm{d}u = F(1) - F(0) = \frac{8}{3} \quad , \tag{23-29}$$

since an indefinite integral of $f(u)$ in this case is $F(u) = u + \frac{1}{2}u^2 + \frac{1}{3}u^3$ such that $F(1) = \frac{8}{3}$ og $F(0) = 0$.

The following is valid for the size of the partial sums that (apart from factors that can be put outside the $\sum -$sign) appear in the last expression for $\mathrm{I}(f, n, [-1,1])$ in Equation (23-28):

$$\begin{aligned} \sum_{i=1}^{i=n}\left(\frac{1}{n^3}\right) &= \frac{1}{n^3}\sum_{i=1}^{i=n} 1 = \frac{1}{n^2} \quad , \\ \sum_{i=1}^{i=n}\left(\frac{n}{n^3}\right) = \sum_{i=1}^{i=n}\left(\frac{1}{n^2}\right) &= \frac{1}{n^2}\sum_{i=1}^{i=n} 1 = \frac{1}{n} \quad , \\ \sum_{i=1}^{i=n}\left(\frac{n^2}{n^3}\right) = \sum_{i=1}^{i=n}\left(\frac{1}{n}\right) &= \frac{1}{n}\sum_{i=1}^{i=n} 1 = 1 \quad , \end{aligned} \tag{23-30}$$

$$\begin{aligned} \sum_{i=1}^{i=n}\left(\frac{i}{n^3}\right) &= \frac{1}{n^3}\sum_{i=1}^{i=n} i = \frac{n+1}{2n^2} \quad , \\ \sum_{i=1}^{i=n}\left(\frac{i\,n}{n^3}\right) = \sum_{i=1}^{i=n}\left(\frac{i}{n^2}\right) &= \frac{1}{n^2}\sum_{i=1}^{i=n} i = \frac{n+1}{2n} \quad , \\ \sum_{i=1}^{i=n}\left(\frac{i\,n}{n^2}\right) = \sum_{i=1}^{i=n}\left(\frac{i}{n}\right) &= \frac{1}{n}\sum_{i=1}^{i=n} i = \frac{n+1}{2} \quad , \end{aligned} \tag{23-31}$$

$$\begin{aligned} \sum_{i=1}^{i=n}\left(\frac{i^2}{n^3}\right) &= \frac{1}{n^3}\sum_{i=1}^{i=n} i^2 = \frac{2n^2+3n+1}{6n^2} \\ \sum_{i=1}^{i=n}\left(\frac{i^2}{n^2}\right) &= \frac{1}{n^2}\sum_{i=1}^{i=n} i^2 = \frac{2n^2+3n+1}{6n} \\ \sum_{i=1}^{i=n}\left(\frac{i^2}{n}\right) &= \frac{1}{n}\sum_{i=1}^{i=n} i^2 = \frac{2n^2+3n+1}{6} \end{aligned} \tag{23-32}$$

Above we have used that

$$\begin{aligned} \sum_{i=1}^{i=n} i &= \frac{1}{2}n^2 + \frac{1}{2}n \quad \text{and} \\ \sum_{i=1}^{i=n} i^2 &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \quad . \end{aligned} \tag{23-33}$$

Prove the two equations in (23-33) – possibly by ***mathematical induction***, see Wikipedia.

If we wish to determine the Riemann integrals of functions that are not as simple as polynomials then it is often simpler to use a computation with indefinite integrals instead of evaluating the limit values of the integral sums.

