eNote 17

Systems of Linear First-Order Differential Equations

This eNote describes systems of linear first-order differential equations with constant coefficients and shows how these can be solved. The eNote is based on eNote 16, which describes linear differential equations in general. Thus it is a good idea to read that eNote first. Moreover eigenvalues and eigenvectors are used in the solution procedure, see eNotes 13 and 14.
(Updated: 9.11.21 by David Brander).

Here we consider coupled homogeneous linear first-order differential equations with constant coefficients (see Explanation 17.1). Such a collection of ***coupled differential equations*** is called a ***system of differential equations***. A system of n first-order differential equations with constant coefficients looks like this:

$$\begin{array}{ccccccccc} x_1'(t) & = & a_{11}x_1(t) & + & a_{12}x_2(t) & + & \ldots & + & a_{1n}x_n(t) \\ x_2'(t) & = & a_{21}x_1(t) & + & a_{22}x_2(t) & + & \ldots & + & a_{2n}x_n(t) \\ \vdots & & \vdots & & \vdots & & & & \vdots \\ x_n'(t) & = & a_{n1}x_1(t) & + & a_{n2}x_2(t) & + & \ldots & + & a_{nn}x_n(t) \end{array} \tag{17-1}$$

On the left hand side of the system the derivatives of the n unknown functions $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ are written. Every right hand side is a linear combination of the n unknown functions. The coefficients (the a's) are real constants. In matrix form the system can be written like this:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{17-2}$$

Even more compactly it can be written like this

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) \tag{17-3}$$

A is called the ***system matrix***. It is now the aim to solve such a system of differential equations, that is, we wish to determine $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$.

Explanation 17.1 What Is a System of Differential Equations?

Systems of differential equations are collections of differential equations. The reason we do not consider the differential equations individually, is that they cannot be solved independently, because the unknown functions are present in more equations, that is, the equations are *coupled*. A single differential equation from a system can e.g. look like this:

$$x_1'(t) = 4x_1(t) - x_2(t) \tag{17-4}$$

It is not possible to determine neither $x_1(t)$ nor $x_2(t)$, since there are two unknown functions, but only one differential equation.

In order to be able to find the full solution to such an equation one should have as many equations as one has unknown equations (with corresponding derivatives). Thus the second equation in the system might be:

$$x_2'(t) = -6x_1(t) + 2x_2(t) \tag{17-5}$$

We now have as many equations (two), as we have unknown functions (two), and it is now possible to determine both $x_1(t)$ and $x_2(t)$.

For greater clarity we write the system of differential equations in matrix form. The system above looks like this:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x}'(t) = \begin{bmatrix} 4 & -1 \\ -6 & 2 \end{bmatrix} \mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) \tag{17-6}$$

Disregarding that are operating with vectors and matrices the system of equations looks like something we have seen before: $x'(t) = A \cdot x(t)$, something we were able to solve in high school. The solution to this differential equation is trivial: $x(t) = ce^{At}$, where c is an arbitrary constant. Below we find that the solution to the corresponding system of differential equations is similar in structure to $x(t) = ce^{At}$.

We now solve the system of differential equations in the following Theorem 17.2. The theorem contains requirements that are not always satisfied. The special cases where the theorem is not valid are investigated later. The proof uses a well-known method, the so-called ***diagonalization method***.

Theorem 17.2

Let $\mathbf{A} \in^{n\times n}$. A system of linear differential equations consisting of n equations with a total of n unknown functions is given by

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad t \in \mathbb{R}\,. \tag{17-7}$$

If $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ corresponding to (not necessarily different) eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the general solution of the system is determined by

$$\mathbf{x}(t) = c_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + c_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2 + \ldots + c_n \mathrm{e}^{\lambda_n t}\mathbf{v}_n, \quad t \in \mathbb{R}\,, \tag{17-8}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary complex constants.

Note that it is not always possible to find n linearly independent eigenvectors. Therefore Theorem 17.2 cannot always be applied to the solution of systems of first-order differential equations.

