eNote 19

Functions of Two Variables

In this and the following eNotes we will extend the concept of functions to include real functions of more variables; we start the extension with 2 *variables, therefore in this eNote we will define ranges, continuity, and differentiability of functions* $f(x, y)$ *of two variables, here* x *and* y*. As for functions of one variable we will use the epsilon function concept (now also of two variables) for the purpose.*

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19.1 Domains

In the description of a real function $f(x, y)$ of two variables one states on the one hand the points (x, y), in the (x, y) plane where the function is defined and on the other hand the values that can be computed by using the function on the domain. We call the ***domain*** $\mathcal{D}(f)$ and the ***range*** we call $\mathcal{R}(f)$. We will in particular focus on the domains here.

As a new thing in relation to functions of one variable the domains in the plane are generally not as simple as the intervals on the real number axis.

|||| Example 19.1 The Domain of a Function of Two Variables

Let us consider a function $f(x,y)$ of two variables:

$$f(x,y) = \ln(\sqrt{5 - x^2 - y^2}) \quad . \tag{19-1}$$

At which points (x,y) in $\mathbb{R}^2$ is this function defined? E.g. we have that $f(0,0) = \ln(\sqrt{5})$, $f(2,0) = f(0,2) = 0$, $f(1,0) = f(0,1) = \ln(2)$ and in fact $f(\cos(t), \sin(t)) = \ln(2)$ for all t, but $f(3,7)$ is not defined for this function. By inspection of the function it is seen that the domain of $f(x,y)$ consists precisely of the points lying entirely inside the circular disc that has its centre at $(0,0)$ and radius $\sqrt{5}$:

$$\mathcal{D}(f) = \{\, (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 5 \,\} \quad . \tag{19-2}$$

Note that the boundary of the circular disk is not part of the domain.

We now introduce some important concepts for the description of domains and in addition in general for the description of arbitrary subsets of the (x,y) plane that can be useful if one needs to draw or sketch the sets:

19.1.1 Open and Closed Sets in the Plane

|||| Definition 19.2 Open Sets in the Plane

A subset M of the plane is called an ***open set*** if every point (x_0, y_0) in the set is the centre for some (possibly very small) circular disc that itself is entirely contained in M.

|||| Example 19.3 A Circular Disc

The set $\mathcal{D} = \{\, (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 5 \,\}$ is an open set. But the set $\{\, (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 5 \,\}$ is *not* an open set.

|||| Definition 19.4 The Boundary of a Set in the Plane, Interior and Exterior

The boundary of a subset M of the plane consists of those points (x_0, y_0) *in the plane* that have the following properties: Every circular disc with a centre at (x_0, y_0) contains both points that belong to M and points that do not belong to M. Note, the boundary points for M need not themselves belong to the set M. The set of boundary points M is denoted ∂M.

The interior of a subset M is all those points in M that do not lie on the boundary of M. The interior of M is denoted $\mathring{M}$.

The exterior of a subset M of the plane is all those points in the plane that do not belong to either M nor ∂M.

|||| Example 19.5 The Boundary of an Open Circular Disc

The set $\mathcal{D} = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 5 \right\}$ has the boundary:

$$\partial\mathcal{D} = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 5 \right\} \quad . \tag{19-3}$$

|||| Definition 19.6 The Closure of a Set in the Plane

If we add the set of boundary points ∂M to a set M we get ***the closure*** of the set:

$$\bar{M} = M \cup \partial M \quad . \tag{19-4}$$

If the boundary points already belong to the set M we do not get an extension of M. The set M is called ***closed*** if $\bar{M} = M$.

|||| Example 19.7 Open and Closed Sets in Figures

The set A in Figure 19.1 is neither open nor closed (there are some points on the boundary that belong to the set and there are other points on the boundary that do not belong to the set). The rule we have used here for drawings are: points that are included in the set are green or lie on a full-drawn segment of the boundary; points that do not belong to the set are not colored or (if they are part of the boundary) marked with red circles. The set B in Figure 19.1 is an open set. The set C in Figure 19.1 is a closed set.

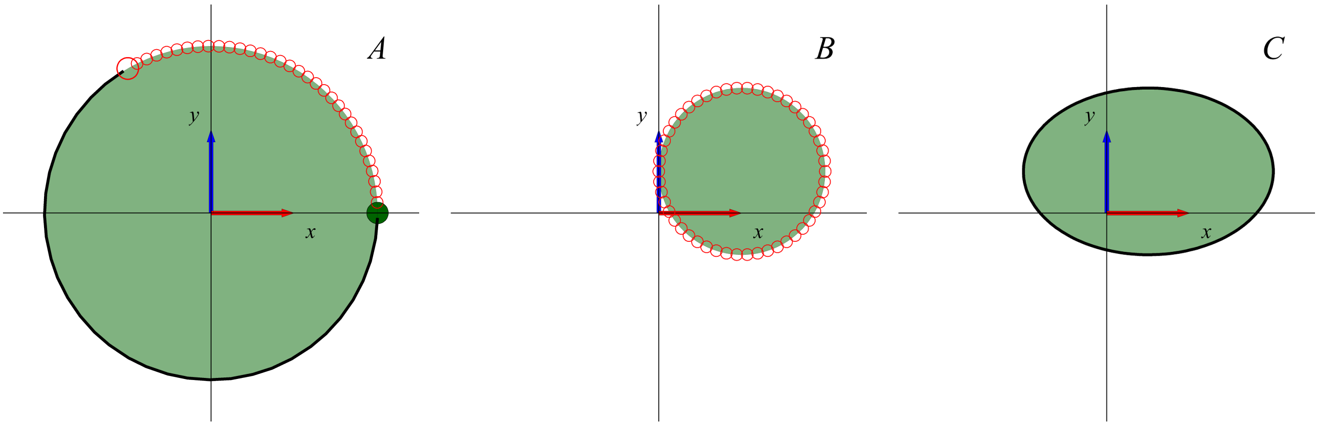

Figure 19.1: Subsets of the plane. The set A is neither open nor closed, B is an open set and C is a closed set.

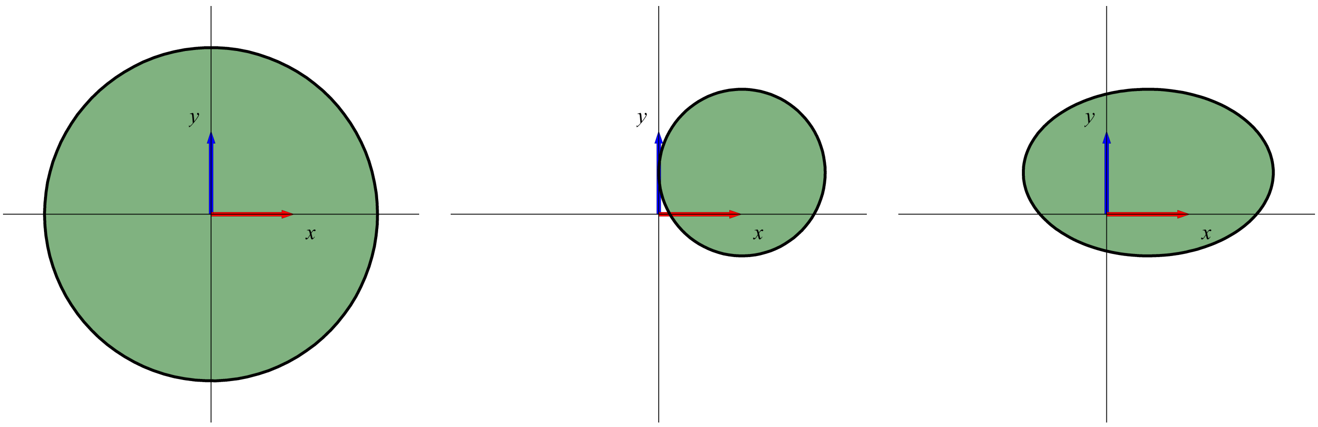

Figure 19.2: The closures $\bar{A}$, $\bar{B}$ and $\bar{C}$ of the subsets A, B and C from Figure 19.1.

19.1.2 Star-Shaped Sets in the Plane

|||| Definition 19.8 Star-Shaped Domains

If every point (x, y) in a set M in the plane can 'be seen' from a point (x_0, y_0) in the set such that the whole line segment of sight from and including (x_0, y_0) to and including (x, y) is contained in the set then M is said to be ***star-shaped*** from the ***star point*** (x_0, y_0). In other words: Every point (x, y) in the set can be connected to the star point by a line segment that is entirely included in the set. See Figure 19.3.

|||| Exercise 19.9 Star-Shape and Double Star-Shape

Which points in the set that is shown to the left in Figure 19.3 can be used as star points for the set? Is the set of all star points closed or open? In what sense can one say that the figure to the right in 19.3 is *doubly* star-shaped?