|||| Example 23.15 Sine as an Integral Sum

We consider a well-known function $f(u) = \cos(u)$ with the indefinite integral:

$$\int f(u)\,\mathrm{d}u = \int \cos(u)\,\mathrm{d}u = \sin(u) \quad , \tag{23-34}$$

such that

$$\int_0^{u_0} \cos(u)\,\mathrm{d}u = \sin(u_0) \tag{23-35}$$

for every $u_0 \in \mathbb{R}$. A 'backward reading' of the fundamental theorem therefore gives $\sin(u_0)$ expressed by values of $\cos(u)$ for $u \in [0, u_0]$:

$$\sin(u_0) = \lim_{n\to\infty} \left(\frac{u_0}{n}\right) \cdot \sum_{i=1}^{i=n} \cos\left(\frac{(i-1)\cdot u_0}{n}\right) \tag{23-36}$$

|||| Example 23.16 Fresnel C og Fresnel S

We will estimate the following sum for $n \to \infty$:

$$S(n) = \frac{1}{n} \sum_{k=1}^{k=n} \cos\left(\frac{(k-1)^2}{n^2}\right) \quad . \tag{23-37}$$

The sum has the form of an integral sum

$$S(n) = \mathrm{I}(f, n, [0,1]) = \sum_{i=1}^{i=n} f(u_i)\cdot\delta_u \tag{23-38}$$

viz. for the function $f(u) = \cos(u^2)$ with $\delta_u = 1/n$ over the interval $[0,1]$. Therefore according to the fundamental theorem we have:

$$S(n) \to \int_0^1 \cos(u^2)\,\mathrm{d}u \quad \text{for} \quad n \to \infty \quad . \tag{23-39}$$

The indefinite integral $F(u)$ to $f(u) = \cos(u^2)$ that has the value $F(0) = 0$ at $u = 0$, can be expressed in terms of a function called $\mathrm{FresnelC}(u)$:

$$F(u) = \int \cos(u^2)\,\mathrm{d}u = \sqrt{\frac{\pi}{2}} \cdot \mathrm{FresnelC}\left(u \cdot \sqrt{\frac{2}{\pi}}\right) \quad , \tag{23-40}$$

such that:

$$\frac{1}{n}\sum_{k=1}^{k=n}\cos\left(\frac{(k-1)^2}{n^2}\right) \rightarrow \sqrt{\frac{\pi}{2}}\cdot \text{FresnelC}\left(\sqrt{\frac{2}{\pi}}\right) = 0.905 \quad \text{for} \quad n\rightarrow\infty \quad . \tag{23-41}$$

Similarly a function denoted FresnelS is the indefinite integral of $\sin(u^2)$ (with the value 0 at $u = 0$), such that

$$\frac{1}{n}\sum_{k=1}^{k=n}\sin\left(\frac{(k-1)^2}{n^2}\right) \rightarrow \sqrt{\frac{\pi}{2}}\cdot \text{FresnelS}\left(\sqrt{\frac{2}{\pi}}\right) = 0.310 \quad \text{for} \quad n\rightarrow\infty \quad . \tag{23-42}$$

23.6 Double Sums and Double Integrals

For functions of two variables we have, corresponding to Theorem 23.5, the following:

|||| Theorem 23.17

Let $f(u,v)$ denote a continuous real function on a rectangle $[a,b] \times [c,d]$ in the (u,v) plane. The interval $[a,b]$ is partitioned equidistantly into n subintervals and the interval $[c,d]$ is partitioned equidistantly into m subintervals. Then every u-subinterval has the length $\delta_u = (b-a)/n$ and every v-subinterval has the length $\delta_v = (d-c)/m$. Similarly we get the coordinates of the division points in the (u,v) parameter region (which equals the rectangle $[a,b] \times [c,d]$ in $\mathbb{R}^2$):

$$\begin{aligned}
(u_1,v_1) &= (a,c), \\
(u_1,v_j) &= (a, c+(j-1)\delta_v), \\
(u_i,v_1) &= (a+(i-1)\delta_u, c), \\
(u_i,v_j) &= (a+(i-1)\delta_u, c+(j-1)\delta_v), \\
&\;\;.... \\
(u_n,v_m) &= (a+(n-1)\delta_u, c+(m-1)\delta_v) \quad .
\end{aligned} \tag{23-43}$$

Thereby the rectangular parameter region will be similarly subdivided into a total of $n \cdot m$ completely identical sub-rectangles each with the area $\delta_u \cdot \delta_v$. The division points so defined are the lower left corner-points in these sub-rectangles.

