eNote 16

First Order Linear Differential Equations

In this eNote we first give a short introduction to differential equations in general and then the main subject is a special type of differential equation the so-called first order differential equations. The eNote is based on knowledge of special functions, differential and integral calculus and linear maps.

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16.1 What Is a Differential Equation?

A *differential equation* is an equation, in which one or more unknown *functions* appear together with one or more of their derivatives. In this eNote we only consider differential equations that contain one unknown function. Differential equations naturally occur in the modelling of physical, mechanical, economic, chemical and manifold other problems, and this is why it is an important subject.

One says that a differential equation has the *order* n if it contains the nth derivative of the unknown function, but no derivatives of order higher than n. The unknown function is in this eNote denoted by x or $x(t)$, if the name of the independent variable t is important in the context.

An example of a differential equation is

$$x'''(t) - 2x'(t) + x(t) = t\,, \quad t \in \mathbb{R}. \tag{16-1}$$

The equation has *order* 3, since the highest number of times the unknown function x is differentiated in the equation is 3. A *solution* to the equation is a function x_0 which, inserted into the equation, makes it true. If we, for example, want to investigate whether the function

$$x_0(t) = \mathrm{e}^t + t + 2\,, \quad t \in \mathbb{R}$$

is a solution to (16-1), we test this by insertion of x_0 in place of x in the equation. Since

$$\begin{aligned} x_0'''(t) - 2x_0'(t) + x_0(t) &= (\mathrm{e}^t + t + 2)''' - 2(\mathrm{e}^t + t + 2)' + ((\mathrm{e}^t + t + 2)) \\ &= \mathrm{e}^t - 2(\mathrm{e}^t + 1) + \mathrm{e}^t + t + 2 \\ &= t, \end{aligned}$$

x_0 is a solution.

This eNote is about an important type of first order differential equation, the so-called *linear* differential equations. In order to be able to investigate these precisely we first express them in a standard way.

16.2 Introduction to First Order Linear Differential Equations

|||| Definition 16.1

By a first order *linear* differential equation we understand a differential equation that can be brought into the standard form

$$x'(t) + p(t)x(t) = q(t)\,, \quad t \in I \tag{16-2}$$

where I is an open interval in $\mathbb{R}$, and p and q are (known) continuous functions defined on I.

The equation is called *homogeneous* if $q(t) = 0$ for all t. Otherwise it is called *inhomogeneous*.

|||| Example 16.2 Standard Form

The first order differential equation

$$x'(t) + 2x(t) = 30 + 8t\,, \quad t \in \mathbb{R}\,. \tag{16-3}$$

is immediately seen to be in standard form (16-2) with $p(t) = 2$ and $q(t) = 30 + 8t\,, \quad t \in \mathbb{R}\,.$ Therefore it is linear.

|||| Example 16.3 Standard form

Let I be an open interval in $\mathbb{R}$. Consider the first order linear differential equation

$$t \cdot x'(t) + 2x(t) - 8t^2 = -10\,, \quad t \in \mathbb{R}\,. \tag{16-4}$$

In order to bring this into standard form we first have to add $8t^2$ to both sides of the equation, since on the left-hand side only terms containing the unknown function $x(t)$ must appear. Then we divide both sides by t, since the coefficient of $x'(t)$ must be 1 in the standard form. To avoid division by 0, we must assume that t is either greater or less than zero: let us choose the first:

$$x'(t) + \frac{2}{t}\,x(t) = 8t - \frac{10}{t}, \quad t > 0\,. \tag{16-5}$$

Now the differential equation is in the standard form with $p(t) = \frac{2}{t}$ and $q(t) = 8t - \frac{10}{t}, \quad t > 0\,.$

|||| Exercise 16.4 Standard Form

Explain why the first order differential equation

$$x'(t) + \frac{1}{2}\,t^2 = 0\,, \quad t \in \mathbb{R}$$

is not homogeneous.

|||| Exercise 16.5 More Solutions

Given the differential equation

$$x'(t) + 2x(t) = 30 + 8t\,, \quad t \in \mathbb{R}.$$

Show that, for any of $c = 1$, $c = 2$ or $c = 3$, the function

$$x_0(t) = 13 + 4t + ce^{-2t}\,, \quad t \in \mathbb{R}$$

is a solution.

From Exercise 16.5 it appears that a differential equation can have more than one solution. We will in what follows investigate in more detail the question about the number of solutions. In order to understand precisely what is meant by a first order differential equation being linear and what this means for its solution set, we will need the following lemma.

|||| Lemma 16.6

Let p be a continuous function defined on an open interval I in $\mathbb{R}$. Then the map $f : C^1(I) \to C^0(I)$ given by

$$f(x(t)) = x'(t) + p(t)x(t) \tag{16-6}$$

is linear.