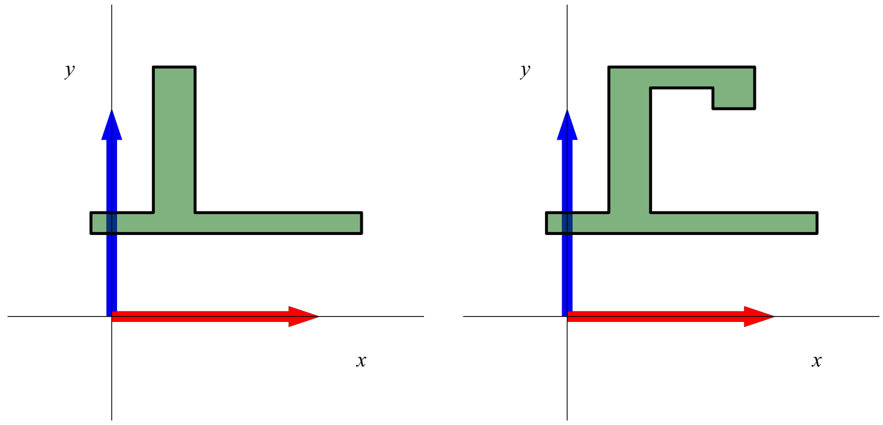

Figure 19.3: The set to the left is star-shaped. The set to the right is *not* star-shaped.

19.1.3 Bounded Sets in the Plane

|||| Definition 19.10 Bounded Sets

A set M in the plane is said to be ***bounded*** if it is entirely contained in a (possibly very large) circular disc centered at $(0,0)$.

|||| Example 19.11

The three sets A, B and C in Figure 19.1 are obviously bounded; they are entirely contained in the circular disc with centre $(0,0)$ and radius 100. The set of points that is constituted by the points on the x-axis is not bounded.

19.1.4 Extensions of Domains to All of $\mathbb{R}^2$

As with the extension of domains of functions of one variable the same thing can be done for functions $f(x,y)$ of two variables:

|||| Definition 19.12 Extension of the Domain

Given the function $f(x,y)$ with the domain $\mathcal{D}(f)$, we define the 0-extension $\widehat{f}(x,y)$ of $f(x,y)$ in the following way:

$$\widehat{f}(x,y) = \begin{cases} f(x,y)\,, & \text{for} \quad (x,y) \in \mathcal{D}(f) \\ 0\,, & \text{for} \quad (x,y) \in \mathbb{R}^2 \setminus \mathcal{D}(f) \end{cases} \quad . \tag{19-5}$$

19.2 Graphs of Functions of Two Variables

In order to illustrate functions of two variables we draw them in 3D space – we plot the set of points that appears by constructing the graphs in an $\{\mathcal{O},x,y,z\}$-coordinate system:

|||| Definition 19.13 Graphs of Functions of Two Variables

Let $f(x,y)$ be a function of two variables with the domain $\mathcal{D}(f)$. Then the ***graph of a function of two variables*** is given by:

$$z = f(x,y) \quad , \quad \text{where} \quad (x,y) \in \mathcal{D}(f) \quad . \tag{19-6}$$

So the graph consists of the set of points in (x,y,z) space that we can also describe in the following way:

$$\mathcal{G}(f) = \{\, (x,y,f(x,y)) \mid (x,y) \in \mathcal{D}(f) \,\} \quad . \tag{19-7}$$

Every individual point on the graph appears in the following way: From the point $(x,y,0)$ in the (x,y) plane we move up the height (with sign) $f(x,y)$ vertically (or down) from the (horizontal) (x,y) plane and mark the graph point $(x,y,f(x,y)$ at the height that the function value $f(x,y)$ states – just above (or under) the point $(x,y,0)$.

19.3 Level Sets and Height Sections

In the (x,y) plane, where the function $f(x,y)$ is defined, we can do something quite different in order to show how the function values vary from point to point.

|||| Definition 19.14 Level Sets

For a function $f(x,y)$ of two variables we define for every real number c the corresponding ***level set*** in the following way:

$$\mathcal{K}_c(f) = \{\, (x,y) \in \mathcal{D}(f) \mid f(x,y) = c \,\} \quad . \tag{19-8}$$

The set $\mathcal{K}_c$ can be empty, the whole plane, a curve or any set of points in the plane.

|||| Example 19.15 Level Sets

We let A denote an arbitrary set in the plane and construct a function $f(x,y)$ on all of the plane in the following way:

$$f(x,y) = \begin{cases} 1 & \text{for } (x,y) \in A \\ 0 & \text{for } (x,y) \in \mathbb{R}^2 \setminus A \end{cases} \quad , \tag{19-9}$$

i.e. f is the 0-extension of the function that is constant 1 on A. Then

$$\mathcal{K}_c(f) = \begin{cases} A & \text{for } c = 1 \\ \mathbb{R}^2 \setminus A & \text{for } c = 0 \\ \emptyset & \text{for } c \neq 1 \text{ and } c \neq 0 \end{cases} \quad . \tag{19-10}$$

Often the level set $\mathcal{K}_c$ is better constructed, though, and consists typically of one or more curves. These curves can rightly be called level curves or contour lines because they comprise precisely those points (x,y) in the domain where the function has the value c and where the graph of f therefore exactly has the height (with sign) c over the (x,y) plane. In other words: if we intersect the graph of f with the horizontal plane $z = c$ at the height c over the (x,y) plane then we get an intersection curve the projection of which onto (or up into) the (x,y) plane precisely is $\mathcal{K}_c$, see Figures 19.4 19.6, 19.5.

Below in section 19.8.1 we will at every point in the domain of $f(x,y)$ define a vector, the gradient vector of $f(x,y)$ that has the special property that if it is not $\mathbf{0}$ on an open set around a given point (x_0,y_0) then the level set containing (x_0,y_0) is a curve through the point. Gradient vectors are only well defined, though, for differentiable functions, so this property we shall have to first introduce for functions of two variables.

|||| Example 19.16 Graphs and Contour Lines

The following functions are all defined in the whole (x,y) plane.

$$\begin{aligned} f(x,y) &= -x + y + 2 \\ f(x,y) &= 1 - \frac{1}{2}\left(x^2 + y^2\right) \\ f(x,y) &= \cos(3x) \cdot \sin(3y) \quad . \end{aligned} \tag{19-11}$$

The graphs of the functions are shown in the Figures 19.4, 19.5 and 19.6 both together with their respective contour lines in the (x,y) plane and together with a figure that indicates how the contour lines in the (x, y) plane can be seen as projections of those height section curves that appear by intersecting the graph surfaces of the functions in different heights with the planes $z = c$, where c is the constant value that the actual function $f(x, y)$ assumes on the contour lines $\mathcal{K}_c$. Note that the contour lines $\mathcal{K}_c$ really *are* curves (or points) in these cases.

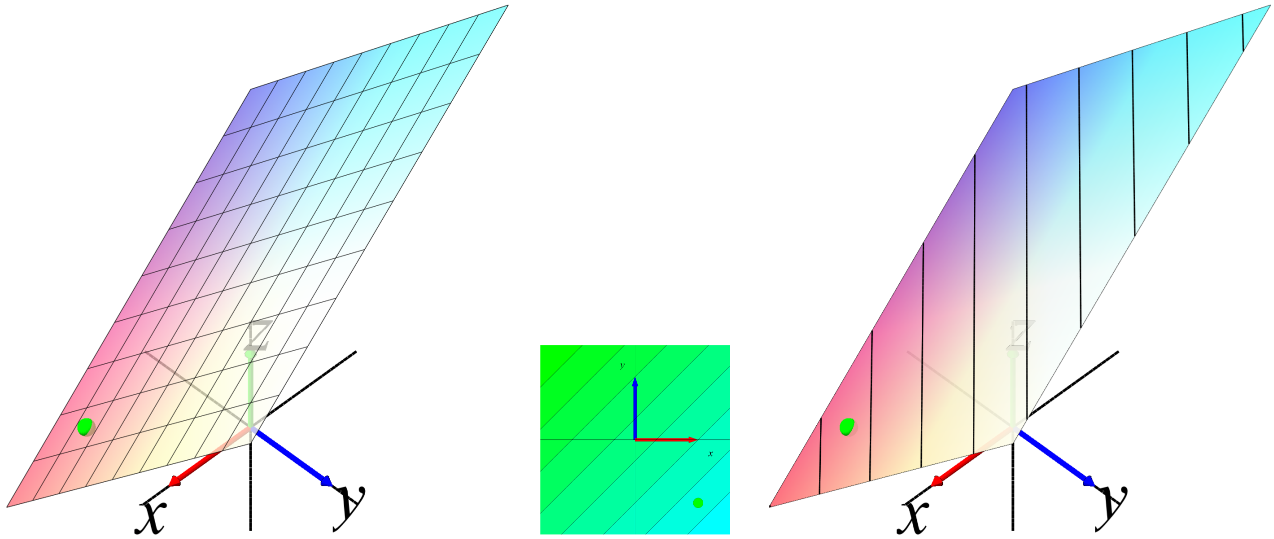

Figure 19.4: The graph in (x, y, z) space, its contour lines in the (x, y) plane and the height section curves for the function $f(x, y) = -x + y + 2$. Note that it is of course not the *whole* graph of the function we can plot. Here is only shown the segment corresponding to $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