Now let $\mathrm{II}(f,n,m,[a,b]\times[c,d])$ denote the following double-sum, where every addend is a weighted area; the weights are the respective values of $f(u,v)$ in each of the above defined lower left corner-points in the rectangles partitioning the parameter region, and every sub-area is the constant area of each sub-rectangle $\delta_u \cdot \delta_v$:

$$\begin{aligned}
&\mathrm{II}(f,n,m,[a,b]\times[c,d]) \\
&= \sum_{j=1}^{j=m}\left(\sum_{i=1}^{i=n} f\left(a+(i-1)\frac{b-a}{n},\;\; c+(j-1)\frac{d-c}{m}\right)\cdot\left(\frac{b-a}{n}\right)\cdot\left(\frac{d-c}{m}\right)\right) \\
&= \sum_{j=1}^{j=m}\left(\sum_{i=1}^{i=n} f\left(a+(i-1)\delta_u,\;\; c+(j-1)\delta_v\right)\cdot\delta_u\cdot\delta_v\right) \\
&= \sum_{j=1}^{j=m}\left(\sum_{i=1}^{i=n} f\left(u_i,\;\; v_j\right)\cdot\delta_u\right)\cdot\delta_v \quad .
\end{aligned} \tag{23-44}$$

Then $\mathrm{II}(f,n,m,[a,b]\times[c,d])$ has a limit value for $n \to \infty$, $m \to \infty$, and it is denoted by:

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\mathrm{II}(f,n,m,[a,b]\times[c,d])\right) = \int_c^d\left(\int_a^b f(u,v)\,\mathrm{d}u\right)\mathrm{d}v \quad . \tag{23-45}$$

Sums of the type $\mathrm{II}(f, n, m, [a,b] \times [c,d])$ we will call *double integral sums* and the limit value $\int_c^d \left(\int_a^b f(u,v)\,\mathrm{d}u \right) \mathrm{d}v$ is again similarly called the Riemann integral of $f(u,v)$ over $[a,b] \times [c,d]$.

Note that now we allow ourselves to write the following – when the Riemann integral of $f(u,v)$ over $[a,b] \times [c,d]$ exists:

$$\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} f\left(u_i,\ v_j\right) \cdot \delta_u \right) \cdot \delta_v = \int_c^d \left(\int_a^b f(u,v)\,\mathrm{d}u \right) \mathrm{d}v \quad , \tag{23-46}$$

so that, to put it simply, the $\sum$ sign becomes the $\int$ sign in the limit and correspondingly δ_u and δ_v become $\mathrm{d}u$ and $\mathrm{d}v$, respectively.

The Riemann integral of $f(u,v)$ over $[a,b] \times [c,d]$

$$\int_c^d \left(\int_a^b f(u,v)\,\mathrm{d}u \right) \mathrm{d}v \tag{23-47}$$

in Theorem 23.17 is *only* a *symbol*, a designation, for the limit value of the double integral sum for $f(u,v)$. The essential statement of the theorem is that the limit value *exsists* when $f(u,v)$ is continuous in the rectangular region. But the reduction of the double sum that takes part in (23-44) and the notation in itself more than hints about the fact that the Riemann integral actually can *be computed* by use of an indefinite integral for suitable functions of one variable. This is of course the content of the fundamental theorem for the double integral sums.