In the theorem we use the general complex solution for the system of differential equations. Therefore the general real solution can be found as the real subset of the complex solution.

Proof

We guess that a solution to the system of differential equations $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ is a vector $\mathbf{v}$ multiplied by $\mathrm{e}^{\lambda t}$, λ being a constant, such that $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$. We then have the derivative

$$\mathbf{x}'(t) = \lambda \mathrm{e}^{\lambda t}\mathbf{v}\,. \tag{17-9}$$

If this expression for $\mathbf{x}'(t)$ is substituted into (17-7) together with the expression for $\mathbf{x}(t)$ we get:

$$\lambda \mathrm{e}^{\lambda t}\mathbf{v} = \mathbf{A}\mathrm{e}^{\lambda t}\mathbf{v} \Leftrightarrow \mathbf{A}\mathbf{v} - \lambda\mathbf{v} = 0 \Leftrightarrow (\mathbf{A} - \lambda\mathbf{E})\mathbf{v} = 0 \tag{17-10}$$

$\mathrm{e}^{\lambda t}$ is non-zero for every $t \in \mathbb{R}$, and can thus be eliminated. The resulting equation is an eigenvalue problem. λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is the corresponding eigenvector. They can both be determined. We have now succeeded in finding that $e^{\lambda t}\mathbf{v}$ is one solution to the system of differential equations, when λ is an eigenvalue and $\mathbf{v}$ the corresponding eigenvector of $\mathbf{A}$.

In order to find the general solution we use the so-called ***diagonalization method***:

We suppose that $\mathbf{A} = \mathbf{A}_{n\times n}$ has n linearly independent (real or complex) eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We now introduce the invertible matrix $\mathbf{V}$, that contains all the eigenvectors:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \tag{17-11}$$

Furthermore we introduce the function $\mathbf{y}$ with $\mathbf{y}(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ such that

$$\mathbf{x}(t) = \mathbf{V}\mathbf{y}(t) \tag{17-12}$$

We then get $\mathbf{x}'(t) = \mathbf{V}\mathbf{y}'(t)$. If these expressions for $\mathbf{x}(t)$ og $\mathbf{x}'(t)$ are substituted into Equation (17-7) we get

$$\mathbf{V}\mathbf{y}'(t) = \mathbf{A}\mathbf{V}\mathbf{y}(t) \Leftrightarrow \mathbf{y}'(t) = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}\mathbf{y}(t) = \mathbf{\Lambda}\mathbf{y}(t), \tag{17-13}$$

where $\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$.

We now get the equation $\mathbf{y}'(t) = \mathbf{\Lambda}\mathbf{y}(t)$, which can be written in the following way:

$$\begin{aligned} y_1'(t) &= \lambda_1 y_1(t) \\ y_2'(t) &= \lambda_2 y_2(t) \\ &\vdots \\ y_n'(t) &= \lambda_n y_n(t) \end{aligned} \tag{17-14}$$

since $\mathbf{\Lambda}$ only has non-zero elements in the diagonal. In this system the single equations are uncoupled: each of the equations only contains one function and its derivative. Therefore we can solve them independently and the general solution for every equation is $y(t) = ce^{\lambda t}$ for all $c \in \mathbb{C}$. In total this yields the general solution consisting of the functions below for all $c_1, c_2, \ldots, c_n \in \mathbb{C}$:

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} \tag{17-15}$$

Since we now have the solution $\mathbf{y}(t)$ we can also find the solution $\mathbf{x}(t) = \mathbf{V}\mathbf{y}(t)$:

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} \\ &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \ldots + c_n e^{\lambda_n t}\mathbf{v}_n\,. \end{aligned} \tag{17-16}$$

Now we have found the general complex solution to the system of equations in Equation (17-7) consisting of the functions $\mathbf{x}(t)$ for all $c_1, c_2, \ldots, c_n \in \mathbb{C}$.

