# eNote 18

# Linear Second-Order Differential Equations with Constant Coefficients

*Following eNotes 16 and 17 about differential equations, we now present this eNote about second-order differential equations. Parts of the proofs closely follow the preceding notes and a knowledge of these notes is therefore a prerequisite. In addition, complex numbers are used.*

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Linear second-order differential equations with constant coefficients look like this:

$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t), \quad t \in I, q : I \to \mathbb{R} \tag{18-1}$$

$a_0, a_1 \in \mathbb{R}$ are constant coefficients of $x(t)$ and $x'(t)$, respectively. The right hand side $q(t)$ is a continuous real function, with the domain being an interval $I$ (which could be all of $\mathbb{R}$). The equation is called homogeneous if $q(t) = 0$ for all $t \in I$ and otherwise inhomogeneous.

The left hand side is linear in $x$, i.e., the map $f : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by

$$f(\,x(t)\,) = x''(t) + a_1 x'(t) + a_0 x(t) \tag{18-2}$$

satisfies the linearity requirements $L_1$ and $L_2$. The method used in this eNote for solving the inhomogeneous equation exploits this linearity.

|||| **Method 18.1 Solutions and their structure**

1. The general solution $L_{hom}$ for a homogeneous linear second-order differential equation
$$x''(t) + a_1 x'(t) + a_0 x(t) = 0, \quad t \in I \tag{18-3}$$
where $a_0, a_1 \in \mathbb{R}$, can be determined using Theorem 18.2.

2. The general solution set $L_{inhom}$ for an inhomogeneous linear second-order differential equation
$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t), \quad t \in I\,,\, q : I \to \mathbb{R}, \tag{18-4}$$
where $a_0, a_1 \in \mathbb{R}$, can, using Theorem 12.14, be split into two:

   (a) First the general solution $L_{hom}$ to the *corresponding homogeneous equation* is determined. This is produced by setting $q(t) = 0$ in (18-4).

   (b) Then a particular solution $x_0(t)$ to (18-4) is determined e.g. by guessing. Concerning this see section 18.2.

   The general solution then has the following structure
$$L_{inhom} = x_0(t) + L_{hom}\,. \tag{18-5}$$

## 18.1 The Homogeneous Equation

We now consider the linear homogeneous second-order differential equation
$$x''(t) + a_1 x'(t) + a_0 x(t) = 0, \quad t \in \mathbb{R}\,, \tag{18-6}$$
where $a_0$ and $a_1$ are real constants. We wish to determine the general solution. This can be accomplished using exact formulas that depend on the appearance of the equation.

### |||| Theorem 18.2 Solution to the Homogeneous Equation

The homogeneous differential equation

$$x''(t) + a_1 x'(t) + a_0 x(t) = 0, \quad t \in \mathbb{R}, \tag{18-7}$$

has the so-called ***characteristic equation***

$$\lambda^2 + a_1 \lambda + a_0 = 0. \tag{18-8}$$

The type of roots to this equation determines how the general solution $L_{hom}$ to the homogeneous differential equation will appear.

- **Two different real roots** $\lambda_1$ and $\lambda_2$ yield the solution

$$x(t) = c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t}, \quad t \in \mathbb{R}. \tag{18-9}$$

- **Two complex roots** $\lambda = \alpha \pm \beta i$, with $\mathrm{Im}(\lambda) = \pm\beta \neq 0$, yield the real solution

$$x(t) = c_1 \mathrm{e}^{\alpha t} \cos(\beta t) + c_2 \mathrm{e}^{\alpha t} \sin(\beta t), \quad t \in \mathbb{R}. \tag{18-10}$$

- **The double root** $\lambda$ yields the solution

$$x(t) = c_1 \mathrm{e}^{\lambda t} + c_2 t \mathrm{e}^{\lambda t}, \quad t \in \mathbb{R}. \tag{18-11}$$

In all three cases the respective functions for all $c_1, c_2 \in \mathbb{R}$ constitute the general solution $L_{hom}$.

In Section 17.4 you find the theory for rewriting this type of differential equation as a system of first-order differential equations.This method works here. The system will then look like this:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{18-12}$$

where $x_1(t) = x(t)$ and $x_2(t) = x_1'(t) = x'(t)$. The problem can now be solved using the theory and methods outlined in that section.

### |||| Proof

The homogeneous second-order linear differential equation (18-7) is rewritten as a system of first-order differential equations:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{18-13}$$

where $x_1(t) = x(t)$ is the wanted solution that constitutes the general solution. The proof begins with the theorems and methods in Section 17.1. For the proof we need the eigenvalues of the system matrix $\mathbf{A}$:

$$\det(\mathbf{A} - \lambda \mathbf{E}) = \begin{vmatrix} -\lambda & 1 \\ -a_0 & -a_1 - \lambda \end{vmatrix} = \lambda^2 + a_1 \lambda + a_0 = 0, \tag{18-14}$$

which is also the characteristic equation for the differential equation. The type of roots of this equation determines the solution $x(t) = x_1(t)$, which gives the following three parts of the proof:

**First part**
The characteristic equation has two different real roots: $\lambda_1$ and $\lambda_2$. By using Method 17.4 we obtain two linearly independent solutions $\mathbf{u}_1(t) = \mathbf{v}_1 \mathrm{e}^{\lambda_1 t}$ and $\mathbf{u}_2(t) = \mathbf{v}_2 \mathrm{e}^{\lambda_2 t}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to the two eigenvalues , respectively. The general solution is then spanned by:

$$\mathbf{x}(t) = k_1 \mathbf{u}_1(t) + k_2 \mathbf{u}_2(t) = k_1 \mathrm{e}^{\lambda_1 t} \mathbf{v}_1 + k_2 \mathrm{e}^{\lambda_2 t} \mathbf{v}_2, \tag{18-15}$$

for all $k_1, k_2 \in \mathbb{R}$. The first coordinate $x_1(t) = x(t)$ is the solution wanted:

$$x_1(t) = x(t) = c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t}, \tag{18-16}$$

which for all the arbitrary constants $c_1, c_2 \in \mathbb{R}$ constitutes the general solution. $c_1$ and $c_2$ are two new arbitrary constants and they are the products of the $k$-constants and the first coordinates of the eigenvectors: $c_1 = k_1 v_{1_1}$ and $c_2 = k_2 v_{2_1}$.

