eNote 22

Quadratic Equations with Two and Three Variables

*In this eNote we will again consider the quadratic polynomials with two and three variables that are also treated and investigated by means of different techniques in eNote 20 and 21 and 19, respectively. In the introductory treatment of functions of two variables we defined the level sets $\mathcal{K}_c(f)$ of functions $f(x,y)$ of two variables, see eNote 19. In this eNote we shall see that for quadratic polynomials with two variables the level sets are typically well-known curves in the (x,y) plane as e.g. ellipses and hyperbolas and it is the purpose of this eNote to show types of level curves that appear for given equations. To do so we will extensively use the method of reduction developed in eNote 21. It works for quadratic polynomials of both two and three (and more) variables and as we shall see, the corresponding level curves and surface can be identified from a short namelist. The level curves for quadratic polynomials with two variables and the level surfaces for quadratic polynomials with three variables are classically known under the names **conic sections** and **quadratic surfaces**.*

Updated: 31.1.2023, shsp.

22.1 Quadratic Equations with Two Variables

From eNote 18 we know from inspection of the level curves shown that ellipses and hyperbolas or ellipsis-like and hyperbola-like curves are typically level curves of functions of two variables – in particular around stationary points. This is no coincidence; quadratic polynomials have exactly such level curves. And suitably chosen quadratic polynomials are at the same time good approximations to given smooth functions of two variables.

In the following we will through some examples go through the standard method for reduction of the equations that appear when we find those points in $\mathbb{R}^2$ for which a given quadratic polynomial is 0, that is, exactly those points that constitute the level set $\mathcal{K}_0(f)$ of $f(x,y)$.

|||| Example 22.1 Ellipsis

For a quadratic polynomial $f(x,y)$ we determine the following ingredients for the reduction of polynomials in exactly the same way as presented in eNote 21. In particular we again need the positive orthogonal substitution matrix $\mathbf{Q}$ that diagonalizes the matrix $\frac{1}{2}\cdot\mathbf{H}f(x,y)$ in order to obtain an expression of $f(x,y)$ that does not include product terms – now in the new coordinates $\widetilde{x}$ and $\widetilde{y}$. The result of the reduction is $\widetilde{f}(\widetilde{x},\widetilde{y})$ as shown below:

$$f(x,y) = 2\cdot x^2 + 2\cdot y^2 + 2\cdot x\cdot y - 8\cdot x - 10\cdot y + 13 \quad .$$

$$\nabla f(x,y) = (4\cdot x + 2\cdot y - 8\,,\, 2\cdot x + 4\cdot y - 10) \quad .$$

$$\frac{1}{2}\cdot\mathbf{H}f(x,y) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad .$$

$$\mathbf{\Lambda} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{22-1}$$

$$\mathbf{Q} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} \quad .$$

$$\begin{aligned} \widetilde{f}(\widetilde{x},\widetilde{y}) &= 3\cdot\widetilde{x}^2 + \widetilde{y}^2 - 9\cdot\sqrt{2}\cdot\widetilde{x} - \sqrt{2}\cdot\widetilde{y} + 13 \\ &= 3\cdot\left(\widetilde{x} - \frac{3}{2}\cdot\sqrt{2}\right)^2 + \left(\widetilde{y} - \frac{1}{2}\cdot\sqrt{2}\right)^2 - 1 \quad . \end{aligned}$$

The last equation in the above computation of the reduced quadratic polynomial $\widetilde{f}(\widetilde{x},\widetilde{y})$ appears through ***completing the square***. This can be done first for the $\widetilde{x}$-terms and then for the $\widetilde{y}$-terms. For the $\widetilde{x}$-terms it goes like this:

$$\begin{aligned} 3\cdot\widetilde{x}^2 - 9\cdot\sqrt{2}\cdot\widetilde{x} &= 3\cdot(\widetilde{x}^2 - 3\cdot\sqrt{2}\cdot\widetilde{x}) \\ &= 3\cdot\left(\left(\widetilde{x} - \frac{3}{2}\sqrt{2}\right)^2 - \frac{9}{2}\right) \quad . \end{aligned} \tag{22-2}$$

The quadratic equation that gives the level curve $\mathcal{K}_0(f)$ is now determined by either of the following equivalent equations:

$$f(x,y) = 0 = \widetilde{f}(\widetilde{x},\widetilde{y})$$

$$\left(\frac{\widetilde{x}-\frac{3}{2}\cdot\sqrt{2}}{\frac{1}{\sqrt{3}}}\right)^2 + \left(\widetilde{y}-\frac{1}{2}\cdot\sqrt{2}\right)^2 = 1 \quad . \tag{22-3}$$

The last equation describes an *ellipsis* with its centre at ${}_{\mathrm{e}}\mathbf{C} = (x_0, y_0) = (1,2)$ (with respect to the old coordinates) corresponding to ${}_{\mathrm{v}}\mathbf{C} = (\widetilde{x}_0,\widetilde{y}_0) = (\frac{3}{2}\cdot\sqrt{2}\,,\,\frac{1}{2}\cdot\sqrt{2})$ (with respect to the new coordinates) and the semi-axes $\frac{1}{\sqrt{3}}$ and 1 , see Figures 22.1 and 22.2. The new coordinate system appears by a rotation of the old coordinate system with rotation angle $\phi = \pi/4$.

The function $f(x,y)$ has a stationary point at the centre of the ellipsis, where the function value is -1. The Hessian matrix is positive definite and therefore the stationary point is a proper local minimum point. Obviously we are talking about a global minimum point.