19.4 Epsilon Functions of Two Variables

A very important class of functions of two variables is the distance functions. For every point (x_0, y_0) in the plane we define the distance to (x_0, y_0) in a well-known fashion:

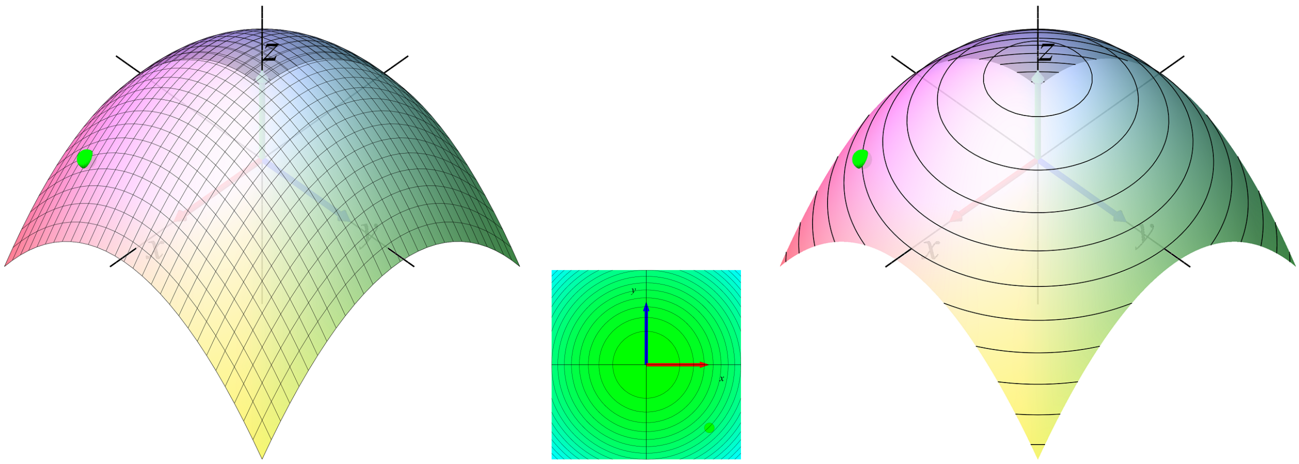

Figure 19.5: The graph in (x, y, z) space, its contour lines in the (x, y) plane and the height section curves for the function $f(x, y) = 1 - \frac{1}{2}\left(x^2 + y^2\right)$.

|||| Definition 19.17 Distance Functions

The distance from a point (x, y) to a point (x_0, y_0) in the (x, y) plane is denoted by

$$\rho_{(x_0,y_0)}(x, y) = \sqrt{(x - x_0)^2 + (y - y_0)^2} \quad . \tag{19-12}$$

This is the ordinary distance between two points (x, y) and (x_0, y_0) in the plane – determined using Pythagoras' Theorem.

It is this function that we will use in the same way as we used the functions $(x - x_0)$ in the definition of epsilon functions of one variable and in the definition of continuity and differerentiability of functions of one variable. Note that $\rho_{(x_0,y_0)}(x, y)$ is always positive or 0; and note that the value 0 only appears for $(x, y) = (x_0, y_0)$. The function $\rho_{(0,0)}(x, y)$, i.e. the distance from (x, y) to $(x_0, y_0) = (0, 0)$, is shown in Figure 19.7. Note that level curves are equidistant and that the graph is 'conically pointed' in the contact point to the (x, y) plane!

We are now ready to define the class of epsilon functions of two variables:

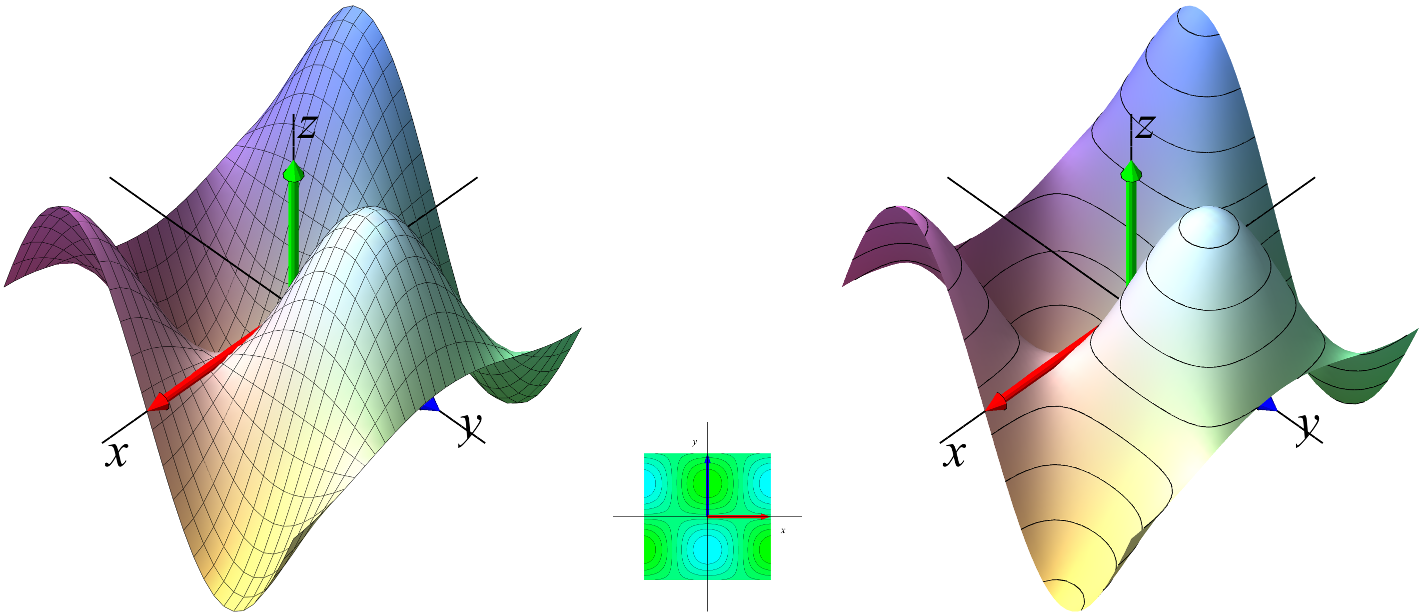

Figure 19.6: The graph in (x, y, z) space, its contour lines in the (x, y) plane and the height section curves for the function $f(x, y) = \cos(3x) \cdot \sin(3y)$.

|||| Definition 19.18 Epsilon Functions of Two Variables

Every function $\varepsilon(x, y)$ that is defined on an open subset of $\mathbb{R}^2$ which contains $(0, 0)$ and that assumes the value 0 at $(x, y) = (0, 0)$ and which furthermore tends towards 0 when (x, y) tends towards (x_0, y_0) is called an ***epsilon function of*** (x, y). Epsilon functions of two variables are thus characterized by the properties:

$$\varepsilon(0,0) = 0 \quad \text{and} \quad \varepsilon(x,y) \to 0 \quad \text{for} \quad (x,y) \to (0,0) \quad . \tag{19-13}$$

The last condition means that the absolute value of the function values $\varepsilon(x, y)$ can be made as small as we want simply by choosing (x, y) *sufficiently close to* $(0, 0)$. To be precise the condition means: For every positive integer k a positive integer K exists such that $|\varepsilon(x, y)| < 1/k$ for all (x, y) with $\rho_{(x_0,y_0)}(x, y) < 1/K$.

It follows directly from the definition that the distance function $\rho_{(x_0,y_0)}(x, y)$ to a point (x_0, y_0) is itself an epsilon function of $x - x_0$ and $y - y_0$.

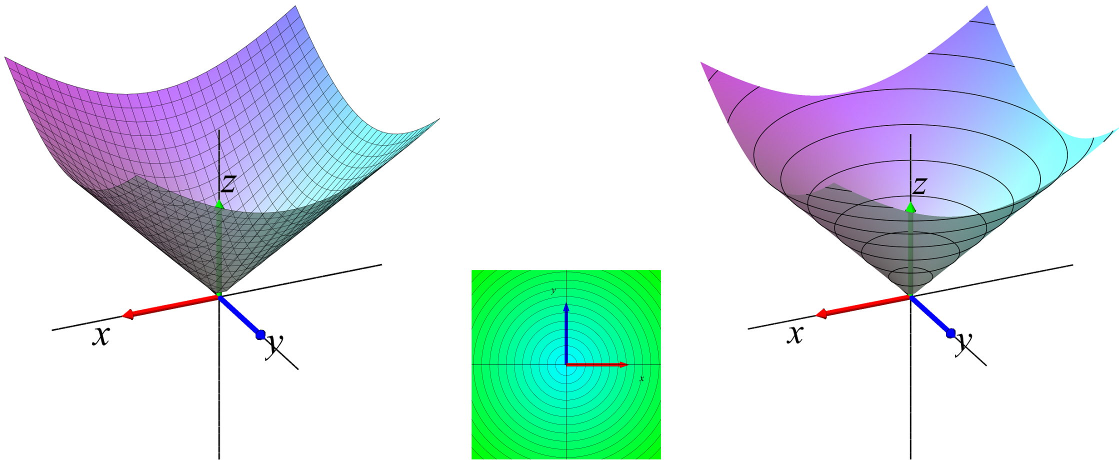

Figure 19.7: The graph of the distance function $\rho_{(0,0)}(x,y)$ to the point $(x_0,y_0) = (0,0)$, the level curves of the function and the height section curves on the graph.