|||| Proof

We will show that f satisfies the two linearity requirements L_1 and L_2. Let $x_1, x_2 \in C^1(I)$ (i.e. the two functions are arbitrary differentiable functions with continuous derivatives on I), and let $k \in \mathbb{R}$. That f satisfies L_1 appears from

$$\begin{aligned} f(x_1(t) + x_2(t)) &= (x_1(t) + x_2(t))' + p(t)(x_1(t) + x_2(t)) \\ &= x_1'(t) + x_2'(t) + p(t)(x_1(t) + p(t)x_2(t) \\ &= (x_1'(t) + p(t)x_1(t)) + (x_2'(t) + p(t)x_2(t)) \\ &= f(x_1(t)) + f(x_2(t))\,. \end{aligned}$$

That f satisfies L_2 appears from

$$\begin{aligned} f(kx_1(t)) &= (kx_1'(t)) + p(t)(kx_1(t)) = k(x_1'(t) + p(t)x_1(t)) \\ &= kf(x_1(t)). \end{aligned}$$

By this the proof is completed.

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From Lemma 16.6 we can deduce important properties for the solution set for first order linear differential equations. First we introduce convenient notations for the solution sets that we will treat.

L_{inhom} denotes all solutions for a given inhomogeneous differential equation. L_{inhom} is briefly known as the *solution set* or the *general solution*.

L_{hom} denotes the solution set for a homogeneous differential equation corresponding to an inhomogeneous equation (where the right-hand side $q(t)$ is replaced by 0).

Theorem 16.7 Three Properties

For a first order linear differential equation $x'(t) + p(t)x(t) = q(t)$, $t \in I$:

1. If the equation is homogeneous (i.e. $q(t)$ is the 0-function), then the solution set is a vector subspace of $C^1(I)$.

2. *Structure Theorem*: If the equation is inhomogeneous the general solution L_{inhom} can be written in the form

$$L_{inhom} = x_0(t) + L_{hom} \tag{16-7}$$

   where $x_0(t)$ is a *particular solution* to the inhomogeneous differential equation, and L_{hom} is the solutions set to the corresponding homogeneous differential equation.

3. *Superposition principle*: If $x_1(t)$ is a solution when the right-hand side of the differential equation is replaced by the function $q_1(t)$, and $x_2(t)$ is a solution when the right-hand side is replaced by the function $q_2(t)$, then $x_1(t) + x_2(t)$ is a solution when the right-hand side is replaced by the function $q_1(t) + q_2(t)$.

|||| Proof

We consider the map between vector spaces, $f : C^1(I) \to C^0(I)$ given by

$$f(x(t)) = x'(t) + p(t)x(t)\,. \tag{16-8}$$

This is, according to Lemma 16.6, linear. Therefore we have:

1. L_{hom} is equal to $\ker(f)$. Since the kernel for every linear map is a subspace of the domain, L_{hom} is a subspace of $C^1(I)$.
2. Since the equation $f(x(t)) = x'(t) + p(t)x(t) = q(t)$ is linear, the structure theorem follows directly from the general structure theorem for linear equations (see eNote 12, Theorem 12.14).
3. The superposition principle follows from the fact that f satisfies the linearity requirement L_1. Assume that $f(x_1(t)) = q_1(t)$ and $f(x_2(t)) = q_2(t)$. Then

$$f(x_1(t) + x_2(t)) = f(x_1(t)) + f(x_2(t)) = q_1(t) + q_2(t)\,.$$

By this the proof is completed.

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When we call a first order differential equation of the form (16-2) linear, it is – as shown above – closely related to the fact that its left-hand side represents a linear map, and that its solution set therefore has the unique properties of Theorem 16.7. In the following example we juggle with the properties in order to decide whether a given differential equation is not linear.

|||| Example 16.8 First Order Differential Equation That Is Nonlinear

We consider a first order differential equation

$$x'(t) - (x(t))^2 = q(t)\,,\ t \in \mathbb{R}\,. \tag{16-9}$$

where we in the usual way have isolated the terms that contain the unknown function on the left-hand side. The left-hand side represents the map $f : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ given by

$$f(x(t)) = x'(t) - (x(t))^2\,. \tag{16-10}$$

Here we will show that one, in different ways, can demonstrate that the differential equation is not linear.

1. We can show directly that f does not satisfy the linearity conditions. To show this we can test L_2, e.g. with $k = 2$. We compute the two sides in L_2 :

$$f(2x(t)) = (2x(t))' - (2x(t))^2 = 2x'(t) - 4(x(t))^2$$
$$2f(x(t)) = 2(x'(t) - (x(t))^2) = 2x'(t) - 2(x(t))^2\,.$$

   By subtraction of the two equation we get:

$$f(2x(t)) - 2f(x(t)) = -2(x(t))^2$$

   where the right-hand side is only the 0-function when $x(t)$ is the 0-function. Since L_2 applies for all $x(t) \in C^1(\mathbb{R})$, L_2 is not satisfied. Therefore the equation is nonlinear.