**Second part**
The characteristic equation has the complex pair of roots $\lambda = \alpha + \beta i$ and $\bar{\lambda} = \alpha - \beta i$. It is possible to find the general solution using Method 17.5.

$$\begin{aligned} \mathbf{x}(t) &= k_1 \mathbf{u}_1(t) + k_2 \mathbf{u}_2(t) \\ &= k_1 \mathrm{e}^{\alpha t} \left( \cos(\beta t) \mathrm{Re}(\mathbf{v}) - \sin(\beta t) \mathrm{Im}(\mathbf{v}) \right) + k_2 \mathrm{e}^{\alpha t} \left( \sin(\beta t) \mathrm{Re}(\mathbf{v}) + \cos(\beta t) \mathrm{Im}(\mathbf{v}) \right) \\ &= \mathrm{e}^{\alpha t} \cos(\beta t) \cdot (k_1 \mathrm{Re}(\mathbf{v}) + k_2 \mathrm{Im}(\mathbf{v})) + \mathrm{e}^{\alpha t} \sin(\beta t) \cdot (-k_1 \mathrm{Im}(\mathbf{v}) + k_2 \mathrm{Re}(\mathbf{v})). \end{aligned} \tag{18-17}$$

$\mathbf{v}$ is an eigenvector corresponding to $\lambda$ and $k_1$ and $k_2$ are arbitrary constants. The first coordinate $x_1(t) = x(t)$ is the wanted solution, and is according to the above given by

$$x_1(t) = x(t) = c_1 \mathrm{e}^{\alpha t} \cos(\beta t) + c_2 \mathrm{e}^{\alpha t} \sin(\beta t). \tag{18-18}$$

For all $c_1, c_2 \in \mathbb{R}$, $x(t)$ constitutes the general solution. $c_1$ and $c_2$ are two new arbitrary constants given by $c_1 = k_1\text{Re}(v_1) + k_2\text{Im}(v_1)$ and $c_2 = -k_1\text{Im}(v_1) + k_2\text{Re}(v_1)$. $v_1$ is the first coordinate of $\mathbf{v}$.

**Third part**
The characteristic equation has the double root $\lambda$. Because of the appearance of the system matrix (the matrix is equivalent to an upper triangular matrix) it is possible to see that the geometric multiplicity of the corresponding eigenvector space is 1, and it is then possible to use Method 17.7 to find the general solution.

$$\mathbf{x}(t) = k_1\mathbf{u}_1(t) + k_2\mathbf{u}_2(t) = k_1 e^{\lambda t}\mathbf{v} + k_2(te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{b}) = e^{\lambda t}(k_1\mathbf{v} + k_2\mathbf{b}) + k_2 te^{\lambda t}\mathbf{v}, \quad (18\text{-}19)$$

where $\mathbf{v}$ is an eigenvector corresponding to $\lambda$, $\mathbf{b}$ is the solution to the system of equations $(\mathbf{A} - \lambda\mathbf{E})\mathbf{b} = \mathbf{v}$, and $k_1, k_2$ are two arbitrary constants. Taking the first coordinate we get

$$x(t) = c_1 e^{\lambda t} + c_2 te^{\lambda t}, \quad (18\text{-}20)$$

which for all $c_1, c_2 \in \mathbb{R}$ constitutes the general solution. $c_1 d$ and $c_2$ are two new arbitrary constants, given by $c_1 = k_1 v_1 + k_2 b_1$ and $c_2 = k_2 v_1$, in which $v_1$ is the first coordinate in $\mathbf{v}$, as $b_1$ is the first coordinate in $\mathbf{b}$.

All the three different cases of roots of the characteristic equation have now been treated thus proving the theorem.

Notice that it is also possible to arrive at the characteristic equation by guessing a solution to the differential equation of the form $x(t) = e^{\lambda t}$. One then gets:

$$x''(t) + a_1 x'(t) + a_0 x(t) = 0 \quad \Rightarrow \quad \lambda^2 e^{\lambda t} + a_1\lambda e^{\lambda t} + a_0 e^{\lambda t} = 0 \quad (18\text{-}21)$$

Dividing this equation by $e^{\lambda t}$, which is non-zero for all values of $t$, yields the characteristic equation.

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## |||| Example 18.3 Solution to the Homogeneous Equation

Given the homogeneous differential equation

$$x''(t) + x'(t) - 20x(t) = 0, \quad t \in \mathbb{R}, \quad (18\text{-}22)$$

which has the characteristic equation

$$\lambda^2 + \lambda - 20 = 0. \quad (18\text{-}23)$$

We wish to determine the general solution $L_{hom}$ to this homogeneous equation.

The characteristic equation has the roots $\lambda_1 = -5$ and $\lambda = 4$, since $-5 \cdot 4 = -20$ and $-(-5+4) = 1$ are the coefficients of the characteristic equation. Therefore the general solution to the homogeneous equation is

$$L_{hom} = \left\{ c_1 e^{-5t} + c_2 e^{4t}, t \in \mathbb{R} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}, \tag{18-24}$$

that has been found using 18.2.

### |||| Example 18.4 Solution to the Homogeneous Equation

A homogeneous second-order differential equation with constant coefficients is given by:

$$x''(t) - 8x'(t) + 16x(t) = 0, \quad t \in \mathbb{R}. \tag{18-25}$$

We wish to determine $L_{hom}$, the general solution to the homogeneous equation. The characteristic equation is

$$\lambda^2 - 8\lambda + 16 = 0 \Leftrightarrow (\lambda - 4)^2 = 0 \tag{18-26}$$

Thus we have the double root $\lambda = 4$, and the general solutions set is composed of the following function for all $c_1, c_2 \in \mathbb{R}$:

$$x(t) = c_1 e^{4t} + c_2 t e^{4t}, \quad t \in \mathbb{R}. \tag{18-27}$$

The result is determined using Theorem 18.2.