19.5 Continuous Functions of Two Variables

Just as for functions of one variable we define continuity of functions of two variables by using the class of epsilon functions:

|||| Definition 19.19 Continuous Functions of Two Variables

A function $f(x,y)$ is continuous at (x_0,y_0) if there exists an epsilon function $\varepsilon_f(x - x_0, y - y_0)$ such that the following applies on an open set containing (x_0,y_0):

$$f(x,y) = f(x_0,y_0) + \varepsilon_f(x - x_0, y - y_0) \quad . \tag{19-14}$$

If $f(x,y)$ is continuous at all (x_0,y_0) in a given open subset of $\mathcal{D}(f)$, then we say that $f(x,y)$ is continuous on the whole subset.

Every epsilon function of $x - x_0$ and $y - y_0$ as e.g. $\rho_{(x_0,y_0)}(x,y)$ is therefore continuous at (x_0,y_0). Graphs and contour lines can often reveal whether a function is continuous at a point.

|||| Theorem 19.20 Inspection of Contours

If a function $f(x,y)$ has two level sets $\mathcal{K}_{c_1}$ and $\mathcal{K}_{c_2}$ (where $c_1 \neq c_2$) that both contain points (x,y) arbitrarily close to (x_0,y_0) then $f(x,y)$ is not continuous at (x_0,y_0).

|||| Proof

Suppose that $f(x,y)$ is continuous at (x_0,y_0). If we in the set $\mathcal{K}_{c_1}$ approach (x_0,y_0) then (due to continuity) $f(x_0,y_0)$ is equal to c_1. If we on the other hand in the set $\mathcal{K}_{c_2}$ approach (x_0,y_0) we get $f(x_0,y_0) = c_2$ which is a contradiction since $c_1 \neq c_2$.

■

Two level curves corresponding to different function values of a function $f(x,y)$ can both be very close to a specific point in the (x,y) plane – but, even so, not arbitrarily close – if the function is continuous.

|||| Example 19.21 A Function that is Not Continuous

We consider the 0-extension $\widehat{f}(x,y)$ of the function

$$f(x,y) = \frac{x^2\,y}{x^4+y^2} \quad , \quad \text{with} \quad (x,y) \in \mathbb{R}^2 \setminus \{\,(0,0)\,\} \quad . \tag{19-15}$$

I.e.:

$$\widehat{f}(x,y) = \begin{cases} \frac{x^2\,y}{x^4+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases} \quad . \tag{19-16}$$

The function can be inspected in Figure 19.8. By the inspection we note that the contour lines 'look like parabolas' through $(0,0)$. We will test the hypothesis that they are in fact parabolas: If we put $y = c\,x^2$ (the equation of a parabola) we get the following calculations:

$$\begin{aligned} f(x,y) &= f(x, c\,x^2) \\ &= \frac{x^2\,c\,x^2}{x^4 + (c\,x^2)^2} \\ &= \frac{c\,x^4}{(1+c^2)\,x^4} \\ &= \frac{c}{1+c^2} \quad , \end{aligned} \tag{19-17}$$

which exactly means that the parabolas $y = c\,x^2$ are (contained in) the level set $\mathcal{K}_{c/(1+c^2)}$. Since all the parabolas go through $(0,0)$ it follows from Theorem 19.20 that the function $f(x,y)$ is not continuous at $(0,0)$.

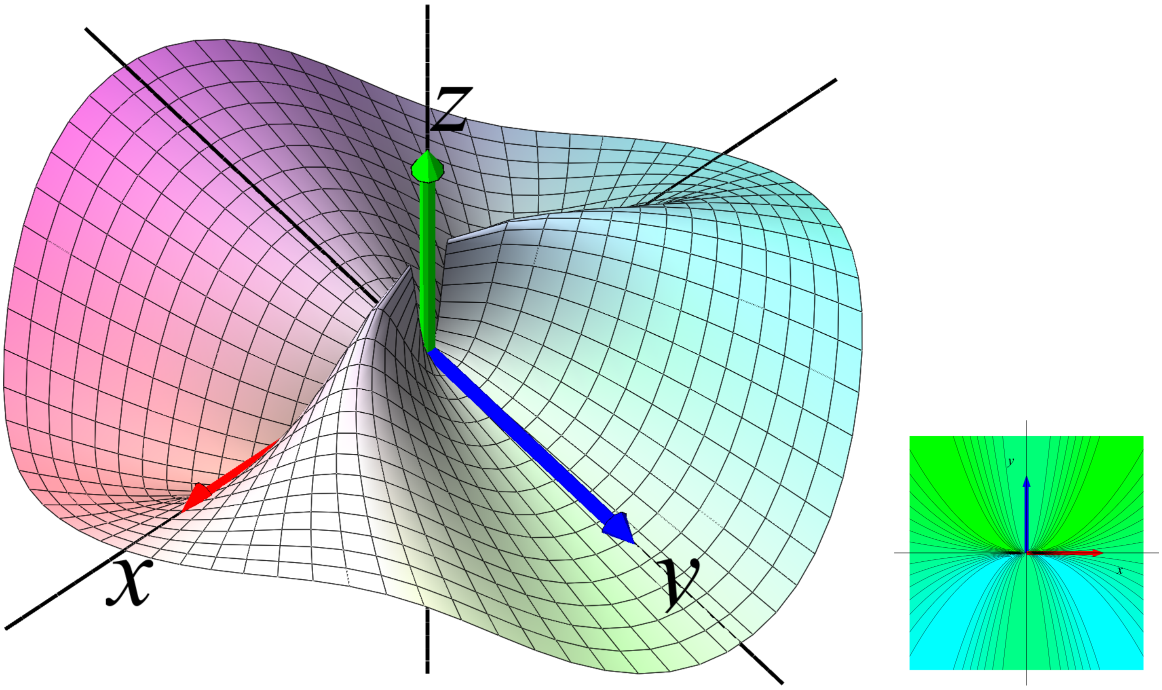

Figure 19.8: The graph in 3D space and the contour lines for the function in Example 19.21.

|||| Exercise 19.22

Show that the 0-extension $\widehat{f}(x,y)$ of the following function is not continuous at $(0,0)$:

$$f(x,y) = \frac{x}{x^2+y^2} \quad . \tag{19-18}$$

|||| Example 19.23 First-Degree Polynomials are Continuous

We will show that $f(x,y) = \alpha x + \beta y + \gamma$ is continuous at (x_0,y_0). It follows directly from the fact that

$$f(x,y) - f(x_0,y_0) = \alpha(x-x_0) + \beta(y-y_0) \to 0 \quad \text{for} \quad (x,y) \to (x_0,y_0)) \quad , \tag{19-19}$$

such that

$$f(x,y) = f(x_0,y_0) + \varepsilon_f(x-x_0, y-y_0) \tag{19-20}$$

with the epsilon function $\varepsilon_f(x-x_0, y-y_0) = \alpha(x-x_0) + \beta(y-y_0)$. Consider why this is an epsilon function.

|||| Exercise 19.24

Show that the following second-degree polynomial is continuous at $(0,0)$:

$$f(x,y) = x^2 + y^2 \quad . \tag{19-21}$$

|||| Exercise 19.25

Show that the 0-extension of the following function is continuous at $(0,0)$:

$$f(x,y) = \frac{x\,y^2}{x^2+y^2} \quad . \tag{19-22}$$

|||| Example 19.26 The Square Root of the Distance Function

The function $f(x,y) = \sqrt{\rho_{(x_0,y_0)}(x,y)}$ is – like $\rho_{(x_0,y_0)}(x,y)$ itself – an epsilon function and is therefore continuous at (x_0,y_0). See Figure 19.9.

19.6 Differentiable Functions of Two Variables

Most by far of the functions that we consider in Advanced Engineering Mathematics 1 are differentiable on their domains and because of this also continuous as we shall see below – cf. similar properties for functions of one variable.