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|||| Example 17.3

Given the system of differential equations

$$\begin{aligned} x_1'(t) &= \ \ x_1(t) + 2x_2(t) \\ x_2'(t) &= 3x_1(t) \end{aligned} \tag{17-17}$$

Which in matrix form is

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \mathbf{x}(t) = \mathbf{A}\mathbf{x}(t). \tag{17-18}$$

It can be shown that $\mathbf{A}$ has the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$ with the eigenvectors $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (2, -3)$ (try for yourself!). Therefore the general real solution to the system of differential equations is given by the functions below for all $c_1, c_2 \in \mathbb{R}$:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \mathrm{e}^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-2t} \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad t \in \mathbb{R} \tag{17-19}$$

The solution is found using Theorem 17.2. Another way of writing the solution is to separate the system of equations so that

$$\begin{aligned} x_1(t) &= c_1 \mathrm{e}^{3t} + 2c_2 \mathrm{e}^{-2t} \\ x_2(t) &= c_1 \mathrm{e}^{3t} - 3c_2 \mathrm{e}^{-2t} \end{aligned} \tag{17-20}$$

constitutes the general solution, where $t \in \mathbb{R}$, for all $c_1, c_2 \in \mathbb{R}$.

17.1 Two Coupled Differential Equations

Given a linear homogeneous first order system of differential equations with constant coefficients with n equations and n unknown functions

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) . \quad t \in \mathbb{R} \tag{17-21}$$

If the system matrix $\mathbf{A}$ has n linearly independent eigenvectors, the real solution can be found using Theorem 17.2. If the eigenvalues are real then the real solution can be written directly following formula (17-8) in the theorem, where the n corresponding linearly independent eigenvectors are real and the arbitrary constants are stated as being real. If the system matrix has eigenvalues that are not real then the real solution can be found by extracting the real subset of the complex solution. Also in this case the solution can be written as a linear combination of n linearly independent real solutions to the system of differential equations.

We are left with the special case in which the system matrix does not have n linearly independent eigenvectors. Also in this case the real solution will be a linear combination of n linearly independent real solutions to the system of differential equations. Here the method of diagonalization obviously cannot be used and one has to resort to other methods.

In this section we show the three cases above for systems consisting of $n = 2$ coupled differential equations with 2 unknown functions.

|||| Method 17.4

The general real solution to the system of differential equations

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad t \in \mathbb{R}, \tag{17-22}$$

consisting of $n = 2$ equations with 2 unknown functions can be written as

$$\mathbf{x}(t) = c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t)\,, \quad t \in \mathbb{R}, \tag{17-23}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are real linearly independent particular solutions and $c_1, c_2 \in \mathbb{R}$.

First determine the eigenvalues of $\mathbf{A}$. For the roots of the characteristic polynomial $\mathbf{A}$ there are three possibilities:

- **Two real single roots.** In this case both of the eigenvalues λ_1 and λ_2 have the algebraic multiplicity 1 and geometric multiplicity 1 and we can put

  $$\mathbf{u}_1(t) = \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 \quad \text{and} \quad \mathbf{u}_2(t) = \mathrm{e}^{\lambda_2 t}\mathbf{v}_2, \tag{17-24}$$

  where $\mathbf{v}_1$ and $\mathbf{v}_2$ are proper eigenvectors of λ_1 and λ_2, respectively.

- **Two complex roots.** The two eigenvalues λ and $\bar{\lambda}$ are then conjugate complex numbers. We then determine $\mathbf{u}_1$ and $\mathbf{u}_2$ using Method 17.5.

- **One double root.** Here the eigenvalue λ has the algebraic multiplicity 2. If the geometric multiplicity of λ is 1, $\mathbf{u}_1$ and $\mathbf{u}_2$ are determined using method 17.7.

In the first case in Method 17.4 with two different real eigenvalues, Theorem 17.2 can be used directly with the arbitrary constants chosen as real, see Example 17.3.