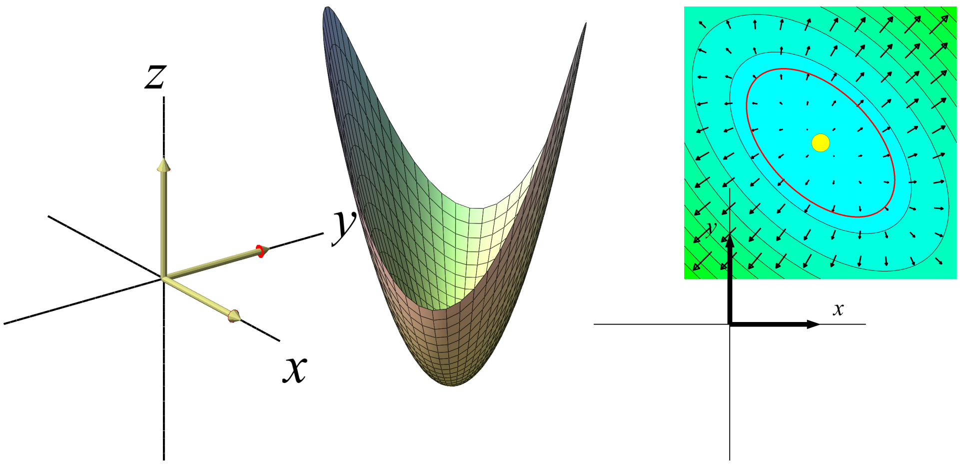
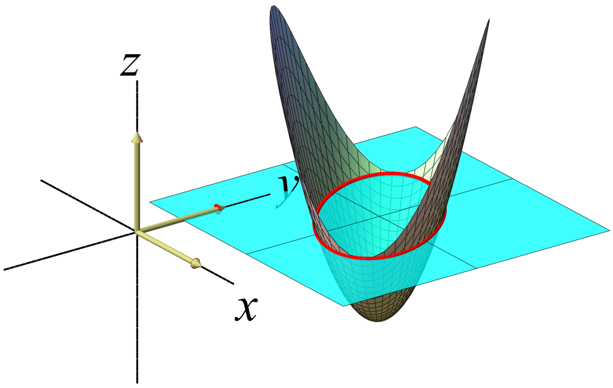


Figure 22.1: The graph of the function $f(x,y) = 2\cdot x^2 + 2\cdot y^2 + 2\cdot x\cdot y - 8\cdot x - 10\cdot y + 13$ together with level curves and the gradient vector field of the function. Note in particular the level curve corresponding to level 0, which is the ellipsis we analyse in Example 22.1.



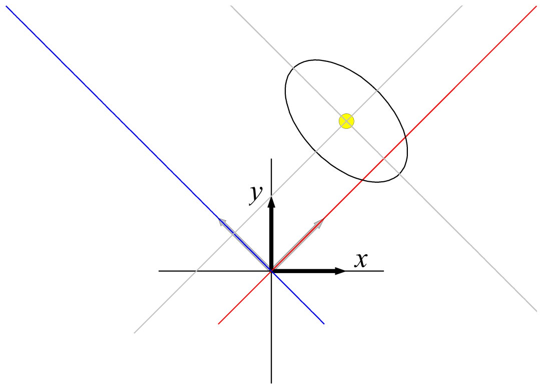


Figure 22.2: The graph of the function with the elliptical level curve at level 0 of the function in Example 22.1.

|||| Example 22.2 Ellipsis

$$f(x,y) = 29 \cdot x^2 + 36 \cdot y^2 + 24 \cdot x \cdot y - 152 \cdot x - 336 \cdot y + 620 \quad .$$

$$\nabla f(x,y) = (58 \cdot x + 24 \cdot y - 152\,,\, 72 \cdot y + 24 \cdot x - 336) \quad .$$

$$\frac{1}{2} \cdot \mathbf{H} f(x,y) = \begin{bmatrix} 29 & 12 \\ 12 & 36 \end{bmatrix} \quad .$$

$$\mathbf{\Lambda} = \begin{bmatrix} 45 & 0 \\ 0 & 20 \end{bmatrix} \quad . \tag{22-4}$$

$$\mathbf{Q} = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \quad .$$

$$\begin{aligned} \widetilde{f}(\widetilde{x},\widetilde{y}) &= 45 \cdot \widetilde{x}^2 + 20 \cdot \widetilde{y}^2 - 360 \cdot \widetilde{x} - 80 \cdot \widetilde{y} + 620 \\ &= 45 \cdot (\widetilde{x} - 4)^2 + 20 \cdot (\widetilde{y} - 2)^2 - 180 \quad . \end{aligned}$$

The quadratic equation that gives the level curve $\mathcal{K}_0(f)$ is therefore given by any of the following equivalent equations:

$$\begin{gathered} f(x,y) = 0 = \widetilde{f}(\widetilde{x},\widetilde{y}) \\ \left(\frac{\widetilde{x}-4}{2}\right)^2 + \left(\frac{\widetilde{y}-2}{3}\right)^2 = 1 \quad . \end{gathered} \tag{22-5}$$

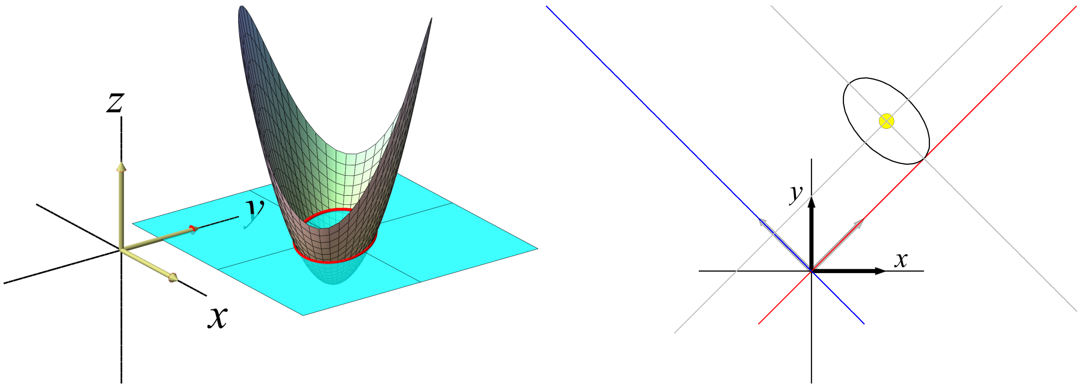


Figure 22.3: The graph of the function that is only lifted $1/2$ as compared to values of the function in Example 22.1. The level curve $\mathcal{K}_0(f)$ at level 0 is correspondingly *smaller* with correspondingly smaller semi-axes, but its orientation with respect to the axis is the same.

This is the equation describing an *ellipsis* with its centre at ${}_{\mathrm{e}}\mathbf{C} = (x_0, y_0) = (4/5, 22/5)$ (with respect to the old coordinates) corresponding to ${}_{\mathrm{v}}\mathbf{C} = (\tilde{x}_0, \tilde{y}_0) = (4, 2)$ (with respect to the new coordinates) and the semi-axes 2 and 3, see Figure 22.4. The new coordinate system appears by rotation of the old coordinate system with the rotation angle $\phi = \arccos(3/5)$.

The quadratic polynomial $f(x, y)$ has a stationary point at the centre with the value -180. The Hessian matrix is positive definite and therefore the stationary point is a proper local minimum point with the minimum value -180. Again it is obviously a global minimum point.