Differentiability is defined as for functions of one variable, but here again using the introduced epsilon functions of two variables:

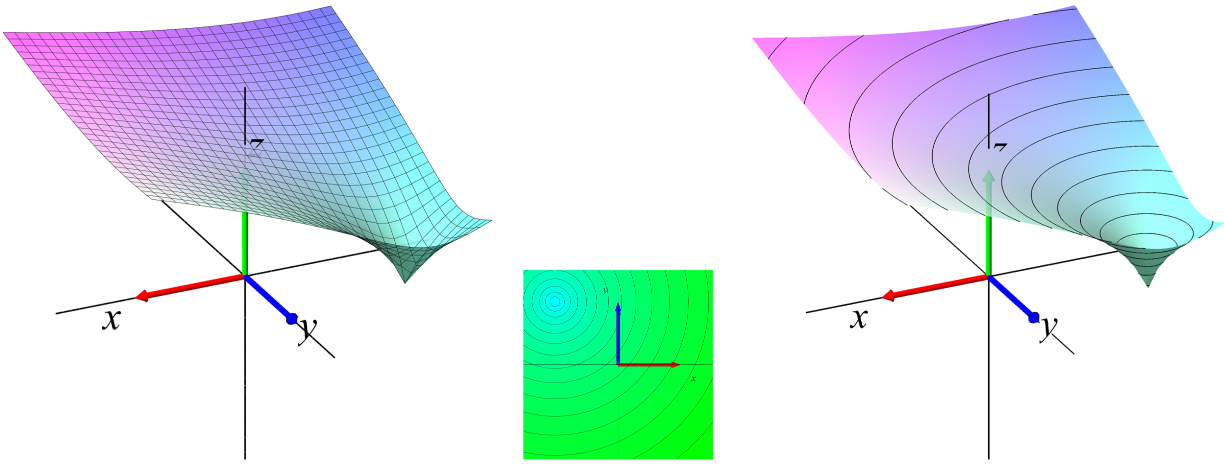

Figure 19.9: The graph, level curves, and height section curves for the square root of the distance function to the point $(x_0, y_0) = (-1, 1)$: $f(x,y) = \sqrt{\rho_{(x_0,y_0)}(x,y)}$.

|||| Definition 19.27 Differentiability and Partial Derivatives

A Function $f(x,y)$ is differentiable at $(x_0, y_0) \in \mathcal{D}(f)$ if *two* constants a and b and an epsilon function $\varepsilon_f(x - x_0, y - y_0)$ can be found such that

$$\begin{aligned} f(x,y) =& f(x_0,y_0) + a \cdot (x - x_0) + b \cdot (y - y_0) \\ &+ \rho_{(x_0,y_0)}(x,y) \cdot \varepsilon_f(x - x_0, y - y_0) \quad . \end{aligned} \tag{19-23}$$

It is the two constants a and b that we shall hereafter (when they exist, that is, if $f(x,y)$ is differentiable) call the ***partial derivatives*** of f at (x_0, y_0). They are denoted:

$$a = f'_x(x_0, y_0) \quad \text{and} \quad b = f'_y(x_0, y_0) \tag{19-24}$$

respectively. With this notation it applies that – when $f(x,y)$ is differentiable at (x_0, y_0):

$$\begin{aligned} f(x,y) =& f(x_0,y_0) + f'_x(x_0,y_0) \cdot (x - x_0) + f'_y(x_0,y_0) \cdot (y - y_0) \\ &+ \rho_{(x_0,y_0)}(x,y) \cdot \varepsilon_f(x - x_0, y - y_0) \quad . \end{aligned} \tag{19-25}$$

|||| Definition 19.28 Partial Derivatives of Partial Derivatives

If $f(x,y)$ is differentiable at all points (x_0, y_0) on a given open subset of $\mathcal{D}(f) \subset \mathbb{R}^2$ we say that $f(x,y)$ is differentiable on the whole subset. We then often write the partial derivatives of $f(x,y)$ in the following way, because they are in themselves functions of the two *variables* (x_0, y_0):

$$f'_x(x,y) = \frac{\partial}{\partial x} f(x,y) = \frac{\partial f}{\partial x}(x,y) \quad \text{and} \quad f'_y(x,y) = \frac{\partial}{\partial y} f(x,y) = \frac{\partial f}{\partial y}(x,y) \quad . \tag{19-26}$$

If these partial derivatives of $f(x,y)$ are themselves differentiable, we can continue and find the corresponding ***partial derivatives of the partial derivatives***, etc. They are named as follows:

$$\begin{aligned} f''_{xx}(x,y) &= \frac{\partial}{\partial x} f'_x(x,y) = \frac{\partial^2 f}{\partial x \partial x}(x,y) \\ f''_{xy}(x,y) &= \frac{\partial}{\partial y} f'_x(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) \\ f''_{yx}(x,y) &= \frac{\partial}{\partial x} f'_y(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ f''_{yy}(x,y) &= \frac{\partial}{\partial y} f'_y(x,y) = \frac{\partial^2 f}{\partial y \partial y}(x,y) \quad . \end{aligned} \tag{19-27}$$

How do we then in practice find the partial derivative of a given function $f(x,y)$, e.g. $f(x,y) = x \cdot \sin(y)$? It is not difficult:

|||| Theorem 19.29 Auxiliary Functions Give Partial Derivatives

The partial derivatives of a function $f(x,y)$ that is differentiable at (x_0,y_0) can be found by ordinary differentiation of two functions of one variable. Put

$$\begin{aligned} f_1(x) &= f(x,y_0) \\ f_2(y) &= f(x_0,y) \quad . \end{aligned} \tag{19-28}$$

Then the functions $f_1(x)$ and $f_2(y)$ are both functions of one variable, x and y respectively, and they are both differentiable at x_0 and y_0, respectively, and the derivatives are exactly the partial derivatives:

$$f'_x(x_0,y_0) = f'_1(x_0) \quad \text{and} \quad f'_y(x_0,y_0) = f'_2(y_0) \tag{19-29}$$

In other words: By introducing the two auxiliary functions of one variable, $f_1(x)$ and $f_2(y)$, we get the partial derivatives of $f(x,y)$ by finding the ordinary derivatives of $f_1(x)$ and $f_2(y)$ at x_0 and y_0, respectively.

|||| Proof

If we put $y = y_0$ everywhere in (19-25) we get:

$$\begin{aligned} f_1(x) = f(x,y_0) = &f_1(x_0) + f'_x(x_0,y_0)\cdot(x-x_0) \\ &+ \rho_{(x_0,y_0)}(x,y_0)\cdot\varepsilon_f(x-x_0,0) \quad , \end{aligned} \tag{19-30}$$

such that the coefficient to the factor $(x-x_0)$ is exactly $f'_x(x_0,y_0)$. Therefore we first read that $f_1(x)$ is differentiable at x_0 and second, that $f'_1(x_0) = f'_x(x_0,y_0)$. And this was the first half of what we should prove; the other half – concerning $f'_2(y_0) = f'_y(x_0,y_0)$ – is proved in an entirely similar way.

■

|||| Example 19.30 Determination of Partial Derivatives

We will determine the partial derivatives of the function $f(x,y) = 3\,x^2 + 7\,y^3 + 10\,x\,y^7$ at every point (x_0,y_0) at $\mathbb{R}^2$. First we state the two auxiliary functions, $f_1(x) = f(x,y_0)$ and $f_2(y) = f(x_0,y)$:

$$\begin{aligned} f_1(x) &= 3\,x^2 + 7\,y_0^3 + 10\,x\,y_0^7 \\ f_2(y) &= 3\,x_0^2 + 7\,y^3 + 10\,x_0\,y^7 \quad . \end{aligned} \tag{19-31}$$

The two auxiliary functions have the derivatives, respectively:

$$\begin{aligned} f_1'(x) &= 6\,x + 0 + 10\,y_0^7 \quad \text{since } y_0 \text{ is a constant here,} \\ f_2'(y) &= 0 + 21\,y^2 + 70\,x_0\,y^6 \quad \text{since } x_0 \text{ is a constant} \quad . \end{aligned} \tag{19-32}$$

From this we then get the differential equations of the auxiliary functions at x_0 and y_0, respectively:

$$\begin{aligned} f_1'(x_0) &= 6\,x_0 + 10\,y_0^7 = f_x'(x_0, y_0) \\ f_2'(y_0) &= 21\,y_0^2 + 70\,x_0\,y_0^6 = f_y'(x_0, y_0) \quad . \end{aligned} \tag{19-33}$$

From this we get generally, i.e. for all (x, y) in $\mathbb{R}^2$:

$$\begin{aligned} f_x'(x, y) &= 6\,x + 10\,y^7 \\ f_y'(x, y) &= 21\,y^2 + 70\,x\,y^6 \quad . \end{aligned} \tag{19-34}$$

|||| Exercise 19.31

Show (in the same way as for differentiable functions of one variable), that if $f(x, y)$ is differentiable at (x_0, y_0) then the two constants a and b, that is the constants $f_x'(x_0, y_0)$ and $f_y'(x_0, y_0)$, are well-defined in the following sense: Two different pairs of constants a_1, b_1 and a_2, b_2 and two epsilon functions ε_1 and ε_2 do *not* exist such that the following apply simultaneously:

$$\begin{aligned} f(x, y) =& f(x_0, y_0) + a_1 \cdot (x - x_0) + b_1 \cdot (y - y_0) \\ & + \rho_{(x_0,y_0)}(x, y) \cdot \varepsilon_1(x - x_0, y - y_0) \\ f(x, y) =& f(x_0, y_0) + a_2 \cdot (x - x_0) + b_2 \cdot (y - y_0) \\ & + \rho_{(x_0,y_0)}(x, y) \cdot \varepsilon_2(x - x_0, y - y_0) \quad . \end{aligned} \tag{19-35}$$

|||| Exercise 19.32

Determine all partial derivatives of the partial derivatives at every point (x, y) of the function $f(x, y) = 3\,x^2 + 7\,y^3 + 10\,x\,y^7$.

|||| Exercise 19.33

Determine all partial derivatives of the partial derivatives at every point (x, y) of the function $f(x, y) = \cos(3x) \cdot \sin(3y)$.