Now follows the method that covers the case with two complex eigenvalues.

|||| Method 17.5

Two linearly independent real solutions to the system of equations

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad t \in \mathbb{R}, \tag{17-25}$$

where $\mathbf{A}$ has the complex pair of eigenvalues $\lambda = \alpha + \beta i$ and $\bar{\lambda} = \alpha - \beta i$ with corresponding eigenvectors $\mathbf{v}$ and $\bar{\mathbf{v}}$, are

$$\begin{aligned} \mathbf{u}_1(t) &= \mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) = \mathrm{e}^{\alpha t}\left(\cos(\beta t)\mathrm{Re}(\mathbf{v}) - \sin(\beta t)\mathrm{Im}(\mathbf{v})\right) \\ \mathbf{u}_2(t) &= \mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) = \mathrm{e}^{\alpha t}\left(\sin(\beta t)\mathrm{Re}(\mathbf{v}) + \cos(\beta t)\mathrm{Im}(\mathbf{v})\right) \end{aligned} \tag{17-26}$$

|||| Example 17.6

Given the system of differential equations

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) \tag{17-27}$$

We wish to determine the general real solution.

The eigenvalues are determined as $\lambda = 1 + i$ and $\bar{\lambda} = 1 - i$, respectively, with the corresponding eigenvectors $\mathbf{v} = (-i, 1)$ and $\bar{\mathbf{v}} = (i, 1)$, respectively. We see that there are two complex eigenvalues and their corresponding complex eigenvectors. With $\lambda = 1 + i$ we get

$$\mathbf{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \mathrm{Re}(\mathbf{v}) + i\mathrm{Im}(\mathbf{v}) \tag{17-28}$$

If we use Method 17.5 we then get the two solutions:

$$\mathbf{u}_1(t) = \mathrm{e}^t\left(\cos(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \mathrm{e}^t\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} \tag{17-29}$$

$$\mathbf{u}_2(t) = \mathrm{e}^t\left(\sin(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos(t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \mathrm{e}^t\begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix} \tag{17-30}$$

The general real solution to the system of differential equations (17-27) is then given by the following functions for all $c_1, c_2 \in \mathbb{R}$:

$$\mathbf{x}(t) = c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t) = \mathrm{e}^t\left(c_1\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + c_2\begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}\right), \quad t \in \mathbb{R} \tag{17-31}$$

found using Method 17.4.

Finally we describe the method that can be used if the system matrix has the eigenvalue λ with $\mathrm{am}(\lambda) = 2$ and $\mathrm{gm}(\lambda) = 1$, that is when diagonalization is not possible.

|||| Method 17.7

If the system matrix $\mathbf{A}$ to the system of differential equations

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad t \in \mathbb{R}, \tag{17-32}$$

has one eigenvalue λ with algebraic multiplicity 2, but the corresponding eigenvector space only has geometric multiplicity 1, there are two linearly independent solutions to the system of differential equations of the form:

$$\begin{aligned} \mathbf{u}_1(t) &= \mathrm{e}^{\lambda t}\mathbf{v} \\ \mathbf{u}_2(t) &= t\mathrm{e}^{\lambda t}\mathbf{v} + \mathrm{e}^{\lambda t}\mathbf{b}, \end{aligned} \tag{17-33}$$

where $\mathbf{v}$ is the eigenvector corresponding to λ and $\mathbf{b}$ is a solution to the following linear system:

$$(\mathbf{A} - \lambda\mathbf{E})\mathbf{b} = \mathbf{v} \tag{17-34}$$

|||| Proof

It is evident that one solution to the system of differential equations is $\mathbf{u}_1(t) = \mathrm{e}^{\lambda t}\mathbf{v}$. The difficulty is to find another solution.

We guess at a solution in the form

$$\mathbf{u}_2(t) = t\mathrm{e}^{\lambda t}\mathbf{v} + \mathrm{e}^{\lambda t}\mathbf{b} = \mathrm{e}^{\lambda t}(t\mathbf{v} + \mathbf{b}), \tag{17-35}$$

where $\mathbf{v}$ is an eigenvector corresponding to λ. We then have

$$\mathbf{u}_2{}'(t) = (\mathrm{e}^{\lambda t} + \lambda t\mathrm{e}^{\lambda t})\mathbf{v} + \lambda\mathrm{e}^{\lambda t}\mathbf{b} = \mathrm{e}^{\lambda t}((1 + \lambda t)\mathbf{v} + \lambda\mathbf{b}) \tag{17-36}$$