2. The solution set to the corresponding homogeneous equation is not a subspace. E.g. it does not satisfy the stability requirement with repsect to multiplication by a scalar which we can show as follows:

   The function $x_0(t) = -\dfrac{1}{t}$ is a solution to the homogeneous eqution because

$$x_0'(t) - (x_0(t))^2 = \frac{1}{t^2} - \frac{1}{t^2} = 0\,.$$

   But $2 \cdot x_0(t) = -\dfrac{2}{t}$ is not, because

$$(2 \cdot x_0(t))' - (2 \cdot x_0(t))^2 = \frac{2}{t^2} - \frac{4}{t^2} = -\frac{2}{t^2} \neq 0\,.$$

   Therefore the differential equation is not linear.

3. The solution set does not satisfy the superposition principle. E.g. we see that

$$f\left(-\frac{1}{t}\right) = 0 \quad \text{and} \quad f\left(\frac{1}{t}\right) = -\frac{2}{t^2}\,, \quad \text{while}$$
$$f\left(-\frac{1}{t} + \frac{1}{t}\right) = 0 \neq 0 - \frac{2}{t^2}\,.$$

   Therefore the differential equation is not linear.

It follows from the structure theorem that homogeneous equations play a special role for linear differential equations. Therefore we treat them separately in the next section.

16.3 Homogeneous First Order Linear Differential Equations

We now establish a solution formula for homogeneous first order linear equations.

|||| Theorem 16.9 Solution of the Homogeneous Equation

Let $p(t)$ be a continuous function defined on an open real interval I, and let $P(t)$ be an arbitrary antiderivative for $p(t)$, i.e., a function satisfying $P'(t) = p(t)$.

The general solution for the homogeneous first order linear differential equation

$$x'(t) + p(t)x(t) = 0\,,\;\; t \in I\,. \tag{16-11}$$

is then given by

$$x(t) = c\,\mathrm{e}^{-P(t)}\,,\;\; t \in I \tag{16-12}$$

where c is an arbitrary real number.

|||| Proof

The theorem follows from the fact that the derivative of a function $g(t)$ is zero on an interval if and only if that function is constant. We apply this to the function $g(t) = x(t)e^{P(t)}$. Using the chain rule and the product rule for differentiation we have:

$$\begin{aligned}\left(x(t)e^{P(t))}\right)' &= e^{P(t)}x'(t) + p(t)e^{P(t)}x(t)\\ &= e^{P(t)}\left(x'(t) + p(t)x(t)\right).\end{aligned}$$

Since $e^{P(t)} \neq 0$, the above expression is zero if and only if the equation (16-11) holds. That is, the differential equation (16-11) is equivalent to the equation:

$$\left(x(t)e^{P(t))}\right)' = 0.$$

As mentioned, this is equivalent to the statement:

$$x(t)e^{P(t)} = c, \tag{16-13}$$

where c is some real constant, i.e. that $x(t) = ce^{-P(t)}$. This shows that not only is $ce^{-P(t)}$ a solution, for any constant c, but that *any* solution to (16-11) must be of this form, since it must satisfy Equation (16-13) for some c.

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We already know that the solution set is a subspace of $C^1(I)$. From the formula (16-12) we now know that the subspace is 1-dimensional, and that the function $\mathrm{e}^{-P(t)}$ is a basis for the solution set.

|||| Remark 16.10

Theorem 16.9 is also valid if p is a continuous *complex*-valued function, with the slight modification that the arbitrary constant c is now a complex constant. The proof is exactly the same, because the product rule is the same for complex-valued functions of t, and, writing $P(t) = u(t) + iv(t)$, one finds that the derivative of $e^{P(t)}$ is still $P'(t)e^{P(t)}$. Finally, by separating the function into real and imaginary parts, one again finds that the derivative of a complex-valued function is zero if and only if the function is equal to a complex constant.

|||| Exercise 16.11

In Theorem 16.9 an arbitrary antiderivative $P(t)$ for $p(t)$ is used. Explain why it is immaterial to the solution set which antiderivative you use when you apply the theorem.

|||| Example 16.12 Solution of a Homogeneous Equation

A homogeneous first order linear differential equation is given by

$$x'(t) + \cos(t)x(t) = 0, \quad t \in \mathbb{R}. \tag{16-14}$$

We see that that the coefficient function $p(t) = \cos(t)$. An antiderivative for $p(t)$ is $P(t) = \sin(t)$. Then the general solution can be written as

$$x(t) = c\mathrm{e}^{-P(t)} = c\mathrm{e}^{-\sin(t)}, \quad t \in \mathbb{R} \tag{16-15}$$

where c is an arbitrary real number.