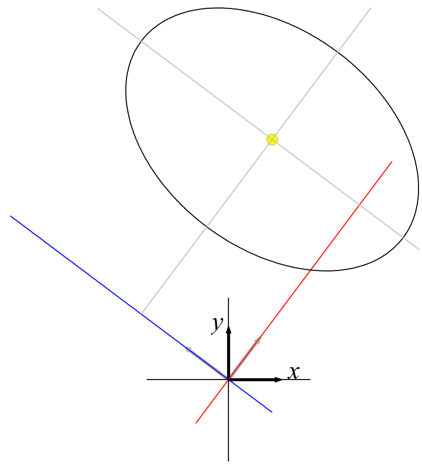


Figure 22.4: The level curve $\mathcal{K}_0(f)$ of the quadratic polynomial $f(x,y) = 29 \cdot x^2 + 36 \cdot y^2 + 24 \cdot x \cdot y - 152 \cdot x - 336 \cdot y + 620$ from Example 22.2.

|||| Example 22.3 Hyperbola

We consider again Example 19.44 from eNote 19 about the quadratic polynomial $f(x,y)$ with the following data:

$$f(x,y) = 11 \cdot x^2 + 4 \cdot y^2 - 24 \cdot x \cdot y - 20 \cdot x + 40 \cdot y - 60 \quad .$$

$$\boldsymbol{\nabla} f(x,y) = (22 \cdot x - 24 \cdot y - 20\,,\, -24 \cdot x + 8 \cdot y + 40) \quad .$$

$$\frac{1}{2} \cdot \mathbf{H} f(x,y) = \begin{bmatrix} 11 & -12 \\ -12 & 4 \end{bmatrix} \quad .$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} 20 & 0 \\ 0 & -5 \end{bmatrix} \quad .$$

$$\mathbf{Q} = \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \quad , \quad \text{where} \quad \varphi = -\arcsin(3/5) \quad .$$

$$\begin{aligned} \widetilde{f}(\widetilde{x},\widetilde{y}) &= 20 \cdot \widetilde{x}^2 - 5 \cdot \widetilde{y}^2 - 40 \cdot \widetilde{x} + 20 \cdot \widetilde{y} - 60 \\ &= 20 \cdot (\widetilde{x} - 1)^2 - 5 \cdot (\widetilde{y} - 2)^2 - 60 \quad . \end{aligned} \tag{22-6}$$

The quadratic equation producing the level curve $\mathcal{K}_0(f)$ is therefore given by any of the following equations:

$$f(x,y) = 0 = \widetilde{f}(\widetilde{x},\widetilde{y})$$
$$\left(\frac{\widetilde{x}-1}{\sqrt{3}}\right)^2 - \left(\frac{\widetilde{y}-2}{2\cdot\sqrt{3}}\right)^2 = 1 \quad . \tag{22-7}$$

The equations describe a *hyperbola* with its centre at ${}_{\mathrm{e}}\mathbf{C} = (x_0, y_0) = (2,1)$ (with respect to the old coordinates) corresponding to ${}_{\mathrm{v}}\mathbf{C} = (\widetilde{x}_0, \widetilde{y}_0) = (1,2)$ (with respect to the new coordinates) and the semi-axes $\sqrt{3}$ and $2\cdot\sqrt{3}$. The new coordinate system appears through a rotation of the old coordinate system with the rotation angle $\phi = -\arcsin(3/5)$.

The function has a stationary point at the centre with the value -60. The Hessian matrix is indefinite and therefore the stationary point is neither a local minimum point nor a local maximum point.

|||| Example 22.4 Hyperbola

Another hyperbola is given by $\mathcal{K}_0(f)$ of the following quadratic polynomial:

$$f(x,y) = -\frac{5}{4}\cdot x^2 + \frac{1}{4}\cdot y^2 + \frac{3}{2}\cdot\sqrt{3}\cdot x\cdot y + 5\cdot x - 3\cdot\sqrt{3}\cdot y - \frac{21}{4} \quad .$$

$$\boldsymbol{\nabla} f(x,y) = (-\frac{5}{2}\cdot x + \frac{3}{2}\cdot\sqrt{3}\cdot y + 5\,,\ \frac{3}{2}\cdot\sqrt{3}\cdot x + \frac{1}{2}\cdot y - 3\cdot\sqrt{3}) \quad .$$

$$\frac{1}{2}\cdot\mathbf{H}f(x,y) = \begin{bmatrix} -5/2 & 3\cdot\sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix} \quad .$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad . \tag{22-8}$$

$$\mathbf{Q} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} \quad .$$

$$\begin{aligned}\widetilde{f}(\widetilde{x},\widetilde{y}) &= \widetilde{x}^2 - 2\cdot\widetilde{y}^2 - 2\cdot\widetilde{x} - 4\cdot\sqrt{3}\cdot\widetilde{y} - \frac{21}{4} \\ &= (\widetilde{x}-1)^2 - 2\cdot(\widetilde{y}+\sqrt{3})^2 - \frac{1}{4} \quad .\end{aligned}$$

The quadratic equation that gives the level curve $\mathcal{K}_0(f)$ is therefore given by any of the following equivalent equations, where the last is found by completing the square:

$$
\begin{gathered}
f(x,y)=0=\widetilde{f}(\widetilde{x},\widetilde{y})=0 \\
\left(\frac{\widetilde{x}-1}{\frac{1}{2}}\right)^2-\left(\frac{\widetilde{y}+\sqrt{3}}{\frac{1}{2\cdot\sqrt{2}}}\right)^2=1 \quad .
\end{gathered}
\tag{22-9}
$$

This is a *hyperbola* with its centre at ${}_{\mathrm{e}}\mathbf{C}=(x_0,y_0)=(2,0)$ (with respect to the old coordinates) corresponding to ${}_{\mathrm{v}}\mathbf{C}=(\widetilde{x}_0,\widetilde{y}_0)=(1,-\sqrt{3})$ (with respect to the new coordinates) and the semi-axes $1/2$ and $1/(2\cdot\sqrt{2})$, see Figure 22.5. The new coordinate system appears through a rotation of the old coordinate system by the rotation angle $\phi=\frac{\pi}{3}$.

The function has a stationary point at the centre with value $-1/4$. The Hessian matrix is indefinite and therefore the stationary point is neither a local minimum point nor a local maximum point, which is also evident from Figure 22.5.