We check whether $\mathbf{u}_2(t)$ is a solution by substitution into $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$:

$$\begin{aligned} \mathbf{u}_2{}'(t) &= \mathbf{A}\mathbf{u}_2(t) \Leftrightarrow \\ (1 + \lambda t)\mathbf{v} + \lambda\mathbf{b} &= \mathbf{A}(t\mathbf{v} + \mathbf{b}) \Leftrightarrow \\ t(\lambda\mathbf{v} - \mathbf{A}\mathbf{v}) + (\mathbf{v} + \lambda\mathbf{b} - \mathbf{A}\mathbf{b}) &= \mathbf{0} \Leftrightarrow \\ \lambda\mathbf{v} - \mathbf{A}\mathbf{v} = \mathbf{0} \wedge \mathbf{v} + \lambda\mathbf{b} - \mathbf{A}\mathbf{b} &= \mathbf{0} \end{aligned} \tag{17-37}$$

The first equation can easily be transformed into $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, which is seen to be true, since $\mathbf{v}$ is an eigenvector corresponding to λ. The other equation is transformed into:

$$\begin{aligned} \mathbf{v} + \lambda\mathbf{b} - \mathbf{A}\mathbf{b} &= \mathbf{0} \Leftrightarrow \\ \mathbf{A}\mathbf{b} - \lambda\mathbf{b} &= \mathbf{v} \Leftrightarrow \\ (\mathbf{A} - \lambda\mathbf{E})\mathbf{b} &= \mathbf{v} \end{aligned} \tag{17-38}$$

If $\mathbf{b}$ satisfies the given system of equations, $\mathbf{u}_2(t)$ will also be a solution to the system of differential equations. We now have found two solutions and we have to find out whether these are linearly independent. This is done by a normal linearity criterion: If the equation $k_1\mathbf{u}_1 + k_2\mathbf{u}_2 = \mathbf{0}$ only has the solution $k_1 = k_2 = 0$ then $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent.

$$\begin{aligned} k_1\mathbf{u}_1 + k_2\mathbf{u}_2 &= \mathbf{0} \Rightarrow \\ k_1 e^{\lambda t}\mathbf{v} + k_2(te^{\lambda t}\mathbf{v} + \mathbf{b}e^{\lambda t}) &= \mathbf{0} \Leftrightarrow \\ t(k_2\mathbf{v}) + (k_1\mathbf{v} + k_2\mathbf{b}) &= \mathbf{0} \Leftrightarrow \\ k_2\mathbf{v} = \mathbf{0} \wedge k_1\mathbf{v} + k_2\mathbf{b} &= \mathbf{0} \end{aligned} \tag{17-39}$$

Since $\mathbf{v}$ is an eigenvector, it is not the zero-vector, and hence $k_2 = 0$ according to the first equation. Thus the other equation is reduced to $k_1\mathbf{v} = \mathbf{0}$, and with the same argument we get $k_1 = 0$. Therefore the two solutions are linearly independent, and thus the method has been proved.

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|||| Example 17.8

Given the system of differential equations

$$\mathbf{x}'(t) = \begin{bmatrix} 16 & -1 \\ 4 & 12 \end{bmatrix}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t). \tag{17-40}$$

The eigenvalues for $\mathbf{A}$ are determined:

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{E}) = \begin{vmatrix} 16-\lambda & -1 \\ 4 & 12-\lambda \end{vmatrix} &= (16-\lambda)(12-\lambda) + 4 \\ &= \lambda^2 - 28\lambda + 196 = (\lambda - 14)^2 = 0 \end{aligned} \tag{17-41}$$

There is only one eigenvalue, viz. $\lambda = 14$, even though it is a 2×2-system. The eigenvectors are determined:

$$\mathbf{A} - 14\mathbf{E} = \begin{bmatrix} 16-14 & -1 \\ 4 & 12-14 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \tag{17-42}$$

We then obtain the eigenvector $(\frac{1}{2}, 1)$ or $\mathbf{v} = (1, 2)$. We can then conclude that the eigenvalue λ has the algebraic multiplicity 2, but that the corresponding eigenvector space has the geometric multiplicity 1. In order to determine two independent solutions to the system of differential equations we can use Method 17.7.