16.4 Inhomogeneous Equations Solved by the Guess Method

Now that we know how to find the general solution for homogeneous first order linear differential equations, it is about time to look at the inhomogeneous ones. If you already know or can guess a particular solution to the inhomogeneous equation, it is obvious to use the structure theorem, see Theorem 16.7. This is demonstrated in the following examples.

|||| Example 16.13 Solution Using a Guess and the Structure Theorem

An inhomogeneous first order linear differential equation is given by

$$x'(t) + tx(t) = t, \quad t \in \mathbb{R}. \tag{16-16}$$

It is easily seen that $x_0(t) = 1$ is a particular solution. Then we solve the corresponding homogeneous differential equation

$$x'(t) + tx(t) = 0, \quad t \in \mathbb{R}. \tag{16-17}$$

Using symbols from Theorem 16.9 we have $p(t) = t$ that has the antiderivative

$$P(t) = \frac{1}{2}t^2 .$$

The general solution therefore consists of the following functions where c is an arbitrary real number:

$$x(t) = ce^{-\frac{1}{2}t^2}, \quad t \in \mathbb{R}. \tag{16-18}$$

In short:

$$L_{hom} = \left\{ ce^{-\frac{1}{2}t^2},\ t \in \mathbb{R} \;\middle|\; c \in \mathbb{R} \right\}. \tag{16-19}$$

Now we can establish the general solution to the inhomogeneous differential equation using the structure theorem as:

$$L_{inhom} = x_0(t) + L_{hom} = \left\{ 1 + ce^{-\frac{1}{2}t^2},\ t \in \mathbb{R} \;\middle|\; c \in \mathbb{R} \right\}.$$

|||| Example 16.14 Solution Using a Guess and the Structure Theorem

An inhomogeneous first order linear differential equation is given by

$$x'(t) + 2x(t) = 30 + 8t, \quad t \in \mathbb{R}. \tag{16-20}$$

First let us try to guess a particular solution. Since the right-hand side is first degree polynomial, one can – with the given left-hand side, where you only differentiate and multiply by 2 – assume that a first degree polynomial could be a solution. Therefore we try to insert an arbitrary first degree polynomial $x_0(t) = b + at$ in the left-hand side of the differential equation:

$$x_0'(t) + 2x_0(t) = (b + at)' + 2(b + at) = a + 2b + 2at\,.$$

We compare the resulting expression with the given right-hand side:

$$a + 2b + 2at = 30 + 8t$$

that is satisfied for all $t \in \mathbb{R}$ eactly when

$$a + 2b = 30 \text{ and } 2a = 8 \Leftrightarrow a = 4 \text{ and } b = 13\,.$$

Thus we have found a particular solution

$$x_0(t) = 13 + 4t\,,\ t \in \mathbb{R}\,.$$

Then we solve the corresponding homogeneous differential equation

$$x'(t) + 2x(t) = 0, \quad t \in \mathbb{R}. \tag{16-21}$$

Using symbols from Theorem 16.9 we have $p(t) = 2$ that has the antiderivative $P(t) = 2t$. Therefore the general solution consists of the following functions where c is an arbitrary real number:

$$x(t) = ce^{-2t}, \quad t \in \mathbb{R}. \tag{16-22}$$

In short:

$$L_{hom} = \left\{ ce^{-2t}\,,\ t \in \mathbb{R} \;\middle|\; c \in \mathbb{R} \right\}. \tag{16-23}$$

Now its is possible to establish the general solution to the inhomogeneous differential equation using the structure theorem:

$$L_{inhom} = x_0(t) + L_{hom} = \left\{ 13 + 4t + ce^{-2t}\,,\ t \in \mathbb{R} \;\middle|\; c \in \mathbb{R} \right\}.$$

|||| Example 16.15 Solution Using a Guess and the Structure Theorem

An inhomogeneous first order linear differential equation is given by

$$x'(t) + x(t) = 1 + \sin(2t), \quad t \geq 0\,. \tag{16-24}$$

First let us try to guess a particular solution. Since the right-hand side consists of constant plus a sine function with the angular frequency 2, it is obvious to guess a solution the type

$$x(t) = k + a\cos(2t) + b\sin(2t)\,.$$

By insertion of this in the differential equation we get:

$$\begin{aligned} &-2a\sin(2t) + 2b\cos(2t) + k + a\cos(2t) + b\sin(2t) = 1 + \sin(2t) \\ &\Leftrightarrow (2b+a)\cos(2t) + (b-2a-1)\sin(2t) + (k-1)\,1 = 0\,. \end{aligned}$$

Since the set $(\cos(2t), \sin(2t), 1)$ is linearly independent, this equation is satisfied exactly when

$$2b + a = 0\,,\ b - 2a - 1 = 0 \text{ and } k = 1 \ \Leftrightarrow a = -\frac{2}{5}\,,\ b = \frac{1}{5} \text{ and } k = 1\,.$$

By this we have found a particular solution

$$x_0(t) = 1 - \frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t)\,,\ t \in \mathbb{R}\,.$$

Since the corresponding homogeneous differential equation

$$x'(t) + x(t) = 0, \quad t \geq 0 \tag{16-25}$$

evidently has the general solution

$$x(t) = ce^{-t}, \quad t \geq 0\,, \tag{16-26}$$

we get the general solution to the given inhomogeneous differential equation by use of the structure theorem:

$$L_{inhom} = x_0(t) + L_{hom} = \left\{ 1 - \tfrac{2}{5}\cos(2t) + \tfrac{1}{5}\sin(2t) + ce^{-t}\,,\ t \in \mathbb{R} \;\middle|\; c \in \mathbb{R} \right\}\,.$$

As demonstrated in the three previous examples it makes sense to use the guess method in the inhomogeneous cases, when you already know a particular solution or easily can find one. It only requires that you can find an antiderivative $P(t)$ for the coefficient function $p(t)$.