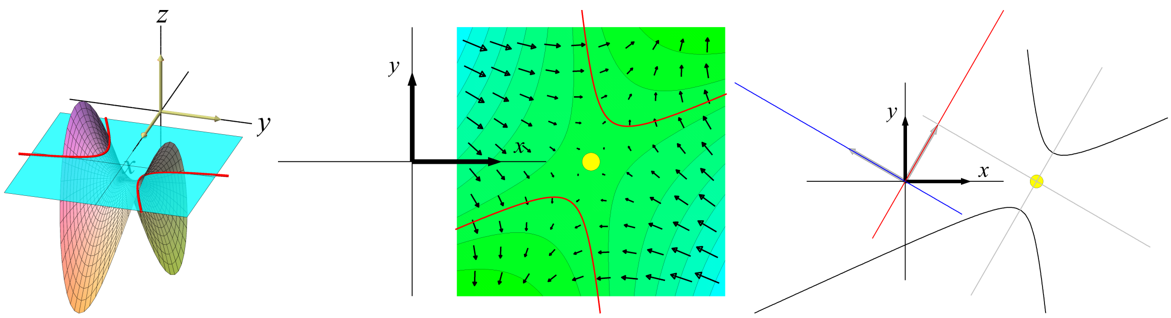


Figure 22.5: The graph of the function from example 22.4 intersected at level 0, the gradient field, level curves and in particular the level curve $\mathcal{K}_0(f)$.

22.2 Quadratic Equations with Three Variables

For quadratic polynomials $f(x,y,z)$ with three variables we can carry through the same analysis as above but now of the level sets in (x,y,z) space, i.e. the level surfaces that appear by setting $f(x,y,z)=0$. We show the method through some examples:

|||| Example 22.5 Ellipsoid

A quadratic polynomial with three variables is investigated:

$$f(x,y,z) = 7 \cdot x^2 + 6 \cdot y^2 + 5 \cdot z^2 - 4 \cdot x \cdot y - 4 \cdot y \cdot z - 2 \cdot x + 20 \cdot y - 10 \cdot z + 15 \quad .$$

$$\nabla f(x,y,z) = (14 \cdot x - 4 \cdot y - 2, -4 \cdot x + 12 \cdot y - 4 \cdot z + 20, -4 \cdot y + 10 \cdot z - 10) \quad .$$

$$\frac{1}{2} \cdot \mathbf{H} f(x,y,z) = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix} \quad .$$

$$\mathbf{\Lambda} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad .$$

$$\mathbf{Q} = \begin{bmatrix} 2/3 & -2/3 & -1/3 \\ -2/3 & -1/3 & -2/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix} \quad .$$

$$\begin{aligned} \widetilde{f}(\widetilde{x},\widetilde{y},\widetilde{z}) &= 9 \cdot \widetilde{x}^2 + 6 \cdot \widetilde{y}^2 + 3 \cdot \widetilde{z}^2 - 18 \cdot \widetilde{x} - 12 \cdot \widetilde{y} - 6 \cdot \widetilde{z} + 15 \\ &= 9 \cdot (\widetilde{x} - 1)^2 + 6 \cdot (\widetilde{y} - 1)^2 + 3 \cdot (\widetilde{z} - 1)^2 - 3 \quad . \end{aligned} \tag{22-10}$$

The quadratic equation that gives the level surface $\mathcal{K}_0(f)$ is therefore given by either of the following equivalent equations:

$$\begin{gathered} f(x,y,z) = 0 = \widetilde{f}(\widetilde{x},\widetilde{y},\widetilde{z}) \\ \left(\frac{\widetilde{x} - 1}{\frac{1}{\sqrt{3}}}\right)^2 + \left(\frac{\widetilde{y} - 1}{\frac{1}{\sqrt{2}}}\right)^2 + (\widetilde{z} - 1)^2 = 1 \quad . \end{gathered} \tag{22-11}$$

This is the equation describing an *ellipsoid* with its centre at ${}_{\mathrm{e}}\mathbf{C} = (x_0, y_0, z_0) = (-1/3, -5/3, 173)$ (with respect to the old coordinates) corresponding to ${}_{\mathrm{v}}\mathbf{C} = (\widetilde{x}_0, \widetilde{y}_0, \widetilde{z}_0) = (1,1,1)$ (with respect to the new coordinates) and the semi-axes $\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{2}}$, and 1, see Figure 22.6 and the Name Table 22.3. The new coordinate system appears through a rotation of the old coordinate system by the positive orthogonal substitution $\mathbf{Q}$.

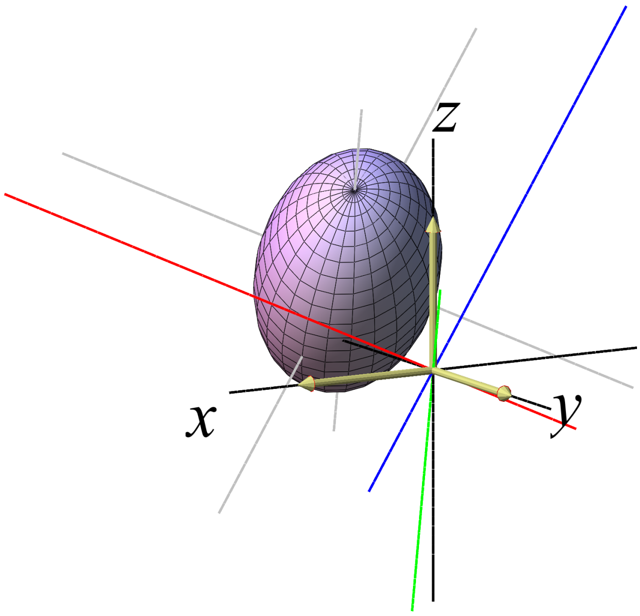


Figure 22.6: The level surface $\mathcal{K}_0(f)$ of the function that is analysed in example 22.5. The new rotated coordinate system in which the ellipsoid has the reduced form is indicated by the red $\widetilde{x}$-axis, blue $\widetilde{y}$-axis, and green $\widetilde{z}$-axis.

The function $f(x,y,z)$ investigated in Example 22.5 has a *stationary point* at the centre found with the value -3. The Hessian matrix is positive definite and therefore the stationary point of $f(x,y,z)$ in (x,y,z) space is a proper local minimum point, a global minimum point.