First we solve the following system of equations:

$$(\mathbf{A}-\lambda\mathbf{E})\mathbf{b}=\mathbf{v}\Rightarrow\begin{bmatrix}2 & -1\\ 4 & -2\end{bmatrix}\mathbf{b}=\begin{bmatrix}1\\ 2\end{bmatrix} \tag{17-43}$$

$$\begin{bmatrix}2 & -1 & 1\\ 4 & -2 & 2\end{bmatrix}\rightarrow\begin{bmatrix}1 & -\frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 0\end{bmatrix} \tag{17-44}$$

This yields $\mathbf{b}=(1,1)$, if the free parameter is put at1. The two solutions then are

$$\begin{aligned}\mathbf{u}_1(t)&=\mathrm{e}^{14t}\begin{bmatrix}1\\ 2\end{bmatrix}\\ \mathbf{u}_2(t)&=t\mathrm{e}^{14t}\begin{bmatrix}1\\ 2\end{bmatrix}+\mathrm{e}^{14t}\begin{bmatrix}1\\ 1\end{bmatrix}.\end{aligned} \tag{17-45}$$

By use of Method 17.4 the general solution can be determined to the following functions for all $c_1, c_2 \in \mathbb{R}$:

$$\mathbf{x}(t)=c_1\mathbf{u}_1(t)+c_2\mathbf{u}_2(t)=c_1\mathrm{e}^{14t}\begin{bmatrix}1\\ 2\end{bmatrix}+c_2\mathrm{e}^{14t}\left(t\begin{bmatrix}1\\ 2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}\right). \tag{17-46}$$

17.2 n-Dimensional Solution Space

In the preceding section we have considered coupled systems consisting of two linear equations with two unknown functions. The solution space is two-dimensional, since it can be written as a linear combination of two linearly independent solutions. This can be generalized to arbitrary systems with $n \geq 2$ coupled linear differential equations with n unknown functions: The solution is a linear combination of exactly n linearly independent solutions. This is formulated in a general form in the following theorem.

|||| Theorem 17.9

Given the linear homogeneous first order system of differential equations with constant real coefficients

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad t \in \mathbb{R}, \tag{17-47}$$

consisting of n equations and with n unknown functions. The general real solution to the system is n-dimensional and can be written as

$$\mathbf{x}(t) = c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t) + \cdots + c_n\mathbf{u}_n(t), \tag{17-48}$$

where $\mathbf{u}_1(t), \mathbf{u}_2(t), \ldots, \mathbf{u}_n(t)$ are linearly independent real solutions to the system of differential equations and $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Below is an example with a coupled system of three differential equations that exemplifies Theorem 17.9.

|||| Example 17.10 Advanced

Given the system of differential equations

$$\mathbf{x}'(t) = \begin{bmatrix} -9 & 10 & 0 \\ -3 & 1 & 5 \\ 1 & -4 & 6 \end{bmatrix} \mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) \tag{17-49}$$

We wish to determine the general real solution to the system of differential equations. Eigenvalues and eigenvectors can be determined and are as follows:

$$\begin{aligned} \lambda_1 = -4 &\quad : \quad \mathbf{v}_1 = (10, 5, 1) \\ \lambda_2 = 1 &\quad : \quad \mathbf{v}_2 = (5, 5, 3) \end{aligned}$$

Moreover λ_2 has the algebraic multiplicity 2, but the corresponding eigenvector space has the geometric multiplicity 1. Because $n = 3$ we need 3 linearly independent solutions to construct the general solution, as seen in 17.9. The eigenvalues are considered separately:

1) The first eigenvalue, $\lambda_1 = -4$, has both geometric and algebraic multiplicity equal to 1. This yields exactly one solution

$$\mathbf{u}_1(t) = \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 = \mathrm{e}^{-4t} \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} \tag{17-50}$$

2) The other eigenvalue, $\lambda_2 = 1$, has algebraic multiplicity 2, but geometric multiplicity 1.