Otherwise if you do not have an immediate particular solution, you must use the general solution formula (see below) instead. Here you get rid of the guesswork, but you must find two antiderivatives, one is $P(t)$ as above, while the other often is somewhat more difficult (if not impossible) to find, since you must integrate a product of functions. In the following section we establish the general solution formula and discuss the said problems.

16.5 The General Solution Formula

Now we consider the general first order linear differential equation in the standard form

$$x'(t) + p(t)x(t) = q(t), \quad t \in I, \tag{16-27}$$

We can determine the general solution using the following general formula.

|||| Theorem 16.16 The General Solution Formula

Let $p(t)$ and $q(t)$ be continuous functions on an open real interval I, and let $P(t)$ be an arbitrary antiderivative to $p(t)$. The differential equation

$$x'(t) + p(t)x(t) = q(t), \quad t \in I \tag{16-28}$$

then has the general solution

$$x(t) = \mathrm{e}^{-P(t)} \int \mathrm{e}^{P(t)} q(t) dt + c\mathrm{e}^{-P(t)}, \quad t \in I \tag{16-29}$$

where c is an arbitrary real number.

|||| Proof

The second term in the solution formula (16-29) we identify as L_{hom}. If we can show that the first term is a particular solution to the differential equation, then it follows from the structure theorem that the solution formula is the general solution to the differential equation.

First we must of course ask ourselves whether the indefinite integral that is part of the solution formula even exists. It does! See a detailed reasoning for this in the proof of the existence and uniqueness Theorem 16.24. That the first term

$$x_0(t) = \mathrm{e}^{-P(t)} \int \mathrm{e}^{P(t)} q(t) dt$$

is a particular solution we show by testing. We insert the term in left-hand side of the differential equation and see that the result is equal to the right-hand side.

$$\begin{aligned} x_0'(t) + p(t)x_0(t) &= \left(\mathrm{e}^{-P(t)} \int \mathrm{e}^{P(t)} q(t)\,\mathrm{d}t\right)' + p(t)\,\mathrm{e}^{-P(t)} \int \mathrm{e}^{P(t)} q(t)\,\mathrm{d}t \\ &= -p(t)\mathrm{e}^{-P(t)} \int \mathrm{e}^{P(t)} q(t)\,\mathrm{d}t + \mathrm{e}^{-P(t)}\mathrm{e}^{P(t)} q(t) + p(t)\mathrm{e}^{-P(t)} \int \mathrm{e}^{P(t)} q(t) dt \\ &= q(t)\,. \end{aligned}$$

By this the proof is completed.

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|||| Remark 16.17

Using Remark 16.10, it is straightforward to show that Theorem 16.16 is also valid if $p(t)$ and $q(t)$ are continuous *complex*-valued functions, with the modification that the arbitrary constant c is a complex constant.

If one inserts $q(t) = 0$ in the general solution formula (16-29), the first term disappears, and what is left is the second term that is the formula (16-12) for homogeneous equation. Therefore the formula (16-29) is a "'general formula"' that covers both the homogeneous and the inhomogeneous case.

|||| Exercise 16.18

The solution formula (16-29) includes the indefinite integral $\int \mathrm{e}^{P(t)} q(t) dt$, that represents an arbitrary antiderivative of $e^{P(t)} q(t)$. Explain why it does not matter to the solution set which antiderivative you choose to use, when you apply the formula.

Now we give a few examples using the general solution formula. Since it contains an indefinite integral of a product of functions you will often need *integration by parts*, which the second example demonstrates.

|||| Example 16.19 Solution Using the General Formula

Given the differential equation

$$x'(t) + \frac{2}{t}x(t) = 8t - \frac{10}{t}, \quad t > 0\,. \tag{16-30}$$

With the symbols in the general solution formula we have $p(t) = \frac{2}{t}$ and $q(t) = 8t - \frac{10}{t}$. An antiderivative for $p(t)$ is given by:

$$P(t) = 2\ln t\,. \tag{16-31}$$

We then have

$$\mathrm{e}^{-P(t)} = \mathrm{e}^{-2\ln t} = \mathrm{e}^{\ln(t^{-2})} = t^{-2} = \frac{1}{t^2}\,. \tag{16-32}$$

From this it follows that $\mathrm{e}^{P(t)} = t^2$. Now we use the general solution formula:

$$\begin{aligned} x(t) &= \mathrm{e}^{-P(t)} \int \mathrm{e}^{P(t)} q(t)\,\mathrm{d}t + c\mathrm{e}^{-P(t)} \\ &= \frac{1}{t^2} \int t^2 \left(8t - \frac{10}{t}\right)\,\mathrm{d}t + c\frac{1}{t^2} \\ &= \frac{1}{t^2} \int (8t^3 - 10t)\,\mathrm{d}t + c\frac{1}{t^2} \\ &= \frac{1}{t^2}\left(2t^4 - 5t^2 + c\right) \\ x(t) &= 2t^2 - 5 + \frac{c}{t^2}\,, \quad t > 0. \end{aligned} \tag{16-33}$$