If we could draw the graph of the function $f(x,y,z)$ in the 4-dimensional (x,y,z,w) space, i.e. if we considered the set of those points in $\mathbb{R}^4$ that can be written in the form $(x,y,z,f(x,y,z))$ when (x,y,z) runs through the (x,y,z) space in $\mathbb{R}^4$, then the level surface of $f(x,y,z)$ will be the set we get in (x,y,z) space by putting $w=0$, i.e. exactly $f(x,y,z)=0$.

|||| Example 22.6 Hyperboloid with One Net

$$f(x,y,z) = x^2 + 2\cdot y^2 + z^2 + 2\cdot x - 4\cdot y + 2\cdot z - 4\cdot x\cdot z - 1 \quad .$$

$$\nabla f(x,y,z) = (2\cdot x - 4\cdot z + 2, -4\cdot y - 4, -4\cdot x + 2\cdot z + 2) \quad .$$

$$\frac{1}{2}\cdot \mathbf{H}f(x,y,z) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad . \tag{22-12}$$

$$\mathbf{\Lambda} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad .$$

$$\mathbf{Q} = \begin{bmatrix} -\sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \quad .$$

$$\begin{aligned} \widetilde{f}(\widetilde{x},\widetilde{y},\widetilde{z}) &= 3\cdot \widetilde{x}^2 + 2\cdot \widetilde{y}^2 - \widetilde{z}^2 - 4\cdot \widetilde{y} - 2\cdot \sqrt{2}\cdot \widetilde{z} - 1 \\ &= 3\cdot \widetilde{x}^2 + 2\cdot (\widetilde{y}-1)^2 - \left(\widetilde{z} + \sqrt{2}\right)^2 - 1 \quad . \end{aligned} \tag{22-13}$$

The quadratic equation that gives the level surface $\mathcal{K}_0(f)$ is therefore given by either of the following equivalent equations:

$$\begin{gathered} f(x,y,z) = 0 = \widetilde{f}(\widetilde{x},\widetilde{y},\widetilde{z}) \\ \left(\frac{\widetilde{x}}{\frac{1}{\sqrt{3}}}\right)^2 + \left(\frac{\widetilde{y}-1}{\frac{1}{\sqrt{2}}}\right)^2 - \left(\widetilde{z} + \sqrt{2}\right)^2 = 1 \quad . \end{gathered} \tag{22-14}$$

The equations represent a *hyperboloid with one net* that has its centre at ${}_{\mathrm{e}}\mathbf{C} = (x_0, y_0, z_0) = (1,1,1)$ (with respect to the old coordinates) corresponding to ${}_{\mathrm{v}}\mathbf{C} = (\widetilde{x}_0, \widetilde{y}_0) = (0, 1, -\sqrt{2})$ (with respect to the new coordinates) and the semi-axes $\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{2}}$, and 1, see Figure 22.7 and the Table of names in 22.3. The new coordinate system appears through a rotation of the old coordinate system by the positive orthogonal substitution $\mathbf{Q}$.

The function $f(x,y,z)$, investigated in 22.6, has a *stationary point* at the centre found with the value -1. The Hessian matrix is indefinite and therefore the stationary point of $f(x,y,z)$ in (x,y,z) space is neither a minimum point nor a maximum point.

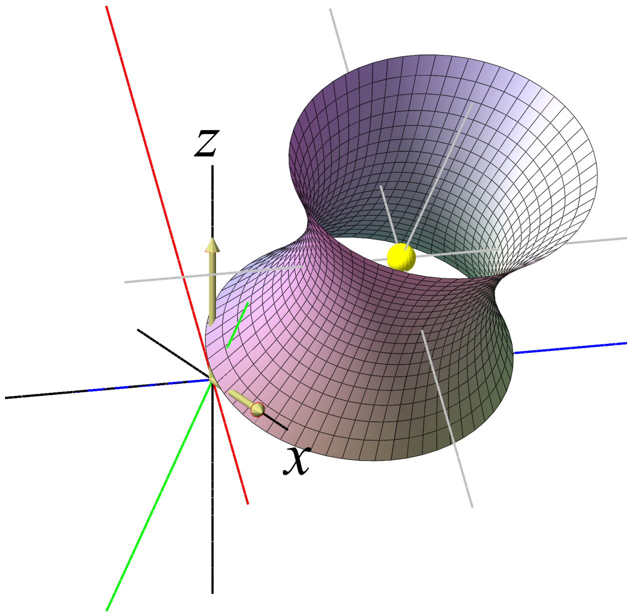


Figure 22.7: The level surface $\mathcal{K}_0(f)$ of the function that is analyzed in Example 22.6.

|||| Example 22.7 Hyperboloid with Two Nets

A quadratic polynomial $f(x,y,z)$ is given by the following data:

$$f(x,y,z) = -x^2 + \frac{1}{2}\cdot y^2 + \frac{1}{2}\cdot z^2 + 2\cdot x + 4\cdot y + 4\cdot z - 5\cdot y\cdot z - 6 \quad .$$

$$\boldsymbol{\nabla} f(x,y,z) = (-2\cdot x + 2, y - 5\cdot z + 4, -5\cdot y + z + 4) \quad .$$

$$\frac{1}{2}\cdot \mathbf{H}f(x,y,z) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/2 & -5/2 \\ 0 & -5/2 & 1/2 \end{bmatrix} \quad .$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad . \tag{22-15}$$

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} \quad .$$

$$\begin{aligned}\widetilde{f}(\widetilde{x},\widetilde{y},\widetilde{z}) &= 3\cdot\widetilde{x}^2 - \widetilde{y}^2 - 2\cdot\widetilde{z}^2 + 2\cdot\widetilde{y} + 4\cdot\sqrt{2}\cdot\widetilde{z} - 6 \\ &= 3\cdot\widetilde{x}^2 - (\widetilde{y}-1)^2 - 2\cdot\left(\widetilde{z}-\sqrt{2}\right)^2 - 1 \quad .\end{aligned}$$

The quadratic equation describing the level surface $\mathcal{K}_0(f)$ is therefore given by:

$$\begin{aligned} f(x,y,z) &= 0 = \widetilde{f}(\widetilde{x},\widetilde{y},\widetilde{z}) \\ \left(\frac{\widetilde{x}}{\frac{1}{\sqrt{3}}}\right)^2 - (\widetilde{y}-1)^2 - \left(\frac{\widetilde{z}-\sqrt{2}}{\frac{1}{\sqrt{2}}}\right)^2 &= 1 \quad . \end{aligned} \tag{22-16}$$

This is the equation of a *hyperboloid with two nets* that has its centre at ${}_{\mathrm{e}}\mathbf{C} = (x_0, y_0, z_0) = (1,1,1)$ (with respect to the old coordinates) corresponding to ${}_{\mathrm{v}}\mathbf{C} = (\widetilde{x}_0, \widetilde{y}_0) = (0, 1, \sqrt{2})$ (with respect to the new coordinates) and semi-axes $\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{2}}$ and 1 – see Figure 22.8.