As can be seen from the two preceding examples it is relatively simple to determine the solution to the homogeneous equation. In addition it is possible to determine the differential equation from the solution, that is "go backwards". This is illustrated in the following example.

### |||| Example 18.5 From Solution to Equation

The solution to a differential equation is known:

$$x(t) = c_1 e^{2t}\cos(7t) + c_2 e^{2t}\sin(7t), \quad t \in \mathbb{R}, \tag{18-28}$$

which with the arbitrary constants $c_1, c_2$ constitute the general solution.

Since the solution only includes terms with arbitrary constants, the equation must be homogeneous. Furthermore it is seen that the solution structure is similar to the solution structure in equation (18-10) in Theorem 18.2. This means that the characteristic equation of the second-order differential equation has two complex roots: $\lambda = 2 \pm 7i$. The characteristic equation given these roots reads:

$$\begin{aligned}(\lambda - 2 + 7i)(\lambda - 2 - 7i) = (\lambda - 2)^2 - (7i)^2 = \\ \lambda^2 - 4\lambda + 4 + 49 = \lambda^2 - 4\lambda + 53 = 0\end{aligned} \tag{18-29}$$

Directly from coefficients of the characteristic equation we can write the differential equation as:

$$x''(t) - 4x'(t) + 53x(t) = 0, \quad t \in \mathbb{R}. \tag{18-30}$$

This can also be seen from Theorem 18.2.

## 18.2 The Inhomogeneous Equation

In this section we wish to determine a particular solution $x_0(t)$ to the inhomogeneous differential equation

$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t), \quad t \in I, q : I \to \mathbb{R}. \tag{18-31}$$

We wish to find a particular solution, because it is part of the general solution $L_{inhom}$ together with the general solution $L_{hom}$ to the corresponding homogeneous equation cf. Method 18.1.

In this eNote we do not use a specific solution formula. Instead we use different methods depending on the form of $q(t)$. In general one might guess that a particular solution $x_0(t)$ has a form that somewhat resembles $q(t)$, as will appear from the following methods. Notice that these methods cover some frequently occurring forms of $q(t)$, but certainly not all.

Furthermore the concept of *superposition* will be treated. Superposition is a basic quality of linear equations and linear differential equations. The point is to split the equation into more equations in which the left hand sides stay the same while the sum of the right hand sides is equal to the right hand side of the original equation. If the original equation has the right hand side $q(t) = \sin(2t) + 2t^2$, it may be a good idea to split the equation into two, where the right hand sides become $q_1(t) = \sin(2t)$ and $q_2(t) = 2t^2$ respectively. It is easier to determine particular solutions to the two equations. A particular solution to the original equation will then be the sum of the two particular solutions.

Finally we will introduce *the complex guess method*. The complex guess method can be used if the right hand side $q(t)$ of the equation is the real part of a simple complex expression, e.g. $q(t) = e^t \sin(3t)$ that is the real part of $-ie^{(1+3i)t}$. Solving an equation with a simple right hand side is easier, and therefore the corresponding complex equation is solved instead. The solutions to the real equation and to the corresponding complex equation are closely related.

### 18.2.1 General Solution Methods

**|||| Method 18.6 Polynomial**

Given the inhomogeneous differential equation

$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t), \quad t \in I, \tag{18-32}$$

where $q$ is an $n$-th degree polynomial. If $a_0 \neq 0$ a polynomial of degree $n$ that is a particular solution to the equation exists. In general a polynomial of degree $n+2$ at the most, that is a particular solution to the equation, exists. A particular solutions of the form mentioned is found by insertion of polynomials of a suitable degree with unknown coefficients in the left-hand side of the equation and tune this to the right-hand side $q$, cf the identity theorem for polynomials, eNote 2, Theorem 2.15.

**|||| Example 18.7 Polynomial**

Given the inhomogeneous second-order differential equation with constant coefficients

$$x''(t) - 3x'(t) + x(t) = 2t^2 - 16t + 25, \quad t \in \mathbb{R}. \tag{18-33}$$

We wish to determine a particular solution $x_0(t)$ to the inhomogeneous equation. Since the right hand side is a second degree polynomial we insert an unknown polynomial of second degree in the left-hand side of the equation and equate this with the right-hand side:

$$x_0(t) = b_2 t^2 + b_1 t + b_0, \quad t \in \mathbb{R}. \tag{18-34}$$

The coefficients are determined by substituting the expression into the differential equation together with $x_0'(t) = 2b_2 t + b_1$ og $x_0''(t) = 2b_2$.

$$\begin{aligned} 2b_2 - 3(2b_2 t + b_1) + b_2 t^2 + b_1 t + b_0 = 2t^2 - 16t + 25 \Leftrightarrow \\ (b_2 - 2)t^2 + (-6b_2 + b_1 + 16)t + (2b_2 - 3b_1 + b_0 - 25) = 0 \Leftrightarrow \\ b_2 - 2 = 0 \text{ og } -6b_2 + b_1 + 16 = 0 \text{ og } 2b_2 - 3b_1 + b_0 - 25 = 0 \end{aligned} \tag{18-35}$$

From the first equation it is evident that $b_2 = 2$, and by substituting this in the second equation we get $b_1 = -4$. Finally the last equation yields $b_0 = 9$. Therefore a particular solution to Equation (18-33) is given by

$$x_0(t) = 2t^2 - 4t + 9, \quad t \in \mathbb{R}. \tag{18-36}$$

## |||| Exercise 18.8 Polynomium

Given the following differential equation where the right-hand side is a first degree polynomial:

$$x''(t) = t + 1, \quad t \in \mathbb{R}. \tag{18-37}$$

Show that you have to go to the third degree in order to find a polynomial that is a particular solution to the equation.