Therefore we can use method 17.7 in order to find two solutions. First $\mathbf{b}$ is determined:

$$(\mathbf{A}-\lambda_2\mathbf{E})\mathbf{b}=\mathbf{v}_2 \Rightarrow \begin{bmatrix} -10 & 10 & 0 \\ -3 & 0 & 5 \\ 1 & -4 & 5 \end{bmatrix}\mathbf{b}=\begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} \tag{17-51}$$

A particular solution to this system of equations is $\mathbf{b}=(0,\frac{1}{2},1)$. With this knowledge we have two additional linearly independent solutions to the system of differential equations:

$$\begin{aligned} \mathbf{u}_2(t) &= \mathrm{e}^{\lambda_2 t}\mathbf{v}_2 = \mathrm{e}^t\begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} \\ \mathbf{u}_3(t) &= t\mathrm{e}^{\lambda_2 t}\mathbf{v}_2+\mathrm{e}^{\lambda_2 t}\mathbf{b} = t\mathrm{e}^t\begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}+\mathrm{e}^t\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{aligned} \tag{17-52}$$

We leave it to the reader to show that all three solutions are linearly independent.

According to Method 17.9 the general real solution consists of the following linear combination for all $c_1, c_2, c_3 \in \mathbb{R}$:

$$\mathbf{x}(t)=c_1\mathbf{u}_1(t)+c_2\mathbf{u}_2(t)+c_3\mathbf{u}_3(t) \tag{17-53}$$

Thus this yields

$$\mathbf{x}(t)=c_1\mathrm{e}^{-4t}\begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}+c_2\mathrm{e}^t\begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}+c_3\left(t\mathrm{e}^t\begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}+\mathrm{e}^t\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}\right) \tag{17-54}$$

where $t \in \mathbb{R}$ and all $c_1, c_2, c_3 \in \mathbb{R}$.

17.3 Existence and Uniqueness of Solutions

According to the Structural Theorem 17.9 the general solution to a system of differential equations with n equations contains n arbitrary constants. If we have n ***initial conditions***, then the constants can be determined, and we then get a unique solution. This is formulated in the following ***existence and uniqueness theorem***.

|||| Theorem 17.11

A first order system of differential equations consisting of n equations in n unknown functions with constant coefficients is given by

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad t \in I\,. \tag{17-55}$$

For every $t_0 \in I$ and every number set $\mathbf{y}_0 = (y_1, y_2, \ldots, y_n)$ exactly one solution exists $\mathbf{x}(t) = (x_1(t), x_2(t) \ldots, x_n(t)\,)$ satisfying the initial conditions

$$\mathbf{x}(t_0) = \mathbf{y}_0\,, \tag{17-56}$$

that is

$$x_1(t_0) = y_1,\ x_2(t_0) = y_2,\ \ldots,\ x_n(t_0) = y_n. \tag{17-57}$$

|||| Example 17.12

In Example 17.3 we found the general solution to the system of differential equations

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \mathbf{x}(t), \quad t \in \mathbb{R}, \tag{17-58}$$

viz.

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad t \in \mathbb{R} \tag{17-59}$$

Now we wish to determine the unique solution $\mathbf{x}(t) = (x_1(t), x_2(t))$ that satisfies the initial condition $\mathbf{x}(0) = (x_1(0), x_2(0)) = (6, 6)$. This yields the system of equations

$$\begin{bmatrix} 6 \\ 6 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{17-60}$$

By ordinary Gauss-Jordan elimination we get

$$\begin{bmatrix} 1 & 2 & 6 \\ 1 & -3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 6 \\ 0 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \end{bmatrix} \tag{17-61}$$

Thus we obtain the solution $(c_1, c_2) = (6, 0)$, and the unique conditional solution is therefore

$$\mathbf{x}(t) = 6e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}\,, \tag{17-62}$$

which is equivalent to

$$x_1(t) = 6e^{3t} \quad x_2(t) = 6e^{3t}\,. \tag{17-63}$$

In this particular case the two functions are identical.