23.7 The Fundamental Theorem for Double Integral Sums

The Riemann double integrals are calculated by the use of the concept of indefinite integrals like this:

|||| **Theorem 23.18**

Let $f(u,v)$ denote a continuous function on $[a,b] \times [c,d]$. Assume that $F(u,v)$ is an (arbitrary) indefinite integral for $f(u,v)$ (considered as a function of the one variable u) for an arbitrary given $v \in [c,d]$. Furthermore let $G(a,v)$ be an arbitrary indefinite integral to $F(a,v)$ and let $G(b,v)$ be an arbitrary indefinite integral to $F(b,v)$, then the following applies:

$$\begin{aligned} \lim_{n\to\infty}\left(\lim_{m\to\infty} \mathrm{II}(f,n,m,[a,b]\times[c,d])\right) &= \int_c^d \left(\int_a^b f(u,v)\,\mathrm{d}u\right)\mathrm{d}v \\ &= \int_c^d \left[\,F(u,v)\,\right]_{u=a}^{u=b}\mathrm{d}v \\ &= \int_c^d (F(b,v)-F(a,v))\,\mathrm{d}v \\ &= \left[\,G(b,v)\,\right]_{v=c}^{v=d} - \left[\,G(a,v)\,\right]_{v=c}^{v=d} \\ &= (G(b,d)-G(b,c)) \\ &\quad -(G(a,d)-G(a,c)) \quad . \end{aligned} \tag{23-48}$$

We illustrate computation of Riemann double integrals by some examples – primarily to show that in concrete cases the computations can be simpler than hinted at in (23-48):

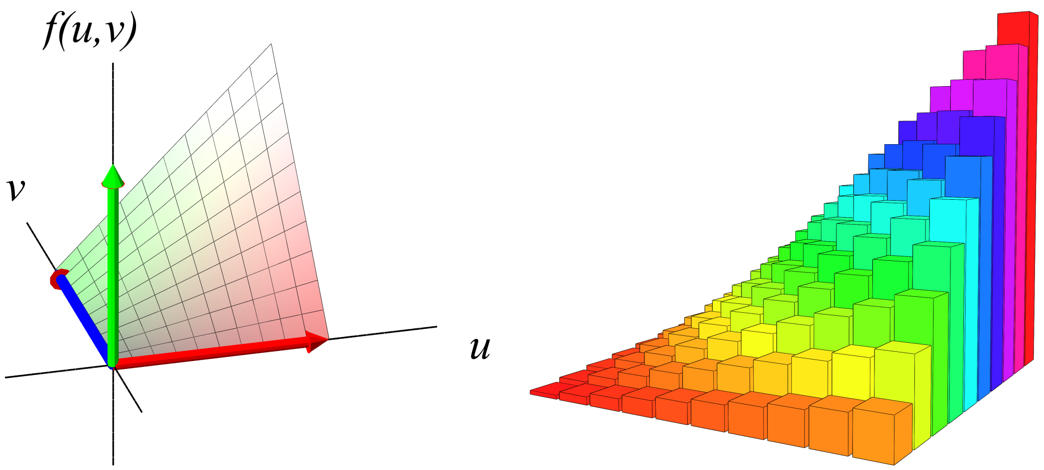

Figure 23.6: Volume representation of the integral sum $\mathrm{II}(f,10,10,[0,1]\times[0,1])$ for the function $f(u,v) = u\,v$.

|||| Example 23.19 A Riemann Double Integral

Let $f(u,v) = u \cdot v^2$ for $u \in [0,1]$ and $v \in [-1,1]$. Then

$$\begin{aligned} \int_{-1}^{1} \left(\int_{0}^{1} v^2 \, u \mathrm{d}u \right) \mathrm{d}v &= \frac{1}{2} \int_{-1}^{1} [\, v^2 \, u^2 \,]_{u=0}^{u=1} \, \mathrm{d}v \\ &= \frac{1}{2} \int_{-1}^{1} v^2 \, \mathrm{d}v \\ &= \frac{1}{6} [\, v^3 \,]_{v=-1}^{v=1} \\ &= \frac{1}{3} \quad . \end{aligned} \tag{23-49}$$