## |||| Method 18.9 Trigonometric

A particular solution $x_0(t)$ to the inhomogeneous differential equation

$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t), \quad t \in I, \tag{18-38}$$

where $q(t) = a\cos(\omega t) + b\sin(\omega t)$, is of the same form:

$$x_0(t) = A\sin(\omega t) + B\cos(\omega t), \quad t \in I, \tag{18-39}$$

where $A$ and $B$ are determined by substitution of the expression for $x_0(t)$ as a solution into the inhomogeneous equation.

It is also possible to determine a particular solution to a differential equation like the one in Method 18.9 using *the complex guess method*, cf. e.g. section 18.2.3.

### |||| Example 18.10 Trigonometric

Given the differential equation

$$x''(t) + x'(t) - x(t) = -20\sin(3t) + 6\cos(3t), \quad t \in \mathbb{R}. \tag{18-40}$$

We wish to determine a particular solution $x_0(t)$. By the use of Method 18.9 a particular solution is

$$x_0(t) = A\sin(\omega t) + B\cos(\omega t) = A\sin(3t) + B\cos(3t). \tag{18-41}$$

In addition we have

$$\begin{aligned} x_0'(t) &= 3A\cos(3t) - 3B\sin(3t) \\ x_0''(t) &= -9A\sin(3t) - 9B\cos(3t) \end{aligned} \tag{18-42}$$

This is substituted into the equation

$$\begin{aligned} (-9A\sin(3t) - 9B\cos(3t)) + (3A\cos(3t) - 3B\sin(3t)) - (A\sin(3t) + B\cos(3t)) \\ = -20\sin(3t) + 6\cos(3t) \Leftrightarrow \\ (-9A - 3B - A + 20)\sin(3t) + (-9B + 3A - B - 6)\cos(3t) = 0 \Leftrightarrow \\ -9A - 3B - A + 20 = 0 \text{ og } -9B + 3A - B - 6 = 0 \end{aligned} \tag{18-43}$$

This is two equations in two unknowns. Substituting $A = -\frac{3}{10}B + 2$ from the first equation in the second yields

$$-9B + 3\left(-\frac{3}{10}B + 2\right) - B - 6 = 0 \Leftrightarrow -10B - \frac{9}{10}B = 0 \Leftrightarrow B = 0 \tag{18-44}$$

From this we get that $A = 2$, and a particular solution to the differential equation is then

$$x_0(t) = 2\sin(3t), \quad t \in \mathbb{R}. \tag{18-45}$$

Note that the number $\omega = 3$ is the same in the arguments of both cosine and sine in Example 18.10, and this is the only case that Method 18.9 facilitates. If two different numbers are present Method 18.9 does not apply, e.g. $q(t) = 3\sin(t) + \cos(10t)$. But either *superposition* or *the complex guess method* can be applied, and they will be described in section 18.2.2 and section 18.2.3, respectively.

### |||| Method 18.11 Exponential Function

A particular solution $x_0(t)$ to the inhomogeneous differential equation

$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t), \quad t \in I, \tag{18-46}$$

where $q(t) = \beta e^{\alpha t}$ og $\alpha, \beta \in \mathbb{R}$, is also an exponential function:

$$x_0(t) = \gamma e^{\alpha t}, \quad t \in I, \tag{18-47}$$

where $\gamma$ is determined by substituting the expression for $x_0(t)$ as a solution into the inhomogeneous equation. We emphasize that $\alpha$ must not be a root of the characteristic equation for the differential equation.

As commented by the end of Method 18.11 the exponent $\alpha$ must not be a root of the characteristic equation. If this is the case the guess will be a solution to the corresponding homogeneous equation c.f. Theorem 18.2. This is a “problem” for all orders of differential equations.

### |||| Example 18.12 Exponential Function

Given the differential equation

$$x''(t) + 11x'(t) + 5x(t) = -20e^{-t}, \quad t \in \mathbb{R}. \tag{18-48}$$

We wish to determine a particular solution $x_0(t)$. According to Method 18.11 a particular solution is given by $x_0(t) = \gamma e^{\alpha t} = \gamma e^{-t}$. We do not yet know whether $\alpha = -1$ is a root in the characteristic equation, but if it is possible to find $\gamma$, it is not a root. We have $x_0'(t) = -\gamma e^{-t}$ and $x_0''(t) = \gamma e^{-t}$, and this is substituted into the differential equation:

$$\gamma e^{-t} + 11(-\gamma e^{-t}) + 5\gamma e^{-t} = -20e^{-t} \Leftrightarrow -5\gamma = -20 \Leftrightarrow \gamma = 4 \tag{18-49}$$

Thus we have succeeded in finding $\gamma$, and therefore we have a particular solution to the differential equation:

$$x_0(t) = 4e^{-t}, \quad t \in \mathbb{R}. \tag{18-50}$$

#### |||| Method 18.13 Exponential Function Belonging to $L_{hom}$

A particular solution $x_0(t)$ to the inhomogeneous differential equation

$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t), \quad t \in I, \tag{18-51}$$

where $q(t) = \beta e^{\lambda t}$, $\beta \in \mathbb{R}$ and $\lambda$ is a root in the characteristic equation of the differential equation, has the following form:

$$x_0(t) = \gamma t e^{\lambda t}, \quad t \in I, \tag{18-52}$$

where $\gamma$ is determined by substitution of the expression for $x_0(t)$ as a solution into the inhomogeneous equation.

#### |||| Example 18.14 Exponential Function Belonging to $L_{hom}$

Given the differential equation

$$x''(t) - 7x'(t) + 10x(t) = -3e^{2t}, \quad t \in \mathbb{R}. \tag{18-53}$$

We wish to determine a particular solution. First we try to use Method 18.11, and guess a solution of the form $x_0(t) = \gamma e^{\alpha t} = \gamma e^{2t}$. One then has $x_0'(t) = 2\gamma e^{2t}$ and $x_0''(t) = 4\gamma e^{2t}$, which by substitution into the equation gives

$$4\gamma e^{2t} - 7 \cdot 2\gamma e^{2t} + 10\gamma e^{2t} = -3e^{2t} \Leftrightarrow 0 = -3 \tag{18-54}$$

It is seen that $\gamma$ does not appear in the last equation, and that the equation otherwise is false. Therefore $\alpha = \lambda$ must be a root in the characteristic equation. The characteristic equation looks like this:

$$\lambda^2 - 7\lambda + 10 = 0 \tag{18-55}$$

This second degree equation has the roots 2 and 5, since $2 \cdot 5 = 10$ and $-(2+5) = -7$. It is true that $\alpha = 2$ is a root.