The careful problem solver will have observed (e.g. in the exercises above) that $f''_{xy}(x,y)$ and $f''_{yx}(x,y)$ are typically equal! This is not a coincidence – as a rule it suffices to calculate only one of the two double derivatives:

|||| Theorem 19.34 The Mixed Double Derivatives Are Equal

If all 4 double derivatives of a given function $f(x,y)$ are continuous on an open set then, on the whole set, we have:

$$f''_{xy}(x,y) = f''_{yx}(x,y) \quad . \tag{19-36}$$

|||| Exercise 19.35

Show (in the same way as for functions of one variable) that: If a function $f(x,y)$ of two variables is differentiable at a point (x_0,y_0), then the function is also continuous at this point. Show also using an example that if a function is continuous at (x_0,y_0) then it need not be differentiable at (x_0,y_0). Consider e.g. $\rho_{(x_0,y_0)}(x,y)$.

The observant reader may also have noticed that one problem has not been addressed: If $f(x,y)$ is differentiable at a point (x_0,y_0) then the partial derivatives as a consequence of the differentiability exist and they can be determined using the differentiable auxiliary functions $f_1(x)$ and $f_2(y)$. We now have every reason to ask: if the two auxiliary functions exist for a given function $f(x,y)$, and if they prove to be differentiable at x_0 and y_0, respectively, does it then mean that $f(x,y)$ is differentiable at (x_0,y_0)?

The following theorem sheds light on this question:

|||| Theorem 19.36 From Partial Derivatives to Differentiability

If $f(x,y)$ has partial derivatives (found using the auxiliary functions $f_1(x)$ and $f_2(y)$) on an open set A containing (x_0,y_0), and if the partial derivatives of $f(x,y)$ both are continuous on A, then $f(x,y)$ is differentiable.

That an extra condition in Theorem 19.36 is needed follows from the example below:

|||| Example 19.37 Differentiable Auxiliary Functions are Not Enough

We consider the 0-extension of the following function:

$$f(x,y) = \frac{x\,y^2}{x^2+y^4} \quad . \tag{19-37}$$

This function is not differentiable at $(0,0)$ – it is not even continuous at $(0,0)$ (!) (Why not?) Nevertheless the two auxiliary functions exist

$$\begin{aligned} f_1(x) &= f(x,0) = 0 \\ f_2(y) &= f(0,y) = 0 \quad , \end{aligned} \tag{19-38}$$

and as can be seen they are both differentiable at $(0,0)$. Even though the auxiliary functions are differentiable the actual function itself needs not be differentiable.

19.7 The Approximating First-Degree Polynomial

As for functions of one variable we can truncate the expression in Equation (19-25) simply by removing the 'epsilon function part' and we are then left with a first-degree polynomial of the two variables x and y:

$$P_{1,(x_0,y_0)}(x,y) = f(x_0,y_0) + f'_x(x_0,y_0)\cdot(x-x_0) + f'_y(x_0,y_0)\cdot(y-y_0) \quad . \tag{19-39}$$

The function $P_{1,(x_0,y_0)}(x,y)$ is called the approximating first-degree polynomial for $f(x,y)$ with the development point (x_0,y_0).

Note that $P_{1,(x_0,y_0)}(x,y)$ really *is* a first-degree polynomial of the two variables x and y because they appear with exponent 1 at the most and all other factors and addends are constants.

|||| Example 19.38 Paraboloid with a Tangent Plane

We consider the function

$$f(x,y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2 \quad . \tag{19-40}$$

Then

$$\begin{aligned} f'_x(x_0,y_0) &= -x_0 \\ f'_y(x_0,y_0) &= -y_0 \quad , \end{aligned} \tag{19-41}$$

such that

$$\begin{aligned} P_{1,(x_0,y_0)}(x,y) &= f(x_0,y_0) - x_0 \cdot (x - x_0) - y_0 \cdot (y - y_0) \\ &= 1 - \frac{1}{2}x_0^2 - \frac{1}{2}y_0^2 - x \cdot x_0 + x_0^2 - y \cdot y_0 + y_0^2 \\ &= 1 + \frac{1}{2}x_0^2 + \frac{1}{2}y_0^2 - x \cdot x_0 - y \cdot y_0 \quad . \end{aligned} \tag{19-42}$$

In particular with the development point $(x_0, y_0) = (1, -1)$ we obtain:

$$P_{1,(1,-1)}(x,y) = y - x + 2 \quad . \tag{19-43}$$

See Figure 19.10, where the graph of this approximating first-degree polynomial for $f(x,y)$ is plotted together with the graph for the function itself. The height-section curves and the contour lines are also shown for both functions. At and around the development point (marked) the two functions are very similar - the graph of the approximating first-degree polynomial with development point (x_0, y_0) evidently deserves to be called the ***tangent plane*** to the graph of $f(x,y)$ at the point $(x_0, y_0, f(x_0, y_0))$.

|||| Definition 19.39 The Tangent Plane to the Graph of a Function of Two Variables

Given a differentiable function $f(x,y)$. The tangent plane to the graph of $f(x,y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is given by the equation:

$$z = P_{1,(x_0,y_0)}(x,y) \quad , \tag{19-44}$$

where the right hand side is the approximating polynomial of the first degree for $f(x,y)$ with the development point (x_0, y_0).

|||| Exercise 19.40

Determine the approximating first-degree polynomial for the following function at every point (x_0, y_0):

$$f(x,y) = \cos(3x) \cdot \sin(3y) \quad . \tag{19-45}$$

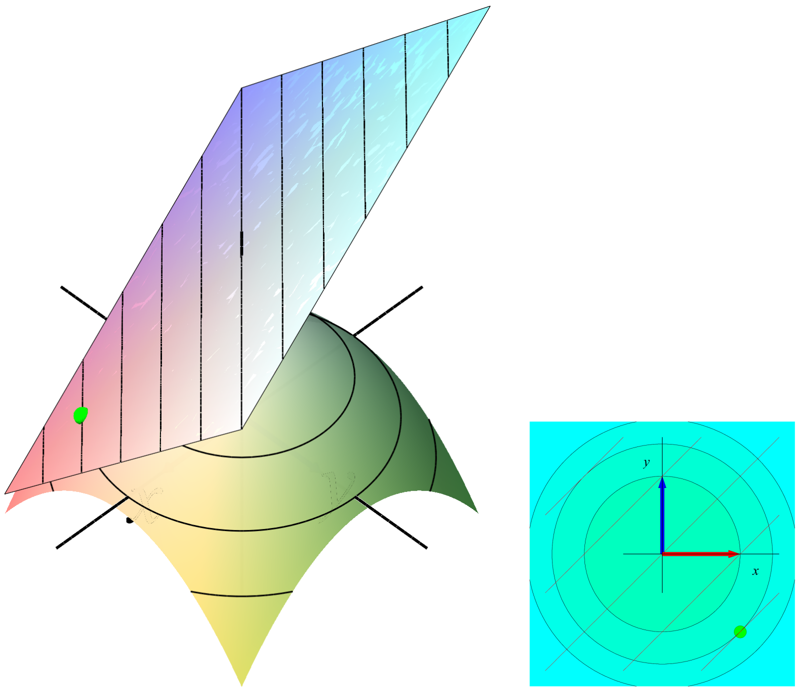

Figure 19.10: The tangent plane approximates the graph above the development point and the contour lines for the approximating first-degree polynomial approximate the contour lines for the function near the development point in the (x, y) plane

19.8 Partial Derivatives of Composite Functions

Composite functions of two variables appear in a great many applications, from GPS technology to geology and thermodynamics.