The general solution consists of these functions where c is an arbitrary real number. In short:

$$L_{inhom} = \left\{ x(t) = 2t^2 - 5 + \frac{c}{t^2}\,,\ t > 0 \;\middle|\; c \in \mathbb{R} \right\}. \tag{16-34}$$

|||| Example 16.20 Solution Using the General Formula

We will solve the differential equation

$$x'(t) - \frac{1}{t}x(t) = t^2\sin(2t), \quad t > 0\,. \tag{16-35}$$

With the symbols in the general solution formula we have $p(t) = -\frac{1}{t}$ and $q(t) = t^2\sin(2t)$. An antiderivative for $p(t)$ is given by:

$$P(t) = -\ln t\,. \tag{16-36}$$

We then have

$$\mathrm{e}^{-P(t)} = \mathrm{e}^{\ln t} = t \quad \text{and} \quad \mathrm{e}^{P(t)} = \mathrm{e}^{-\ln t} = (\mathrm{e}^{\ln t})^{-1} = \frac{1}{t}\,. \tag{16-37}$$

Now we use the general solution formula::

$$\begin{aligned} x(t) &= \mathrm{e}^{-P(t)} \int \mathrm{e}^{P(t)} q(t)\,\mathrm{d}t + c\mathrm{e}^{-P(t)} \\ &= t \int \frac{1}{t} t^2 \sin(2t)\,\mathrm{d}t + ct \\ &= t \int t \sin(2t)\,\mathrm{d}t + ct\,. \end{aligned}$$

Now we perform an intermediate computation where we use integration by parts to find the antiderivative.

$$\begin{aligned} \int t \sin(2t) dt &= -\frac{1}{2} t \cos(2t) - \int -\frac{1}{2} \cos(2t)\,\mathrm{d}t \\ &= -\frac{1}{2} t \cos(2t) + \frac{1}{2} \int \cos(2t)\,\mathrm{d}t \\ &= -\frac{1}{2} t \cos(2t) + \frac{1}{4} \sin(2t)\,. \end{aligned}$$

And return to the computation

$$\begin{aligned} x(t) &= t \int t \sin(2t)\,\mathrm{d}t + ct \\ &= t \left(-\frac{1}{2} t \cos(2t) + \frac{1}{4} \sin(2t) \right) + ct \\ x(t) &= -\frac{1}{2} t^2 \cos(2t) + \frac{1}{4} t \sin(2t) + ct \quad t > 0\,. \end{aligned}$$

The general solution consists of these functions where c is an arbitrary real number. In short:

$$L_{inhom} = \left\{ x(t) = -\tfrac{1}{2} t^2 \cos(2t) + \tfrac{1}{4} t \sin(2t) + ct\,,\ t > 0 \;\middle|\; c \in \mathbb{R} \right\}. \tag{16-38}$$

Until now we have considered the general solution to the differential equation. Often one is interested in a particular solution that for a given value of t assumes a desired functional value, a so-called initial value problem. We treat this in the next section.

16.6 Initial Value Problems

We consider a first order linear differential equation in its standard form

$$x'(t) + p(t)x(t) = q(t), \quad t \in I\,. \tag{16-39}$$

If we need a solution to the equation that for a given value of t assumes a desired functional value, the following questions arise: 1) Is there even a solution that satisfies the desired properties and 2) If yes, how many solutions are there? Before we answer these question generally, we consider a couple of examples.

|||| Example 16.21 An Initial Value Problem

In the Example 16.13 we found the general solution to the differential equation

$$x'(t) + tx(t) = t, \quad t \in \mathbb{R}\,. \tag{16-40}$$

viz.

$$x(t) = 1 + c\mathrm{e}^{-\frac{1}{2}t^2}\,,\ t \in \mathbb{R}$$

where c is an arbitrary real number.

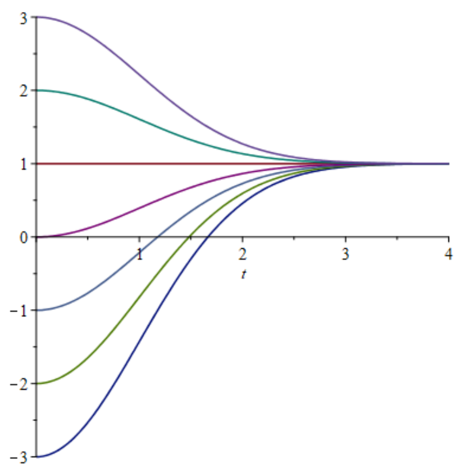
Now we will find the solution $x_0(t)$ that satisfies the initial value condition $x_0(0) = 3$. This is done by insertion of the initial value in the general solution, whereby we determine c:

$$x_0(0) = 1 + c\mathrm{e}^{-\frac{1}{2}\cdot 0^2} = 1 + c = 3 \Leftrightarrow\ c = 2\,. \tag{16-41}$$