Note that the computation of the double integral is done ’from within’ – the innermost integral is an integral over the u-interval, here $[0,1]$, and is computed first, i.e. with *maintained* v. An indefinite integral to $v^2 \cdot u$ with constant v is $v^2 \cdot u^2/2$ such that:

$$\int_{0}^{1} v^2 \cdot u \, \mathrm{d}u = \frac{1}{2} \cdot [\, v^2 \, u^2 \,]_{u=0}^{u=1} = \frac{1}{2} \cdot v^2 \quad . \tag{23-50}$$

In the computation we could alternatively have used that $F_v(u) = v^2 u^2/2$ and therefore that $G_a(v) = v^3 a^2/6$, $G_b(v) = v^3 b^2/6$ and substituted directly into the last expression in (23-48).

|||| Example 23.20 A Double Integral with Symmetry

Let $f(u,v) = u \cdot v$ for $u \in [-1,1]$ and $v \in [-1,1]$. Then

$$\int_{-1}^{1} \left(\int_{-1}^{1} v \, u \, \mathrm{d}u \right) \mathrm{d}v = \frac{1}{2} \int_{-1}^{1} [\, v \, u^2 \,]_{u=-1}^{u=1} \, \mathrm{d}v = 0 \quad , \tag{23-51}$$

while

$$\begin{aligned} \int_{0}^{1} \left(\int_{0}^{1} v \, u \mathrm{d}u \right) \mathrm{d}v &= \frac{1}{2} \int_{0}^{1} [\, v \, u^2 \,]_{u=0}^{u=1} \, \mathrm{d}v \\ &= \frac{1}{2} \int_{0}^{1} v \, \mathrm{d}v \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot [\, v^2 \,]_{v=0}^{v=1} \\ &= \frac{1}{4} \quad . \end{aligned} \tag{23-52}$$

|||| Example 23.21 A Double Integral Sum

The volume representation of the integral sum $\mathrm{II}(f, 10, 10, [0,1] \times [0,1])$ for the function $f(u,v) = u\,v$ is shown in Figure 23.6. The 100 addends in the sum are to the right represented by columns with the same square sections and with heights that are given by the respective values of the function $f(u,v) = u\,v$ at the division points of the (u,v) square. By this we obtain an approximation to the volume of the region in (x,y,z) space that is bounded by the (x,y) plane and the graph surface for the function $f(x,y) = xy$ over the square $(x,y) \in [0,1] \times [0,1]$, as shown to the left(-hand side). The exact volume is $\frac{1}{4}$.

23.8 Triple Sums and Triple Integrals

|||| **Theorem 23.22**

Let $f(u,v,w)$ denote a continuous real function on a box-formed parameter region $[a,b] \times [c,d] \times [h,l]$ in (u,v,w) space. The interval $[a,b]$ is equidistantly partitioned into n subintervals, the interval $[c,d]$ is equidistantly partitioned into m subintervals, and the interval $[h,l]$ is equidistantly partitioned into q subintervals. Then every u-subinterval has the length $\delta_u = (b-a)/n$, every v-subinterval has the length $\delta_v = (d-c)/m$ and every w-subinterval has the length $\delta_w = (h-l)/q$. Similarly the coordinates of the division points in the (u,v,w)-parameter region $[a,b] \times [c,d] \times [h,l]$ in $\mathbb{R}^3$:

$$
\begin{aligned}
(u_1, v_1, w_1) &= (a, c, h), \\
&\;\;\ldots. \\
(u_n, v_m, w_q) &= (a + (n-1)\delta_u, c + (m-1)\delta_v, h + (q-1)\delta_w) \quad .
\end{aligned}
\tag{23-53}
$$

Now let $\mathrm{III}(f, n, m, q, [a,b] \times [c,d] \times [h,l])$ denote the following **triple sum**:

$$
\begin{aligned}
&\mathrm{III}(f, n, m, q, [a,b] \times [c,d] \times [h,l]) \\
&= \sum_{k=1}^{k=q} \left(\sum_{j=1}^{j=m} \left(\sum_{i=1}^{i=n} f\left(u_i,\; v_j,\; w_k\right) \delta_u \right) \delta_v \right) \delta_w \quad .
\end{aligned}
\tag{23-54}
$$