A composite function of two variables is typically constructed as follows: Let $f(x, y)$ be a function of two variables where $(x, y) \in \mathcal{D}(f) \subset \mathbb{R}^2$, and let $p(u, v)$ and $q(u, v)$ be *two* other functions of two variables, where we then assume that $(u, v) \in \mathcal{D}(p) \cap \mathcal{D}(q) \subset \mathbb{R}^2$. If we further assume that (u, v) belongs to a subset A of $\mathbb{R}^2$ where it holds that the *values* of $p(u, v)$ and $q(u, v)$ lie in the domain of $f(x, y)$ in the sense that $(p(u, v), q(u, v)) \in \mathcal{D}(f)$ for $(u, v) \in A$, then the composite function

$$h(u, v) = f(p(u, v), q(u, v)) \quad \text{is well-defined for all} \quad (x, y) \in A \quad . \tag{19-46}$$

Usually we only consider composite functions that are defined on the whole plane, so that $A = \mathbb{R}^2$.

|||| Example 19.41 Composite Functions of Two Variables

Let $f(x,y)$, $p(u,v)$ and $q(u,v)$ be determined by the functions stated below. Then the corresponding composite functions $h(u,v)$ are found by setting $x = p(u,v)$, $y = q(u,v)$ and $h(u,v) = f(x,y) = f(p(u,v), q(u,v))$ in the respective domains (they are not stated here):

$$
\begin{array}{llll}
f(x,y) = x + y & , \; p(u,v) = 2u \cdot v & , \; q(u,v) = u^2 + v^2 & , \; h(u,v) = (u+v)^2 \\
f(x,y) = y \cdot \mathrm{e}^x & , \; p(u,v) = \ln(uv) & , \; q(u,v) = 1/uv & , \; h(u,v) = 1 \\
f(x,y) = \sqrt{x+y} & , \; p(u,v) = u^4 & , \; q(u,v) = 8\,u^4 & , \; h(u,v) = 3 \cdot u^2
\end{array}
\tag{19-47}
$$

|||| Theorem 19.42 The Chain Rule in the Plane

Let $f(x,y)$, $p(u,v)$ and $q(u,v)$ be three differentiable functions - each a function of two variables. Let $h(u,v)$ denote the composite function

$$h(u,v) = f(p(u,v), q(u,v)) \quad . \tag{19-48}$$

Then the partial derivatives of $h(u,v)$ can be expressed using the partial derivatives of $f(x,y)$, the partial derivatives of $p(u,v)$ and the partial derivatives of $q(u,v)$. We will express the partial derivatives of $h(u,v)$ at (u_0, v_0), so we put $x_0 = p(u_0, v_0)$ and $y_0 = q(u_0, v_0)$.
We then have:

$$
\begin{aligned}
h'_u(u_0,v_0) &= f'_x(x_0,y_0) \cdot p'_u(u_0,v_0) + f'_y(x_0,y_0) \cdot q'_u(u_0,v_0) \\
h'_v(u_0,v_0) &= f'_x(x_0,y_0) \cdot p'_v(u_0,v_0) + f'_y(x_0,y_0) \cdot q'_v(u_0,v_0) \quad .
\end{aligned}
\tag{19-49}
$$

|||| Proof

The result follows precisely the same recipe as the proof for differentiation of the composite function $f(g(x))$ of one variable – there are only somewhat more constants and functions to manage.

■

|||| Exercise 19.43

Determine the partial derivatives at every point (u, v) of every single one of the composite functions below – partly by computing these directly from the given explicit expression of $h(u, v)$ and partly by using the chain rule and the partial derivatives of the ingredients in $h(u, v)$, that is, the partial derivatives of $f(x, y)$, $p(u, v)$ and $q(u, v)$.

$$\begin{array}{llllllll} f(x,y) = x + y & , & p(u,v) = 2u \cdot v & , & q(u,v) = u^2 + v^2 & , & h(u,v) = (u+v)^2 \\ f(x,y) = y \cdot e^x & , & p(u,v) = \ln(uv) & , & q(u,v) = 1/uv & , & h(u,v) = 1 \\ f(x,y) = \sqrt{x+y} & , & p(u,v) = u^4 & , & q(u,v) = 8\,u^4 & , & h(u,v) = 3 \cdot u^2 \end{array} \tag{19-50}$$

Note that each of the partial derivatives of the composite function $h(u, v) = f(p(u, v), q(u, v))$ at a point (u_0, v_0) can efficiently be expressed by the dot product in which a common factor appears, viz. the vector $\left(f'_x(x_0, y_0)\,,\; f'_y(x_0, y_0)\right)$ where $x_0 = p(u_0, v_0)$ and $y_0 = q(u_0, v_0)$:

$$\begin{aligned} h'_u(u_0, v_0) &= \left(f'_x(x_0, y_0)\,,\; f'_y(x_0, y_0)\right) \cdot \left(p'_u(u_0, v_0)\,,\; q'_u(u_0, v_0)\right) \\ h'_v(u_0, v_0) &= \left(f'_x(x_0, y_0)\,,\; f'_y(x_0, y_0)\right) \cdot \left(p'_v(u_0, v_0)\,,\; q'_v(u_0, v_0)\right) \quad . \end{aligned}$$

19.8.1 Gradient Vectors

|||| Definition 19.44 Gradient Vectors

Let $f(x, y)$ denote a differentiable function of two variables. The partial derivatives of $f(x, y)$ at a point (x_0, y_0) define the ***gradient vector of*** $f(x, y)$ at (x_0, y_0) in the following way:

$$\nabla f(x_0, y_0) = \left(f'_x(x_0, y_0)\,,\; f'_y(x_0, y_0)\right) \quad . \tag{19-51}$$

Therefore in this way we have defined a quite special vector at every point, where $f(x, y)$ is differentiable:

$$\nabla f(x, y) = \left(f'_x(x, y)\,,\; f'_y(x, y)\right) \quad . \tag{19-52}$$

If we at any point (x_0, y_0) draw the gradient vector $\nabla f(x_0, y_0)$ of $f(x,y)$ in the plane $\mathbb{R}^2$ we have constructed the ***gradient vector field*** of $f(x,y)$.

|||| Exercise 19.45

Determine the gradient vector field at every point (x,y) of each of the following functions in their respective definition sets: $f(x,y) = x + y$, $f(x,y) = y \cdot e^x$ and $f(x,y) = \rho^2_{x_0,y_0}(x,y)$.

Using the gradient vector field $\nabla f(x,y)$ of $f(x,y)$ we can now formulate the partial derivative of the composite function $h(u,v) = f(p(u,v), q(u,v))$ a bit smarter:

|||| Theorem 19.46 The Chain Rule Expressed Using the Gradient of $f(x,y)$

Let $f(x,y)$, $p(u,v)$, and $q(u,v)$ be three differentiable functions - each of two variables. Let $h(u,v)$ denote the composite function

$$h(u,v) = f(p(u,v), q(u,v)) \quad . \tag{19-53}$$

Then the partial derivatives of $h(u_0, v_0)$ with respect to u and v at (u_0, v_0) can be expressed by the gradient of $f(x,y)$ at $(x_0, y_0) = (\,p(u_0,v_0), q(u_0,v_0)\,)$:

$$\begin{aligned} h'_u(u_0,v_0) &= \nabla f(x_0,y_0) \cdot (p'_u(u_0,v_0), q'_u(u_0,v_0)) \\ h'_v(u_0,v_0) &= \nabla f(x_0,y_0) \cdot (p'_v(u_0,v_0), q'_v(u_0,v_0)) \quad . \end{aligned} \tag{19-54}$$

19.8.2 The Chain Rule 'Along' Curves in the Plane

If the functions $p(u,v)$ and $q(u,v)$ only depend on one of the variables u we can of course denote the functions by $p(u)$ and $q(u)$, respectively. The composite function $h(u) = f(p(u), q(u))$ – where $f(x,y)$ is a given differentiable function in the plane – then states the function values that $f(x,y)$ assumes *along the curve* in the plane given by the two functions $p(u)$ and $q(u)$:

|||| Definition 19.47 Parametrized Curves in the Plane

A ***curve in the plane*** consists of a set of points $\mathcal{C}$ in the plane, which we assume to be given by two functions $p(u)$ and $q(u)$ in the following way:

$$\mathcal{C}_{\mathbf{r}} \quad : \quad \mathbf{r}(u) = (p(u), q(u)) \quad \text{where} \quad u \in]\alpha, \beta[\subset \mathbb{R} \quad . \tag{19-55}$$

This means that the curve is the parametrized set of points with the position vector $\mathbf{r}(u)$ and the parameter u in a given interval. The curve is ***differentiable*** if the two functions $p(u)$ and $q(u)$ are both differentiable on the whole interval $]\alpha, \beta[$.

The parametrized curve is said to be ***regular*** if $\mathbf{r}'(u) \neq \mathbf{0}$ for all $u \in]\alpha, \beta[$.

|||| Theorem 19.48 Tangent to a Curve

Let $\mathcal{C}_{\mathbf{r}}$ denote a differentiable curve with the parametric representation $\mathbf{r}(u)$ and suppose that $\mathbf{r}'(u_0) \neq (0, 0)$. The tangent L_{u_0} (through the point $\mathbf{r}(u_0)$) to $\mathcal{C}_{\mathbf{r}}$ is then given by the following parametric representation:

$$L_{u_0} \quad : \quad \mathbf{T}(t) = \mathbf{r}(u_0) + t \cdot \mathbf{r}'(u_0) \quad \text{for} \quad t \in \mathbb{R} \quad . \tag{19-56}$$

|||| Proof

We content ourselves with considering the case where $p(u)$ is very elementary: $p(u) = u$. Here $\mathcal{C}_{\mathbf{r}}$ is simply the graph of the function $q(x)$ in the (x, y) plane. The graph of the function has a tangent at the point $(x_0, q(x_0))$, given by the well-known expression: $y = q(x_0) + q'(x_0)(x - x_0)$. A parametric expression of this tangent is therefore:

$$\begin{aligned} \mathbf{T}(t) &= (x_0, q(x_0)) + t \cdot (1, q'(x_0)) \\ &= (p(u_0), q(u_0)) + t \cdot (p'(u_0), q'(u_0)) \quad \text{because } x = p(u) = u \\ &= \mathbf{r}(u_0) + t \cdot \mathbf{r}'(u_0) \quad , \end{aligned} \tag{19-57}$$

and this is what we should see.