Consequently we use Method 18.13, and we guess a solution of the form $x_0(t) = \gamma t e^{\lambda t} = \gamma t e^{2t}$. We then have

$$\begin{aligned} x_0'(t) &= \gamma e^{2t} + 2\gamma t e^{2t} \\ x_0''(t) &= 2\gamma e^{2t} + 2\gamma e^{2t} + 4\gamma t e^{2t} = 4\gamma e^{2t} + 4\gamma t e^{2t} \end{aligned} \tag{18-56}$$

This is substituted into the equation in order to determine $\gamma$.

$$\begin{aligned} 4\gamma e^{2t} + 4\gamma t e^{2t} - 7(\gamma e^{2t} + 2\gamma t e^{2t}) + 10\gamma t e^{2t} = -3e^{2t} &\Leftrightarrow \\ (4\gamma - 14\gamma + 10\gamma)t + (4\gamma - 7\gamma + 3) = 0 &\Leftrightarrow \\ \gamma = 1 & \end{aligned} \tag{18-57}$$

We have now succeeded in finding $\gamma$, and therefore a particular solution to the equation is

$$x_0(t) = te^{2t}, \quad t \in \mathbb{R}. \tag{18-58}$$

## 18.2.2 Superposition

Within all types of linear equations the concept of ***superposition*** exists. We present the concept here for second-order linear differential equations with constant coefficients. Superposition is here used in order to determine a particular solution to the inhomogeneous equation, when the right hand side ($q(t)$) is a combination (addition) of more types of functions, e.g. a sine function added to a polynomial.

**|||| Theorem 18.15 Superposition**

Let $q_1, q_2, \ldots, q_n$ be continuous functions on an interval $I$. If $x_{0_i}(t)$ is a particular solution to the inhomogeneous differential equation

$$x''(t) + a_1 x'(t) + a_0 x(t) = q_i(t) \tag{18-59}$$

for every $i = 1, \ldots, n$, then

$$x_0(t) = x_{0_1}(t) + x_{0_2}(t) + \ldots + x_{0_n}(t) \tag{18-60}$$

is a particular solution to

$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t) = q_1(t) + q_2(t) + \ldots + q_n(t), \tag{18-61}$$

**|||| Proof**

Superposition is a consequence of the differential equation being linear. We will here give a general proof for all types of linear differential equations.

The left hand side of a differential equation is called $f(x(t))$. We now posit $n$ differential equations:

$$f(x_{0_1}(t)) = q_1(t), \qquad f(x_{0_2}(t)) = q_2(t), \qquad \ldots, \qquad f(x_{0_n}(t)) = q_n(t) \tag{18-62}$$

where $x_{0_1}, x_{0_2}, \ldots, x_{0_n}$ are particular solutions to the respective inhomogeneous differential equations. Define $x_0 = x_{0_1} + x_{0_2} + \ldots + x_{0_n}$ and substitute this into the left hand side:

$$\begin{aligned} f(x_0(t)) &= f(x_{0_1}(t) + x_{0_2}(t) + \ldots + x_{0_n}(t)) \\ &= f(x_{0_1}(t)) + f(x_{0_2}(t)) + \ldots + f(x_{0_n}(t)) \\ &= q_1(t) + q_2(t) + \ldots + q_n(t) \end{aligned} \tag{18-63}$$

On the right hand side we get the sum of the functions $q_1, q_2, \ldots, q_n$, which sum we call $q$. The Theorem is thus proven.

■

### |||| Example 18.16 Superposition

Given the inhomogeneous differential equation

$$x''(t) - x'(t) - 3x(t) = 9e^{4t} + 3t - 14, \quad t \in \mathbb{R}. \tag{18-64}$$

We wish to determine a particular solution $x_0(t)$. It is seen that the right hand side is a combination of an exponential function ($q_1(t) = 9e^{4t}$) and a polynomial ($q_2(t) = 3t - 14$). Therefore we use superposition 18.15 and the equation is split into two parts.

$$x''(t) - x'(t) - 3x(t) = 9e^{4t} = q_1(t) \tag{18-65}$$

$$x''(t) - x'(t) - 3x(t) = 3t - 14 = q_2(t) \tag{18-66}$$

First we treat (18-65), for which we use Method 18.11. A particular solution then has the form $x_{0_1}(t) = \gamma e^{\alpha t} = \gamma e^{4t}$. We have $x'_{0_1}(t) = 4\gamma e^{4t}$ and $x''_{0_1}(t) = 16\gamma e^{4t}$. This is inserted into the equation.

$$16\gamma e^{4t} - 4\gamma e^{4t} - 3\gamma e^{4t} = 9e^{4t} \Leftrightarrow \gamma = 1 \tag{18-67}$$

Therefore $x_{0_1}(t) = e^{4t}$.

Now we treat Equation (18-66), where a particular solution is a polynomial of at the most first degree, cf. Method 18.6, thus $x_{0_2}(t) = b_1 t + b_0$. Hence $x'_{0_2}(t) = b_1$ and $x''_{0_2}(t) = 0$. This is substituted into the differential equation.

$$0 - b_1 - 3(b_1 t + b_0) = 3t - 14 \Leftrightarrow (-3b_1 - 3)t + (-b_1 - 3b_0 + 14) = 0 \tag{18-68}$$

Thus we have two equations in two unknowns, and we find that $b_1 = -1$, and therefore that $b_0 = 5$. Thus a particular solution is $x_{0_2}(t) = -t + 5$. The general solution to (18-64) is then found as the sum of the already found particular solutions to the two split equations:

$$x_0(t) = x_{0_1}(t) + x_{0_2}(t) = e^{4t} - t + 5, \quad t \in \mathbb{R}. \tag{18-69}$$

### 18.2.3 The Complex Guess Method

***The complex guess method*** is used when it is easy to rewrite the right hand side of the differential equation as a complex expression, such that the given real right hand side is the real part of the complex.