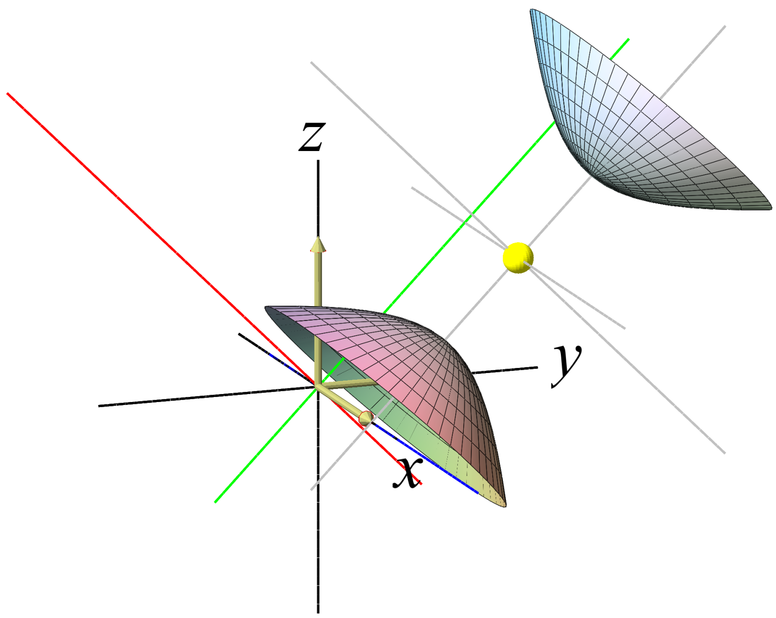


Figure 22.8: A hyperboloid with two nets is the level surface $\mathcal{K}_0(f)$ of the function analyzed in Example 22.7.

22.3 Table of Names of Level Surfaces of Quadratic Polynomials

The maximallt reduced expressions of the level surfaces of Quadratic polynomials are here presented with the assumption that a possible centre (or so-called vertex) occurs in${}_{\mathrm{v}}\mathbf{C} = (0,0,0)$. For a given concrete level surface the name can be read from the table; the location is stated by the coordinates of the centre $\mathbf{C}$ found of the level surface; the orientation is stated by the substitution matrix $\mathbf{Q}$ found and the size of the level surface is stated by the individual semi-axes a, b, and c, or p and k in each case if they appear in the reduced equation of the level surface.

Equation	Name	Point $(0,0,0)$	Example
$\frac{\tilde{x}^2}{a^2}+\frac{\tilde{y}^2}{b^2}+\frac{\tilde{z}^2}{c^2}=1$	ellipsoid	centre	22.5
$\frac{\tilde{x}^2}{a^2}+\frac{\tilde{y}^2}{b^2}-\frac{\tilde{z}^2}{c^2}=1$	hyperboloid with one net	centre	22.6
$\frac{\tilde{x}^2}{a^2}-\frac{\tilde{y}^2}{b^2}-\frac{\tilde{z}^2}{c^2}=1$	hyperboloid with two nets	centre	22.7
$\frac{\tilde{x}^2}{a^2}+\frac{\tilde{y}^2}{b^2}-\frac{\tilde{z}^2}{c^2}=0$	conic surface	centre	fig 22.9
$\frac{\tilde{x}^2}{a^2}+\frac{\tilde{y}^2}{b^2}=\tilde{z}$	elliptic paraboloid	vertex	fig 22.9
$\frac{\tilde{x}^2}{a^2}-\frac{\tilde{y}^2}{b^2}=\tilde{z}$	hyperbolic paraboloid	vertex	fig 22.9
$\frac{\tilde{x}^2}{a^2}=p\cdot\tilde{z}$	parabolic cylinder surface	vertex	fig 22.9
$\frac{\tilde{x}^2}{a^2}+\frac{\tilde{y}^2}{b^2}=1$	elliptic cylinder surface	centre	
$\frac{\tilde{x}^2}{a^2}-\frac{\tilde{y}^2}{b^2}=1$	hyperbolic cylinder surface	vertex	
$\frac{\tilde{x}^2}{a^2}+\frac{\tilde{y}^2}{b^2}+\frac{\tilde{z}^2}{c^2}=0$	the point $(0,0,0)$	"centre"	
$\frac{\tilde{x}^2}{a^2}+\frac{\tilde{y}^2}{b^2}=0$	the $\tilde{z}$-axis		
$\frac{\tilde{x}^2}{a^2}-\frac{\tilde{y}^2}{b^2}=0$	two planes through the $\tilde{z}$-axis		
$\tilde{x}^2=k>0$	two planes parallel to the $(\tilde{y},\tilde{z})$ plane		
$\tilde{x}^2=0$	the $(\tilde{y},\tilde{z})$ plane		

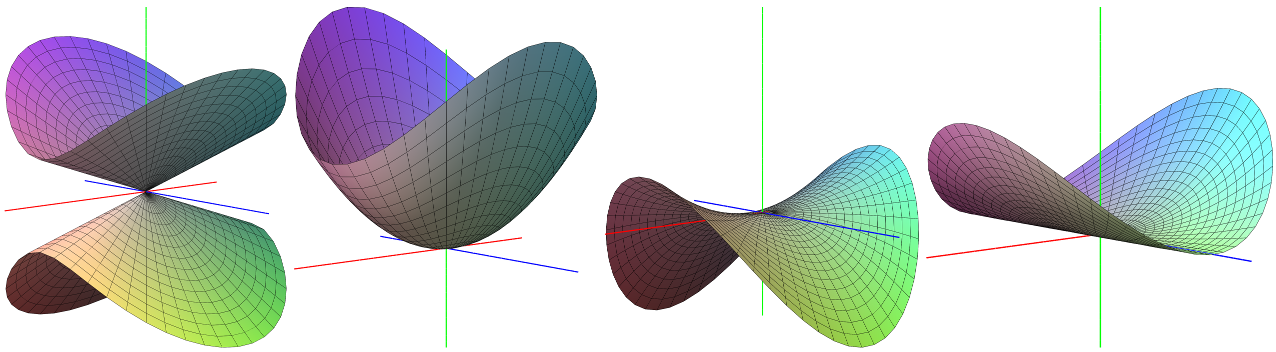


Figure 22.9: A conic surface, an elliptic paraboloid, a hyperbolic paraboloid, and a parabolic cylinder surface.

|||| Exercise 22.8

Show that the list in the Table of Names of level surfaces of quadratic polynomials with three variables is complete; i.e. every fully reduced quadratic equation with three variables (containing at least one of the variables $\widetilde{x}$, $\widetilde{y}$, or $\widetilde{z}$ with degree 2) can be found in the list.

|||| Exercise 22.9

Show that every fully reduced quadratic equation with two variables appears in the table with fully reduced quadratic equations with three variables by setting $\widetilde{z} = 0$.