17.4 Transformation of Linear n'th Order Homogeneous Differential Equations to a First Order System of Differential Equations

With a bit of ingenuity it is possible to transform a homogeneous nth order differential equation with constant coefficients to a system of differential equations that can be solved using the methods in this eNote.

|||| Method 17.13

An nth order linear differential equation

$$x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + a_{n-2}x^{(n-2)}(t) + \cdots + a_1 x'(t) + a_0 x(t) = 0 \tag{17-64}$$

for $t \in \mathbb{R}$, can be transformed into a first order system of differential equations and the system will look like this:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_{n-1}'(t) \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} \tag{17-65}$$

and $x_1(t) = x(t)$.

The proof of this rewriting is simple but gives a good understanding of the transformation.

|||| Proof

Given an nth order differential equation as in Equation (17-64). We introduce n functions in this way:

$$\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= x_1'(t) &&= x'(t) \\ x_3(t) &= x_2'(t) &&= x''(t) \\ &\vdots &&\vdots \\ x_{n-1}(t) &= x_{n-2}'(t) &&= x^{(n-2)}(t) \\ x_n(t) &= x_{n-1}'(t) &&= x^{(n-1)}(t) \end{aligned} \tag{17-66}$$

These new expressions are substituted into the differential equation (17-64):

$$x_n'(t) + a_{n-1}x_n(t) + a_{n-2}x_{n-1}(t) + \ldots + a_1x_2(t) + a_0x_1(t) = 0 \tag{17-67}$$

Now this equation can together with equations (17-66) be written in matrix form.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_{n-1}'(t) \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} \tag{17-68}$$

The method is thus proved.

■

|||| Example 17.14

Given a linear differential equation of third order with constant coefficients:

$$x'''(t) - 4x''(t) - 7x'(t) + 10x(t) = 0, \quad t \in \mathbb{R}. \tag{17-69}$$

We wish to determine the general solution. Therefore the following functions are introduced

$$\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= x_1'(t) = x'(t) \\ x_3(t) &= x_2'(t) = x''(t) \end{aligned} \tag{17-70}$$

In this way we can rewrite the differential equation as

$$x_3'(t) - 4x_3(t) - 7x_2(t) + 10x_1(t) = 0 \tag{17-71}$$

And we can then gather the last three equations in a system of equations.

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= x_3(t) \\ x_3'(t) &= -10x_1(t) + 7x_2(t) + 4x_3(t) \end{aligned} \tag{17-72}$$

This is written in matrix form in this way:

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 7 & 4 \end{bmatrix} \mathbf{x}(t) \tag{17-73}$$

The eigenvalues are determined to be $\lambda_1 = -2$, $\lambda_2 = 1$ and $\lambda_3 = 5$. The general solution to the system of differential equations according to Theorem 17.2 is given by the following functions for all the arbitrary constants $c_1, c_2, c_3 \in \mathbb{R}$:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{-2t}\mathbf{v}_1 + c_2 \mathrm{e}^{t}\mathbf{v}_2 + c_3 \mathrm{e}^{5t}\mathbf{v}_3, \quad t \in \mathbb{R}, \tag{17-74}$$

where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the respective eigenvectors.

But we need only the solution of $x_1(t) = x(t)$, and we isolate this from the general solution to the system. Furthermore we introduce three new arbitrary constants $k_1, k_2, k_3 \in \mathbb{R}$, that are equal to the product of the c's and the first coordinates of the eigenvectors. The result is

$$x(t) = x_1(t) = k_1 \mathrm{e}^{-2t} + k_2 \mathrm{e}^{t} + k_3 \mathrm{e}^{5t}, \quad t \in \mathbb{R} \tag{17-75}$$

This constitutes the general solution to the differential equation (17-69). If the first coordinate in $\mathbf{v}_1$ is 0 , we put $k_1 = 0$; otherwise k_1 can be an arbitrary real number. Similarly for k_2 and k_3.