If e.g. the original right hand side is $2e^{2t}\cos(3t)$ , adding $i(-2e^{2t}\sin(3t))$ , we get

$$2e^{2t}(\cos(3t) - i\sin(3t)) = 2e^{(2-3i)t} . \tag{18-70}$$

Here it is evident that $\text{Re}(2e^{(2-3i)t}) = 2e^{2t}\cos(3t)$ . One then finds a complex particular solution with complex right hand side. The wanted real particular solution to the original equation is then the real part of the found complex solution.

Note that this method can be used because the equation is linear. It is exactly the linearity that secures that the real part of the complex solution found is the wanted real solution. This is shown by interpreting the left hand side of the equation as linear map $f(z(t))$ in the set of complex functions of one real variable and using the following general theorem:

**|||| Theorem 18.17**

Given a linear map $f : (C^\infty(\mathbb{R}), \mathbb{C}) \to (C^\infty(\mathbb{R}), \mathbb{C})$ and the equation

$$f(z(t)) = s(t) . \tag{18-71}$$

If we state $z(t)$ and $s(t)$ in rectangular form as $z(t) = x(t) + i \cdot y(t)$ and $s(t) = q(t) + i \cdot r(t)$ , then (18-71) is true and if and only if

$$f(x(t)) = q(t) \quad \text{and} \quad f(y(t)) = r(t) . \tag{18-72}$$

### |||| Proof

Given the function $z(t)$ and letting the linear map $f$ and the functions $z(t)$ and $s(t)$ be given as in Theorem 18.17. As a consequence of the qualities of a linear map, cf. Definition **??**, the following applies:

$$\begin{aligned} f(z(t)) &= s(t) \Leftrightarrow \\ f(x(t) + i \cdot y(t)) &= q(t) + i \cdot r(t) \Leftrightarrow \\ f(x(t)) + i \cdot f(y(t)) &= q(t) + i \cdot r(t) \Leftrightarrow \\ f(x(t)) = q(t) \ \text{ and } \ f(y(t)) &= r(t)\,. \end{aligned} \tag{18-73}$$

Thus the theorem is proven.

■

### |||| Method 18.18 The Complex Guess Method

A particular solution $x_0(t)$ to the real inhomogeneous differential equation

$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t), \quad t \in \mathbb{R}, \tag{18-74}$$

where $a_0$ og $a_1$ are real coefficients and

$$q(t) = \text{Re}\Big((a+bi)\mathrm{e}^{(\alpha+\omega i)t}\Big) = a\mathrm{e}^{\alpha t}\cos(\omega t) - b\mathrm{e}^{\alpha t}\sin(\omega t), \tag{18-75}$$

is initially determined by the corresponding complex particular solution to the following complex equation

$$z''(t) + a_1 z'(t) + a_0 z(t) = (a+bi)\mathrm{e}^{(\alpha+\omega i)t}, \quad t \in \mathbb{R}, \tag{18-76}$$

The complex particular solution has the form $z_0(t) = (c+di)\mathrm{e}^{(\alpha+\omega i)t}$, where $c$ and $d$ are determined by substitution of $z_0(t)$ into Equation (18-76).

Then a particular solution to equation (18-74) is given by

$$x_0(t) = \text{Re}(z_0(t))\,. \tag{18-77}$$

A decisive reason for using the complex guess method is that it is so easy to determine the derivative of the exponential function, even when it is complex.

### |||| Example 18.19 The Complex Guess Method

Given a second-order inhomogeneous differential equation:

$$x''(t) - 2x'(t) - 2x(t) = 19e^{4t}\cos(t) - 35e^{4t}\sin(t), \quad t \in \mathbb{R}. \tag{18-78}$$

We wish to determine a particular solution. It is evident that we can use *the complex guess method* in Method 18.18. Initially the following is true for the right hand side:

$$q(t) = 19e^{4t}\cos(t) - 35e^{4t}\sin(t) = \mathrm{Re}\Big((19+35i)e^{(4+i)t}\Big). \tag{18-79}$$

We shall now instead of the original problem find a complex particular solution to the differential equation

$$z''(t) - 2z'(t) - 2z(t) = (19+35i)e^{(4+i)t}, \quad t \in \mathbb{R}. \tag{18-80}$$

by guessing that $z_0(t) = (c+di)e^{(4+i)t}$ is a solution. We also have

$$\begin{aligned} z_0'(t) &= (c+di)(4+i)e^{(4+i)t} = (4c-d+(c+4d)i)\,e^{(4+i)t} \quad \text{and} \\ z_0''(t) &= (4c-d+(c+4d)i)(4+i)e^{(4+i)t} = (15c-8d+(8c+15d)i)e^{(4+i)t} \end{aligned} \tag{18-81}$$

These expressions are substituted into the complex equation in order to determine $c$ and $d$:

$$\begin{aligned} &(15c-8d+(8c+15d)i)e^{(4+i)t} - 2(4c-d+(c+4d)i)e^{(4+i)t} - 2(c+di)e^{(4+i)t} \\ &\qquad = (19+35i)e^{(4+i)t} \Leftrightarrow \\ &15c-8d+(8c+15d)i - 2(4c-d+(c+4d)i) - 2(c+di) = 19+35i \Leftrightarrow \\ &5c-6d+(6c+5d)i = 19+35i \Leftrightarrow \\ &5c-6d = 19 \ \text{ og } \ 6c+5d = 35 \end{aligned} \tag{18-82}$$

These are two equations in two unknowns. The augmented matrix of the system of equations is written:

$$\left[\begin{array}{cc|c} 5 & -6 & 19 \\ 6 & 5 & 35 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{6}{5} & \frac{19}{5} \\ 0 & \frac{61}{5} & \frac{61}{5} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 1 \end{array}\right]. \tag{18-83}$$

thus we have that $c = 5$ and $d = 1$, which yields $z_0(t) = (5+i)e^{(4+i)t}$. Therefore a particular solution to the equation (18-78) is

$$x_0(t) = \mathrm{Re}(z_0(t)) = \mathrm{Re}\Big((5+i)e^{(4+i)t}\Big) = 5e^{4t}\cos(t) - e^{4t}\sin(t), \quad t \in \mathbb{R}. \tag{18-84}$$

## 18.3 Existence and Uniqueness

Here we formulate a theorem about ***existence and uniqueness*** for differential equations of the second order with constant coefficients. We need two *initial value conditions*: The value of the function and its first derivative at the chosen initial point.