22.4 Parametrization of an Ellipsoid

Here we will outline how to *parametrize* a level surface of a quadratic polynomial with three variables in preparation for plotting the surface and investigating its other properties. As an example we consider an ellipsoid with given (or found) centre $\mathbf{C}$, given semi-axes (read from the diagonal matrix Λ), and given orientation vectors ${}_{\mathrm{e}}\mathbf{v_1}$, ${}_{\mathrm{e}}\mathbf{v_2}$, and ${}_{\mathrm{e}}\mathbf{v_3}$ (from the positive orthogonal substitution matrix $\mathbf{Q}$) – i.e. exactly from the data determined through the reduction-analysis exemplified above.

We assume for the sake of the example that we are talking about an ellipsoid so that $\mathbf{A} = \frac{1}{2} \cdot \mathbf{H}f(x, y, z)$ is positive definite, i.e. all three eigenvalues λ_1, λ_2, and λ_3 are positive for $\mathbf{A} = \frac{1}{2} \cdot \mathbf{H}f(x, y, z)$. A similar construction can be carried out for any of the other level surfaces.

We assume that the equation of the ellipsoid in reduced form is given by

$$\lambda_1 \cdot (\widetilde{x} - \widetilde{x}_0)^2 + \lambda_2 \cdot (\widetilde{y} - \widetilde{y}_0)^2 + \lambda_3 \cdot (\widetilde{z} - \widetilde{z}_0)^2 = d^2 \quad , \tag{22-17}$$

where d is a positive constant and where ${}_{\mathrm{v}}\mathbf{C} = (\widetilde{x}_0, \widetilde{y}_0, \widetilde{z}_0)$ are the coordinates of the centre of the ellipsoid with respect to the new coordinate system, such that centre-coordinates with respect to the old coordinate system are given by: ${}_{\mathrm{e}}\mathbf{C} = \mathbf{Q} \cdot {}_{\mathrm{v}}\mathbf{C}$. Thus we have the following ingredients at our disposal for the construction of the ellipsoid:

$$\begin{aligned} {}_{\mathrm{e}}\mathbf{C} &= (C_1, C_2, C_3) \\ a &= \frac{d}{\sqrt{\lambda_1}} \\ b &= \frac{d}{\sqrt{\lambda_2}} \\ c &= \frac{d}{\sqrt{\lambda_3}} \\ \mathbf{Q} &= \left[\ {}_{\mathrm{e}}\mathbf{v_1} \quad {}_{\mathrm{e}}\mathbf{v_2} \quad {}_{\mathrm{e}}\mathbf{v_3}\ \right] \quad . \end{aligned} \tag{22-18}$$

We will only consider coordinates refering to the ordinary basis e in $(\mathbb{R}^n, \cdot)$.

A spherical surface $\mathcal{S}_1$ with radius 1 and centre at $(0,0,0)$ can be written as the set of points (x, y, z) having the distance 1 to $(0,0,0)$:

$$\begin{aligned} \rho_{(0,0,0)}(x, y, z) &= 1 \quad , \quad \text{equivalent to} \\ \sqrt{x^2 + y^2 + z^2} &= 1 \quad , \quad \text{or} \\ x^2 + y^2 + z^2 &= 1 \quad . \end{aligned} \tag{22-19}$$

The spherical surface can also be presented by the use of *geographical coordinates* u and v, where $u \in [0, \pi]$ and $v \in [-\pi, \pi]$:

$$\mathcal{S} \quad : \quad (x, y, z) = \mathbf{r}(u, v) = (\sin(u) \cdot \cos(v), \sin(u) \cdot \sin(v), \cos(u)) \quad . \tag{22-20}$$

When u and v run through their respective intervals $u \in [0, \pi]$ and $v \in [-\pi, \pi]$ we get points $(x, y, z) = \mathbf{r}(u, v)$ on the spherical surface – and all points are represented. Let us look at the first of the two statements, i.e. all points $\mathbf{r}(u, v)$ lie on the spherical surface. Substitute $x = \sin(u) \cdot \cos(v)$, $y = \sin(u) \cdot \sin(v)$, and $z = \cos(u)$ into Equation 22-19:

$$\begin{aligned} x^2 + y^2 + z^2 &= \sin^2(u) \cdot \cos^2(v) + \sin^2(u) \cdot \sin^2(v) + \cos^2(u) \\ &= \sin^2(u) \cdot \left(\sin^2(u) + \cos^2(u)\right) + \cos^2(u) \\ &= \sin^2(u) + \cos^2(u) = 1 \quad , \end{aligned} \tag{22-21}$$

so all points that are given in the form $\mathbf{r}(u, v)$ lie on the spherical surface.

|||| Exercise 22.10

Show that the other statement about $\mathbf{r}(u,v)$ is also true, i.e. that all points on the spherical surface are represented by $\mathbf{r}(u,v)$ when u and v run through the intervals $u \in [0,\pi]$ and $v \in [-\pi,\pi]$. Do points on the spherical surface exist that are hit more than once? In that case describe the set of points and 'how many times' the points are hit.

We now scale the spherical surface with the eigenvalues in each coordinate axis direction and by this we get a description with the correct, sought-for semi-axes:

$$\mathbf{r}_1(u,v) = \left(\frac{d}{\sqrt{\lambda_1}} \cdot \sin(u) \cdot \cos(v), \frac{d}{\sqrt{\lambda_2}} \cdot \sin(u) \cdot \sin(v), \frac{d}{\sqrt{\lambda_3}} \cdot \cos(u)\right) \quad , \tag{22-22}$$

where $u \in [0,\pi]\,, v \in [-\pi,\pi]$ such that every point x, y, z represented by the map $\mathbf{r}_1(u,v)$ now satisfies:

$$\lambda_1 \cdot x^2 + \lambda_2 \cdot y^2 + \lambda_3 \cdot z^2 = d^2$$

$$\left(\frac{x}{\frac{d}{\sqrt{\lambda_1}}}\right)^2 + \left(\frac{y}{\frac{d}{\sqrt{\lambda_2}}}\right)^2 + \left(\frac{z}{\frac{d}{\sqrt{\lambda_3}}}\right)^2 = 1 \tag{22-23}$$

or, if we use the names a, b, and c for the semi-axes:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \quad , \tag{22-24}$$

which is exactly the equation of a standard ellipsoid with the wanted semi-axes but constructed in the (x,y,z) system. The points that are described by $\mathbf{r}_1(u,v)$ all lie on this ellipsoid.