Then

$$
\begin{aligned}
&\lim_{n\to\infty} \left(\lim_{m\to\infty} \left(\lim_{q\to\infty} \mathrm{III}(f, n, m, q, [a,b] \times [c,d] \times [h,l]) \right) \right) \\
&= \int_h^l \left(\int_c^d \left(\int_a^b f(u,v,w)\,\mathrm{d}u \right) \mathrm{d}v \right) \mathrm{d}w \quad .
\end{aligned}
\tag{23-55}
$$

Sums of the type $\mathrm{III}(f, n, m, q, [a,b] \times [c,d] \times [h,l])$ we will naturally call *triple integral sums* and the limit value $\int_h^l \left(\int_c^d \left(\int_a^b f(u,v,w)\,\mathrm{d}u \right) \mathrm{d}v \right) \mathrm{d}w$ is called the *Riemann integral* of $f(u,v,w)$ over $[a,b] \times [c,d] \times [h,l]$.

23.9 The Fundamental Theorem for Triple Integral Sums

The Riemann Triple Integrals are computed like this:

|||| Theorem 23.23

Let $f(u, v, w)$ denote a continuous function on $[a, b] \times [c, d] \times [h, l]$.
Suppose that $F(u, v, w)$ is an (arbitrary) indefinite integral for $f(u, v, w)$ (considered to be a function of the one variable u) for arbitrary given $v \in [c, d]$ and $w \in [h, l]$.

- Let $G(a, v, w)$ be an arbitrary indefinite integral to $F(a, v, w)$ (considered to be a function of the one variable v) for arbitrary given $w \in [h, l]$.

- Let $G(b, v, w)$ be an arbitrary indefinite integral to $F(b, v, w)$ (again considered to be a function of the one variable v) for arbitrary given $w \in [h, l]$.

- Finally let $H(a, c, w)$ be an arbitrary indefinite integral $G(a, c, w)$, and similarly $H(b, c, w)$, $H(a, d, w)$, and $H(b, d, w)$ indefinite integrals for $G(b, c, w)$, $G(a, d, w)$, and $G(b, d, w)$.

Then:

$$\begin{aligned}
&\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\lim_{q\to\infty}\mathrm{III}(f, n, m, q, [a, b]\times[c, d]\times[h, l])\right)\right) \\
&= \int_h^l \left(\int_c^d \left(\int_a^b f(u, v, w)\,\mathrm{d}u\right)\mathrm{d}v\right)\mathrm{d}w \qquad (23\text{-}56)\\
&= H(b, d, l) - H(b, d, h) - (H(b, c, l) - H(b, c, h)) \\
&\quad - ((H(a, d, l) - H(a, d, h)) - (H(a, c, l) - H(a, c, h))) \quad .
\end{aligned}$$

We illustrate with a couple of simple computations:

|||| Example 23.24 Triple Integration

Let $f(u, v, w) = u\,v\,\sin(w)$ for $u \in [0, 1]$, $v \in [0, 2]$ and $w \in [0, \pi/2]$. Then

$$\begin{aligned}
\int_0^{\pi/2}\left(\int_0^2\left(\int_0^1 u\,v\,\sin(w)\,\mathrm{d}u\right)\mathrm{d}v\right)\mathrm{d}w &= \int_0^{\pi/2}\left(\int_0^2 v\,\sin(w)\,[\,u^2/2\,]_{u=0}^{u=1}\,\mathrm{d}v\right)\mathrm{d}w \\
&= \frac{1}{2}\int_0^{\pi/2}\left(\int_0^2 v\,\sin(w)\,dv\right)\mathrm{d}w \\
&= \frac{1}{2}\int_0^{\pi/2}\sin(w)\,[\,v^2/2\,]_{v=0}^{v=2}\,\mathrm{d}w \\
&= \int_0^{\pi/2}\sin(w)\,\mathrm{d}w \\
&= [-\cos(w)]_{w=0}^{w=\pi/2} \\
&= 1 \quad .
\end{aligned}$$