**|||| Theorem 18.20 Existence and Uniqueness**

For every 3-tuple $(t_0, x_0, v_0)$ (*double initial value condition*), there exists exactly one solution $x(t)$ to the differential equation

$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t), \quad t \in I, q : I \to \mathbb{R}, \tag{18-85}$$

such that

$$x(t_0) = x_0 \quad \text{and} \quad x'(t_0) = v_0, \tag{18-86}$$

where $t_0 \in I$, $x_0 \in \mathbb{R}$ and $v_0 \in \mathbb{R}$.

**|||| Example 18.21 Exsistence and Uniqueness**

Given the differential equation

$$x''(t) - 5x'(t) - 36x(t) = 0, \quad t \in \mathbb{R}. \tag{18-87}$$

It is seen that the equation is homogeneous. It has the characteristic equation

$$\lambda^2 - 5\lambda - 36 = 0. \tag{18-88}$$

We wish to determine a function $x(t)$ that is a solution to the differential equation and has the initial value condition $(t_0, x_0, v_0) = (0, 5, 6)$. The characteristic equation has the roots $\lambda_1 = -4$ and $\lambda_2 = 9$, since $-4 \cdot 9 = -36$ and $-(9 + (-4)) = 5$ are the coefficients of the equation. Therefore the general solution for the homogeneous equation (using Theorem 18.2) is spanned by the following functions for all $c_1, c_2 \in \mathbb{R}$:

$$x(t) = c_1 e^{-4t} + c_2 e^{9t}, \quad t \in \mathbb{R}. \tag{18-89}$$

One then has

$$x'(t) = -4c_1 e^{-4t} + 9c_2 e^{9t} \tag{18-90}$$

if the initial value condition ($x(0) = 5$ and $x'(0) = 6$) is substituted into the two equations one can solve for $(c_1, c_2)$.

$$\begin{aligned} 5 &= c_1 + c_2 \\ 6 &= -4c_1 + 9c_2 \end{aligned} \tag{18-91}$$

since $e^0 = 1$. If $c_2 = 5 - c_1$ is substituted into the second equation one gets

$$6 = -4c_1 + 9(5 - c_1) = -13c_1 + 45 \Leftrightarrow c_1 = \frac{6-45}{-13} = 3 \qquad (18\text{-}92)$$

Therefore $c_2 = 5 - 3 = 2$ and the conditional solution is

$$x(t) = 3e^{-4t} + 2e^{9t}, \quad t \in \mathbb{R} \qquad (18\text{-}93)$$

Note that one can determine a unique and conditional solution to a homogeneous differential equation, as in this case. The right hand side needs not be different from zero. The general solution for the equation is $L_{inhom} = L_{hom}$, since $x_0(t) = 0$.

Below is an example going through the whole solution procedure for an inhomogeneous equation with a double initial value condition. After that an example is presented where the purpose is to find the differential equation given the general solution. It is analogous to example 18.5, but now we have a right hand side different from zero.

### |||| Example 18.22 Accumulated Example

Given the differential equation

$$x''(t) + 6x'(t) + 5x(t) = 20t^2 + 48t + 13, \quad t \in \mathbb{R}. \qquad (18\text{-}94)$$

We determine the general solution $L_{inhom}$. Then the conditional solution $x(t)$ that satisfies the initial value condition $(t_0, x_0, v_0) = (0, 5, -8)$ , will be determined.

First we solve the corresponding homogeneous equation, and the characteristic equation looks like this:

$$\lambda^2 + 6\lambda + 5 = 0 \qquad (18\text{-}95)$$

This has the roots $\lambda_1 = -5$ and $\lambda_2 = -1$, since $(\lambda + 5)(\lambda + 1) = \lambda^2 + 6\lambda + 5$. Because these roots are real and different, cf. Theorem 18.2, the general homogeneous solution set is given by

$$L_{hom} = \left\{ c_1 e^{-5t} + c_2 e^{-t} \,, t \in \mathbb{R} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}. \qquad (18\text{-}96)$$

Now we determine a particular solution to the inhomogeneous equation. Since the right hand side is a second degree polynomial we guess that $x_0(t) = b_2 t^2 + b_1 t + b_0$, using Method 18.6. We then have that $x_0'(t) = 2b_2 t + b_1$ and $x_0''(t) = 2b_2$. This is substituted into the differential equation.

$$\begin{aligned} 2b_2 + 6(2b_2 t + b_1) + 5(b_2 t^2 + b_1 t + b_0) = 20t^2 + 48t + 13 \Leftrightarrow \\ (5b_2 - 20)t^2 + (12b_2 + 5b_1 - 48)t + (2b_2 + 6b_1 + 5b_0 - 13) = 0 \Leftrightarrow \\ 5b_2 - 20 = 0 \text{ og } 12b_2 + 5b_1 - 48 = 0 \text{ og } 2b_2 + 6b_1 + 5b_0 - 13 = 0. \end{aligned} \qquad (18\text{-}97)$$

The first equation easily yields $b_2 = 4$. If this is substituted into the second equation we get $b_1 = 0$. Finally in the third equation we get $b_0 = 1$. A particular solution to the inhomogeneous equation is therefore

$$x_0(t) = 4t^2 + 1, \quad t \in \mathbb{R}. \tag{18-98}$$

Following the structural theorem, e.g. Method 18.1, the general solution to the inhomogeneous equation is given by

$$L_{inhom} = x_0(t) + L_{hom} = \left\{ 4t^2 + 1 + c_1 e^{-5t} + c_2 e^{-t}, t \in \mathbb{R} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\} \tag{18-99}$$