■

Now we can investigate how the chain rule looks and is simplified along the parametrized curves – we only have to 'use' the general chain rule on the two v-independent functions $p(u)$ and $q(u)$:

|||| Theorem 19.49 The Chain Rule Along Curves

Let $\mathbf{r}(u) = (p(u), q(u))$ be a differentiable parametrized curve in the (x, y) plane. A differentiable function $f(x, y)$ then assumes the values $h(u) = f(p(u), q(u)) = f(\mathbf{r}(u))$ along the curve. The composite function $h(u)$ of the one variable u is differentiable, and

$$h'(u) = \boldsymbol{\nabla} f(p(u), q(u)) \cdot (p'(u), q'(u)) = \boldsymbol{\nabla} f(\mathbf{r}(u)) \cdot \mathbf{r}'(u) \quad . \tag{19-58}$$

|||| Proof

The result follows directly from the upper equation in (19-54). Note that since the functions $h(u)$, $p(u)$ and $q(u)$ do not depend on v, the lower equation in (19-54) is reduced to $0 = 0$.

■

Motivated by this simple expression for the derivative of the function $f(x, y)$ along a parametrized curve $\mathbf{r}(u)$ we introduce the ***directional derivative of a function*** in the direction from a given point (x_0, y_0) given by the unit vector $\mathbf{e}$:

|||| Definition 19.50 The Directional Derivative

The ***directional derivative*** of $f(x, y)$ at the point (x_0, y_0) in the direction with the unit directional vector $\mathbf{e}$ is denoted by $f'((x_0, y_0); \mathbf{e})$ and is given by the dot product:

$$f'((x_0, y_0); \mathbf{e}) = \boldsymbol{\nabla} f(x_0, y_0) \cdot \mathbf{e} \quad . \tag{19-59}$$

The chain rule along curves can now be formulated using the directional derivative:

|||| Theorem 19.51 The Chain Rule Along Curves Expressed by the Directional Derivative

The function $f(x,y)$ assumes the values $h(u) = f(p(u), q(u)) = f(\mathbf{r}(u))$ along $\mathbf{r}(u)$. If $\mathbf{r}(u)$ is a regular parametric representation of the curve we get the derivative of $h(u)$:

$$h'(u) = \boldsymbol{\nabla} f(p(u), q(u)) \cdot (p'(u), q'(u)) = f'((x_0, y_0); \mathbf{e}) \cdot |\mathbf{r}'(u)| \quad . \tag{19-60}$$

Note that the derivative of the composite function $h(u) = f(\mathbf{r}(u))$ at a point on the curve $\mathbf{r}(u)$ only depends on the tangent vector to the curve at the point – $h'(u)$ is independent of the 'rest' of the curve.

|||| Example 19.52 Directional Derivative

The function $f(x,y) = 2\,x^2 + 3\,y^2$ has the partial derivative

$$f'_x(x,y) = 4\,x \quad , \quad f'_y(x,y) = 6\,y \quad , \tag{19-61}$$

and therefore the gradient field

$$\boldsymbol{\nabla} f(x,y) = (4\,x\,,\; 6\,y) \tag{19-62}$$

The directional derivative of $f(x,y)$ at the point $(x_0, y_0) = (1,1)$ in the direction determined by $\mathbf{e}(\theta) = (\cos(\theta), \sin(\theta))$, where $\theta \in [0, 2\pi]$, is therefore:

$$f'((1,1); (\cos(\theta), \sin(\theta))) = 4\cos(\theta) + 6\sin(\theta) \quad . \tag{19-63}$$

19.8.3 The Chain Rule 'Along' Contour Lines of a Function

A curve $(p(u), q(u))$ that is a level curve of a function $f(x,y)$ gives rise to a particularly simple – and very important – version of the chain rule:

|||| Theorem 19.53 The Chain Rule Along Contour Lines

Let $\mathbf{r}(u) = (p(u), q(u))$ be a parametrized curve in the (x, y) plane. Assume that the curve is a contour line of $f(x, y)$, i.e. $f(x, y)$ is equal to a constant c along the whole curve,

$$f(p(u), q(u)) = c \quad \text{for all } u \quad . \tag{19-64}$$

Then

$$\nabla f(\mathbf{r}(u)) \cdot \mathbf{r}'(u) = 0 \quad . \tag{19-65}$$

In other words: The gradient of a function $f(x, y)$ is, at every point where it is not $\mathbf{0}$, perpendicular to the level curve of $f(x, y)$ passing through the point:

$$\nabla f(\mathbf{r}(u)) \perp \mathbf{r}'(u) \quad . \tag{19-66}$$

|||| Proof

The function $h(u) = f(p(u), q(u)) = c$ evidently has $h'(u) = 0$ and since Theorem 19.49 gives $h'(u) = \nabla f(p(u), q(u)) \cdot (p'(u), q'(u))$ we arrive at the predicted result.

■

Note that Theorem 19.53 more than indicates another theorem, which we will not show here though: If a differentiable function $f(x, y)$ has a proper gradient vector field – i.e. if $\nabla f(x, y) \neq (0, 0)$ for all (x, y) in a given open subset A of the domain – then all the contour lines of $f(x, y)$ in A consist of nice curves, i.e. they can be parametrized with differentiable vector functions $\mathbf{r}(u)$ as above.

19.9 Summary

Functions of two variables are treated in this eNote after the same 'procedure' as functions of one variable: The fact of the domains here being subsets of the plane thus results in new concepts and symbols that can be used to describe the new general sets in the plane. We have introduced and illustrated the concepts: Open, closed, bounded, and star-shaped sets together with the definition of the interior of a set and the exterior of a set. Graphs and contour lines of a function are important tools for the understanding of how function values $f(x,y)$ 'behave' depending on the position of the point (x,y) within the domain. Continuity and differentiability (or lack of these properties) for a function can often be inspected by constructing or drawing the graph of the function or by drawing the contour lines of the function.

- The level set corresponding to the value c of the function $f(x,y)$ is given by

$$\mathcal{K}_c(f) = \{ \, (x,y) \in \mathcal{D}(f) \mid f(x,y) = c \, \} \quad . \tag{19-67}$$

- A function $f(x,y)$ is continuous (x_0, y_0) if $f(x,y) - f(x_0,y_0)$ is an epsilon function of $x - x_0, y - y_0)$, i.e.

$$f(x,y) = f(x_0,y_0) + \varepsilon_f(x - x_0, y - y_0) \quad . \tag{19-68}$$

- A function $f(x,y)$ is differentiable with the partial derivatives $f'_x(x_0,y_0)$ and $f'_x(x_0,y_0)$ at (x_0,y_0) if

$$\begin{aligned} f(x,y) = & f(x_0,y_0) + f'_x(x_0,y_0)\cdot(x - x_0) + f'_y(x_0,y_0)\cdot(y-y_0) \\ & + \rho_{(x_0,y_0)}(x,y)\cdot\varepsilon_f(x-x_0,y-y_0) \quad . \end{aligned} \tag{19-69}$$

- The partial derivative of $f(x,y)$ at a point (x_0,y_0) can be found by computing the ordinary derivatives of the two auxiliary functions $f_1(x) = f(x,y_0)$ and $f_2(y) = f(x_0,y)$ at x_0 and y_0, respectively:

$$f'_x(x_0,y_0) = f'_1(x_0) \quad \text{and} \quad f'_y(x_0,y_0) = f'_2(y_0) \tag{19-70}$$

- The approximating first-degree polynomial for $f(x,y)$ with development point (x_0,y_0) is given by:

$$P_{1,(x_0,y_0)}(x,y) = f(x_0,y_0) + f'_x(x_0,y_0)\cdot(x-x_0) + f'_y(x_0,y_0)\cdot(y-y_0) \quad . \tag{19-71}$$

- The tangent plane to the graph of $f(x,y)$ at the point $(x_0, y_0, f(x_0,y_0)$ is given by:

$$z = P_{1,(x_0,y_0)}(x,y) \quad . \tag{19-72}$$

- The gradient vector field of a function $f(x,y)$ is given by:

$$\nabla f(x_0,y_0) = \Big(f'_x(x_0,y_0)\,,\; f'_y(x_0,y_0)\Big) \quad . \tag{19-73}$$

- The directional derivative of $f(x,y)$ at the point (x_0,y_0) in the direction given by a unit vector $\mathbf{e}$ is given by:

$$f'((x_0,y_0);\mathbf{e}) = \nabla f(x_0,y_0)\cdot\mathbf{e} \quad . \tag{19-74}$$