Therefore the conditioned solution function to the differential equation is given by

$$x_0(t) = 1 + 2\mathrm{e}^{-\frac{1}{2}t^2}\,,\ t \in \mathbb{R}\,. \tag{16-42}$$

The figure below shows the graphs for the seven solutions that correspond to initial value conditions $x_0(0) = b$ where $b \in \{-3, -2, -1, 0, 1, 2, 3\}$. The solution we just found is the uppermost. The others are found in a similar way.



|||| Example 16.22 An Initial Value Problem

In Example 16.20 we found the general solution to the differential equation

$$x'(t) + \frac{2}{t}x(t) = 8t - \frac{10}{t}, \quad t > 0, \tag{16-43}$$

viz.

$$x(t) = 2t^2 - 5 + \frac{c}{t^2}, \quad t > 0$$

where c is an arbitrary real number.

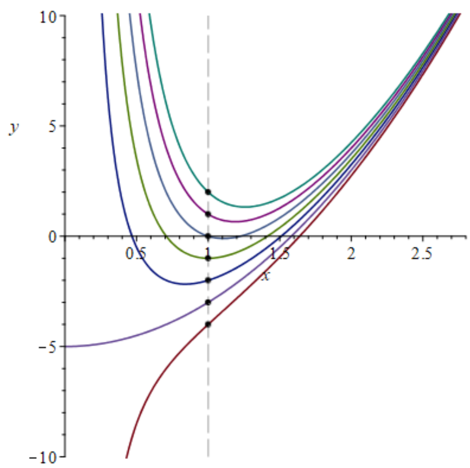
Now we will find the particular solution $x_0(t)$ that satisfies the initial value condition $x_0(1) = 2$. It is done by insertion of initial value in the general solution, whereby we determine c:

$$x_0(1) = 2 \cdot 1^2 - 5 + \frac{c}{1^2}2 - 5 + c = 2 \Leftrightarrow c = 5\,. \tag{16-44}$$

Therefore the conditioned solution function to the differential equation is given by

$$x_0(t) = 2t^2 - 5 + \frac{5}{t^2}\,, \quad t > 0\,. \tag{16-45}$$

The figure below shows the graphs for the seven solutions that correspond to initial value conditions $x_0(0) = b$ where $b \in \{-4, -3, -2, -1, 0, 1, 2\}$. The solution we just found is the uppermost. The others are found in a similar way.



|||| Example 16.23 The Stationary Response

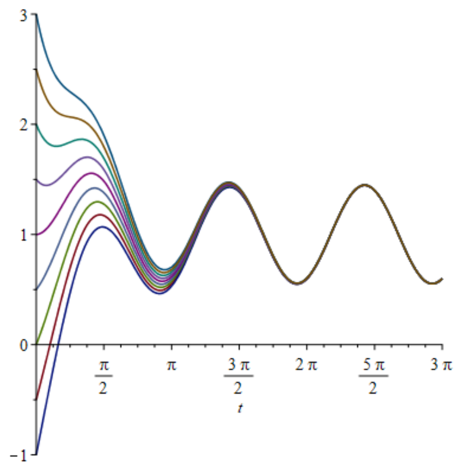
In Example 16.15 we found the general solution to the differential equation

$$x'(t) + x(t) = 1 + \sin(2t), \quad t \geq 0 \tag{16-46}$$

viz.

$$x(t) = 1 - \frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t) + ce^{-t}, \quad t \geq 0\,. \tag{16-47}$$

Here we show a series of solutions with the initial values from -1 to 3 for $t = 0$:



The figure indicates that all solutions approach a periodic oscillation when $t \to \infty$. That this is the case is seen from the general solution of the differential equation where the fourth term ce^{-t} regardless of the choice for c is negligible due to the negative exponent. The first three terms constitute the *the stationary response*.

In the three preceding examples we did not have any difficulties in finding a solution to the differential equation that satisfied a given initial condition. In fact we saw that, for each of the initial value conditions considered, exactly one solution that satisfied the condition exists. That this applies in general we show in the following theorem.

|||| Theorem 16.24 Existence and Uniqueness of Solutions

Given the differential equation

$$x'(t) + p(t)x(t) = q(t), \quad t \in I \tag{16-48}$$

where I is an open interval and $p(t)$ and $q(t)$ are continuous functions on I.

Then: for every number pair (t_0, b) exactly one (particular) solution $x_0(t)$ to the differential equation exists that satisfies the *inital value condition*

$$x_0(t_0) = b\,. \tag{16-49}$$

|||| Proof

From Theorem 16.16 we know that the set of solutions to the differential equation (16-48) is given by

$$x(t) = \mathrm{e}^{-P(t)} \int \mathrm{e}^{P(t)} q(t)dt + c\mathrm{e}^{-P(t)} \tag{16-50}$$

where c is an arbitrary real number.