We now determine the solution that satisfies the given initial value conditions. An arbitrary solution has the form

$$x(t) = 4t^2 + 1 + c_1 e^{-5t} + c_2 e^{-t}, \quad t \in \mathbb{R}. \tag{18-100}$$

We now determine the derivative

$$x'(t) = 8t - 5c_1 e^{-5t} - c_2 e^{-t}, \quad t \in \mathbb{R}. \tag{18-101}$$

If $x(0) = 5$ and $x'(0) = -8$ are substituted we get two equations

$$\begin{aligned} 5 &= \phantom{-5}c_1 + c_2 + 1 \\ -8 &= -5c_1 - c_2 \end{aligned} \tag{18-102}$$

Substituting $c_1 = 4 - c_2$ from the first equation into the second we get

$$-8 = -5(4 - c_2) - c_2 \Leftrightarrow -8 + 20 = 4c_2 \Leftrightarrow c_2 = 3. \tag{18-103}$$

This yields $c_1 = 1$ and therefore the conditional solution is

$$x(t) = e^{-5t} + 3e^{-t} + 4t^2 + 1, \quad t \in \mathbb{R}. \tag{18-104}$$

### |||| Example 18.23 From the Solution to the Equation

Given the general solution to a linear second-order differential equation with constant coefficients:

$$L_{inhom} = \left\{ c_1 e^{-2t} + c_2 e^{2t} - \tfrac{1}{2}\sin(2t), t \in \mathbb{R} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\} \tag{18-105}$$

It is now the aim to find the differential equation, which in general looks like this:

$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t) \tag{18-106}$$

Thus we have to determine $a_1$, $a_0$ and $q(t)$.

First we split the solution into a particular solution and the general homogeneous solution set:

$$x_0(t) = -\tfrac{1}{2}\sin(2t),\ t \in \mathbb{R} \quad \text{and} \quad L_{hom} = \left\{\, c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{2t}\ t \in \mathbb{R} \,\middle|\, c_1, c_2 \in \mathbb{R} \,\right\} \tag{18-107}$$

Now we consider the general homogeneous solution. The looks of this complies with the first case of in Theorem 18.2. The characteristic equation has two real roots and they are $\lambda_1 = -2$ and $\lambda_2 = 2$. Therefore the characteristic equation is

$$(\lambda + 2)(\lambda - 2) = \lambda^2 - 4 = 0 \tag{18-108}$$

This determines the coefficients on the left hand side of the differential equation: $a_1 = 0$ and $a_0 = -4$. The differential equation so far looks like this:

$$x''(t) - 4x(t) = q(t), \quad t \in \mathbb{R}. \tag{18-109}$$

Since $x_0(t)$ is a particular solution to the inhomogeneous equation the right hand side $q(t)$ can be determined by substituting $x_0(t)$. We have that $x_0''(t) = 2\sin(2t)$.

$$\begin{aligned} x_0''(t) - 4x_0(t) &= q(t) \ \Leftrightarrow \\ 2\sin(2t) - 4(-\tfrac{1}{2}\sin(2t)) &= q(t) \ \Leftrightarrow \\ 4\sin(2t) &= q(t) \end{aligned} \tag{18-110}$$

Now all unknowns in the differential equation are determined:

$$x''(t) - 4x(t) = 4\sin(2t), \quad t \in \mathbb{R}. \tag{18-111}$$

In these eNotes we do not consider systems of second-order homogeneous linear differential equations with constant coefficients. We should mention, however, that with the presented theory and a bit of cleverness we can solve such problems. If we have a system of second-order homogeneous differential equations then we can consider each equation individually. By use of Section 17.4 such an equation be rewritten as two equations of first order. If this is done with all the equations in the system, we end up with double the number of equations, but those now of first-order equations. We can solve this new system with the theory presented in eNote 16. Systems of second-order homogeneous linear differential equations are seen in many places in mechanical physics, chemistry, electro-magnetism etc.

## 18.4 Summary

In this note linear second-order differential equations with constant coefficients are written as:

$$x''(t) + a_1 x'(t) + a_0 x(t) = q(t) \tag{18-112}$$

- This equation is solved by first determining the general solution to the corresponding homogeneous equation and then adding this to a particular solution to the inhomogeneous equation, see Method 18.1.
- The general solution to the corresponding homogeneous differential equation is determined by finding the roots of the *characteristic equation*:

  $$\lambda^2 + a_1\lambda + a_0 = 0. \tag{18-113}$$

  There are in principle three cases, see Theorem 18.2.
- A particular solution is determined by "guessing" a solution that has the same appearance as the right hand side $q(t)$. If e.g. $q(t)$ is a polynomial then $x_0(t)$ is also a polynomial of at the most same degree. In the note many examples are given, see Section 18.2.
- In particular we have *the complex guess method* for the determination of the particular solution $x_0(t)$. The complex guess method can be used when the right hand side has this appearance:

  $$q(t) = \mathrm{Re}\Big((a+bi)\mathrm{e}^{(\alpha+\omega i)t}\Big) = a\mathrm{e}^{\alpha t}\cos(\omega t) - b\mathrm{e}^{\alpha t}\sin(\omega t). \tag{18-114}$$

  The solution is then determined by rewriting the differential equation in the corresponding exponential form, see Method 18.18.
- Furthermore *superposition* is introduced. Superposition is a general principle that applies to all types of linear equations. The idea is that two particular solutions can be added. When they are substituted into the differential equation they will not influence each other, and hence the right hand side can also be split into two terms, each corresponding to one of the two solutions. This can be used to determine a particular solution, when the right hand side is the sum of e.g. a sine function and a polynomial. See e.g. Example 18.16.
- Furthermore an *existence and uniqueness theorem* is formulated, see Theorem 18.20. According to this theorem a unique conditional solution that must satisfy two particular *initial value conditions* to a second-order differential equation can be determined.