Finally we rotate the ellipsoid with the rotation matrix $\mathbf{Q}$ and use a parallel displacement to the wanted centre:

$$\mathbf{r}_2(u,v) = \mathbf{Q} \cdot \mathbf{r}_1(u,v) + \mathbf{C} \quad . \tag{22-25}$$

All the points given by the positional vector $\mathbf{r}_2(u,v)$ now lie on the wanted ellipsoid and all points on the ellipsoid are presented when $u \in [0,\pi]\,, v \in [-\pi,\pi]$.

|||| Example 22.11 Ellipsoid Parametrization

A concrete ellipsoid is given by the following data resulting from an investigation of the quadratic polynomial

$$f(x,y,z) = 2\cdot x^2 + 2\cdot y^2 + 2\cdot z^2 - 2\cdot x - 4\cdot y - 2\cdot z - 2\cdot x\cdot z + 3 \quad : \tag{22-26}$$

$$\begin{aligned}
{}_{\mathrm{e}}\mathbf{C} &= (1,1,1) \\
a &= \frac{1}{\sqrt{3}} \\
b &= \frac{1}{\sqrt{2}} \\
c &= 1 \\
\mathbf{Q} &= \begin{bmatrix} -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \quad .
\end{aligned} \tag{22-27}$$

We will construct a parametric representation in the form $\mathbf{r}_2(u,v)$ as in (22-25) for the given ellipsoid. The scaled spherical surface is given by:

$$\mathbf{r}_1(u,v) = \left(\frac{1}{\sqrt{3}}\cdot\sin(u)\cdot\cos(v), \frac{1}{\sqrt{2}}\cdot\sin(u)\cdot\sin(v), \cos(u)\right) \tag{22-28}$$

and the $\mathbf{Q}$-rotated scaled spherical surface translated to the point $(1,1,1)$ then gets the parametric representation:

$$\begin{aligned}
\mathbf{r}_2(u,v) = \\
= (&-\frac{1}{\sqrt{6}}\cdot\sin(u)\cdot\cos(v) - \frac{1}{\sqrt{2}}\cdot\cos(u) + 1\,, \\
&\frac{1}{\sqrt{2}}\cdot\sin(u)\cdot\sin(v) + 1\,, \\
&\frac{1}{\sqrt{6}}\cdot\sin(u)\cdot\cos(v) - \frac{1}{\sqrt{2}}\cdot\cos(u) + 1) \quad .
\end{aligned} \tag{22-29}$$

The construction is shown in Figure 22.10.

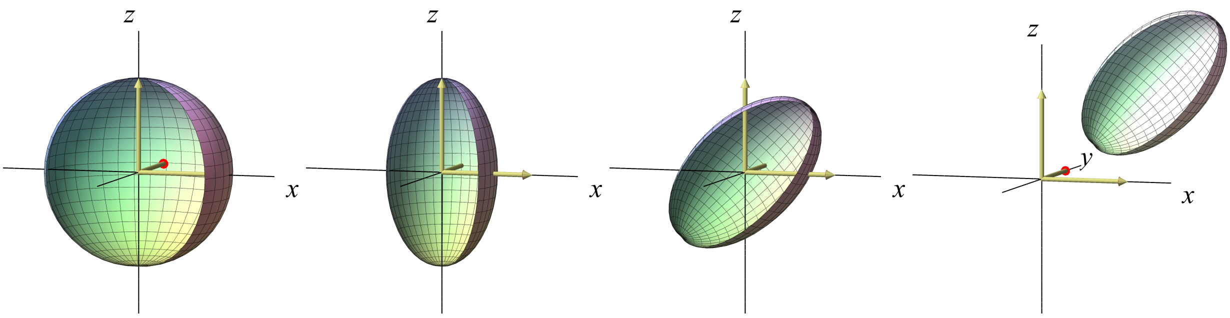


Figure 22.10: Construction of Example 22.11 ellipsoid from data about placement, axes, and rotation matrix.

|||| Example 22.12 Ellipsoid Data

The ellipsoid constructed in example 22.11 is the 0-level surface of the quadratic polynomial with the following data:

$$f(x,y,z) = 2\cdot x^2 + 2\cdot y^2 + 2\cdot z^2 - 2\cdot x - 4\cdot y - 2\cdot z - 2\cdot x\cdot z + 3 \quad .$$

$$\boldsymbol{\nabla} f(x,y,z) = (4\cdot x - 2\cdot z - 2,\ 4\cdot y - 4,\ -2\cdot x + 4\cdot z - 2) \quad .$$

$$\frac{1}{2}\cdot \mathbf{H}f(x,y,z) = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \quad .$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \tag{22-30}$$

$$\mathbf{Q} = \begin{bmatrix} -\sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \quad .$$

$$\begin{aligned} \widetilde{f}(\widetilde{x},\widetilde{y},\widetilde{z}) &= 3\cdot \widetilde{x}^2 + 2\cdot \widetilde{y}^2 + \widetilde{z}^2 - 4\cdot \widetilde{y} + 2\cdot \sqrt{2}\cdot \widetilde{z} + 3 \\ &= 3\cdot \widetilde{x}^2 + 2\cdot (\widetilde{y}-1)^2 + \left(\widetilde{z} + \sqrt{2}\right)^2 - 1 \quad . \end{aligned}$$

The quadratic equation that gives the level surface $\mathcal{K}_0(f)$ is therefore given by the equation:

$$\left(\frac{\widetilde{x}}{\frac{1}{\sqrt{3}}}\right)^2 + \left(\frac{\widetilde{y}-1}{\frac{1}{\sqrt{2}}}\right)^2 + \left(\widetilde{z}+\sqrt{2}\right)^2 = 1 \quad , \tag{22-31}$$

from which we read the above given semi-axes $a = \frac{1}{\sqrt{3}}$, $b = \frac{1}{\sqrt{2}}$, and $c = 1$ together with the centre coordinates ${}_{\mathrm{v}}(\mathbf{C}) = (0, 1, -\sqrt{2})$ corresponding , via the substitution matrix $\mathbf{Q}$, to the coordinates ${}_{\mathrm{e}}(\mathbf{C}) = (1, 1, 1)$ with respect to the standard-basis e.

22.5 Summary

The main outcome in this eNote is an identification – via a series of concrete examples – of the possible level curves and surfaces of quadratic polynomials with two and three variables. The method is the reduction method for quadratic forms and quadratic polynomials introduced in eNote 18. Based on the same method a strategy for the parametrization and presentation of the individual level surfaces of functions of three variables is given.