|||| Example 23.25 A Triple Integral with Symmetry

Let $f(u,v,w) = u \cdot v \cdot w$ for $u \in [-1,1]$ and $v \in [-1,1]$, and $w \in [-1,1]$. Then

$$\int_{-1}^{1} \left(\int_{-1}^{1} \left(\int_{-1}^{1} w\, v\, u \, \mathrm{d}u \right) \mathrm{d}v \right) \mathrm{d}w = 0 \quad , \tag{23-57}$$

while

$$\int_{0}^{1} \left(\int_{0}^{1} \left(\int_{0}^{1} w\, v\, u \, \mathrm{d}u \right) \mathrm{d}v \right) \mathrm{d}w = \frac{1}{8} \quad , \tag{23-58}$$

|||| Example 23.26 The Volume of the Unit Sphere

As shown in the eNote about integration over spatial regions the volume of the solid unit sphere is computed by the following triple Riemann integral (that will be motivated in that eNote). Thus Archimedes' result is verified:

$$\begin{aligned} \mathrm{Vol}(\text{unitsphere}) &= \int_{0}^{1} \left(\int_{-\pi}^{\pi} \left(\int_{0}^{\pi} w^2 \sin(u) \, \mathrm{d}u \right) \mathrm{d}v \right) \mathrm{d}w \\ &= \int_{0}^{1} \left(\int_{-\pi}^{\pi} w^2 \left[-\cos(u) \right]_{u=0}^{u=\pi} \, \mathrm{d}v \right) \mathrm{d}w \\ &= 2 \int_{0}^{1} \left(\int_{-\pi}^{\pi} w^2 \, \mathrm{d}v \right) \mathrm{d}w \\ &= 2 \int_{0}^{1} w^2 \left[v \right]_{v=-\pi}^{v=\pi} \, \mathrm{d}w \\ &= 4\pi \int_{0}^{1} w^2 \, \mathrm{d}w \\ &= 4\pi \left[w^3/3 \right]_{w=0}^{w=1} \\ &= \frac{4\pi}{3} \quad . \end{aligned} \tag{23-59}$$

23.10 Summary

In this eNote we have considered the basis for the methods and results that are necessary tools when finding lengths, areas, volumes, centres of mass (or mass mid-points), moments of inertia etc. of curves, and regions in the 2D plane and in 3D space, respectively.

- For every given continuous function $f(u)$ of one variable u on an interval $[a, b]$ we state integral sums $\mathrm{I}(f, n, [a, b])$ that in the limit for $n \rightarrow \infty$ define the Riemann integral of the function over the interval. These Riemann integrals can then be computed (via the fundamental theorem) by the use of an indefinite integral $F(u)$ for $f(u)$ like this:

$$\begin{aligned} \mathrm{I}(f, n, [a, b]) &= \sum_{i=1}^{i=n} f(u_i)\ \delta_u \\ \sum_{i=1}^{i=n} f(u_i)\ \delta_u &\rightarrow \int_a^b f(u)\,\mathrm{d}u \quad \text{for} \quad \mathrm{n} \rightarrow \infty \\ \int_a^b f(u)\,\mathrm{d}u &= [F(u)]_{u=a}^{u=b} = F(b) - F(a) \quad , \end{aligned} \tag{23-60}$$

  where

$$u_i = a + (i-1) \cdot \delta_u \quad \text{og} \quad \delta_u = \frac{b-a)}{n} \quad . \tag{23-61}$$

- Similar limit values for sums for continuous functions of two and three variables are stated and exemplified.