Let us first investigate the indefinite integral that is included in the formula. Does it exist? This is equivalent to asking: does an antiderivative for the function under the integration sign exist? We must start with $p(t)$. Since it is continuous, it has an antiderivative which we call $P(t)$. Being an antiderivative, $P(t)$ is differentiable and thus continuous. Since the exponential function is also continuous the composite function $\mathrm{e}^{P(t)}$ is continuous. Finally since $q(t)$ is continuous, the product $\mathrm{e}^{P(t)}q(t)$ is continuous.

By this we have shown that the function under the integration sign is continuous. Therefore it has an antiderivative, in fact infinitely many antiderivatives that only differ from each other by constants. We choose an arbitrary antiderivative and call it $F(t)$. Now we can reformulate the solution formula as

$$x(t) = \mathrm{e}^{-P(t)}F(t) + c\mathrm{e}^{P(t)} \tag{16-51}$$

where c is an arbitrary real number. Then we insert the initial value condition:

$$x(t_0) = \mathrm{e}^{-P(t_0)}F(t_0) + c\mathrm{e}^{-P(t_0)} = b \;\Leftrightarrow\; c = F(t_0) + b\mathrm{e}^{-P(t_0)}$$

where we first multiplied by $e^{P(t_0)}$ on both sides of the equality sign and then isolated c. Thus in the general solution set exactly one solution exists that satisfies the initial value condition, viz. the one that emerge when we in (16-51) insert the found value of c.

By this the proof is completed.

■

|||| Exercise 16.25

Again let us consider the linear map $f : C^1(I) \to C^0(I)$ that represents the left-hand side of a first order linear differential equation:

$$f(x(t)) = x'(t) + p(t)x(t) \tag{16-52}$$

We know that $\ker(f)$ is one dimensional and has the basis vector $e^{-P(t)}$. But what is the image space (the range) for f?

We end this section by an example that shows how it is possible to "go backwards" from a given general solution to the differential equation it solves.

|||| Example 16.26 From Solution to the Differential Equation

The general solution to a first order inhomogeneous differential equation is given by

$$L_{inhom} = \left\{\, x(t) = te^{-5t} + ct\,,\ t > 0 \;\middle|\; c \in \mathbb{R} \,\right\}. \tag{16-53}$$

Determine the corresponding differential equation that has the form

$$x'(t) + p(t)x(t) = q(t)\,. \tag{16-54}$$

(That is, determine $p(t)$ and $q(t)$).

First we consider the corresponding homogeneous differential equation. With the structure theorem in mind we immediately see that

$$L_{hom} = \left\{\, x(t) = ct\,,\ t > 0 \;\middle|\; c \in \mathbb{R} \,\right\}$$

By insertion of $x(t) = ct$ in the homogeneous equation $x'(t) + p(t)x(t) = 0$ we get

$$c + p(t)ct = 0\,, \tag{16-55}$$

and since this equation must hold for all c

$$p(t) = -\frac{1}{t}\,. \tag{16-56}$$

Since we now know $p(t)$, it only remains to determine the right-hand side $q(t)$. We find this by insertion of the particular solution $x(t) = t\mathrm{e}^{-5t}$ into the left-hand side of the equation.

$$\mathrm{e}^{-5t} - 5t\mathrm{e}^{-5t} - \frac{1}{t}\cdot t\mathrm{e}^{-5t} = -5t\mathrm{e}^{-5t} = q(t)\,. \tag{16-57}$$

Now since both $p(t)$ and $q(t)$ are determined, the whole differential equation is determined as:

$$x'(t) - \frac{1}{t}x(t) = -5t\mathrm{e}^{-5t}\,,\ t > 0\,. \tag{16-58}$$

16.7 Finite Dimensional Domain

In some cases we know in advance what type of solutions to the differential equation are of interest. Therefore one can choose to restrict the domain $C^1(\mathbb{R})$. We end this eNote with an example where the domain is a finite dimensional subset of $C^1(\mathbb{R})$ which leads to the introduction of matrix methods.

|||| Example 16.27 Solution by Matrix Compution

Consider the differential equation

$$x'(t) + (1-2t)x(t) = 7t - 4t^3\,. \tag{16-59}$$

In this example we are only interested in solutions that belong to the polynomial space $P_2(\mathbb{R})$, i.e. the subset of $C^1(\mathbb{R})$ that has the monomial base $(1, t, t^2)$.

To find the range $f\left(P_2(\mathbb{R})\right)$ of the linear map f that represents the left-hand side of the differential equation, we first determine the images of the basis vectors:

$$f(1) = 1 - 2t\,,\ f(t) = 1 + t - 2t^2 \ \text{ and } \ f(t^2) = 2t + t^2 - 2t^3\,.$$

Since $P_3(\mathbb{R})$ has the monomial base $(1, t, t^2, t^3)$, and the found images lie in their span, we see that the range $f(P_2(\mathbb{R}))$ is a subspace of $P_3(\mathbb{R})$.

We want to solve the equation

$$f(x(t)) = 7t - 4t^3,$$

which can be expressed in matrix form as

$$\mathbf{F}\,\mathbf{x} = \mathbf{b},$$

where $\mathbf{F}$ is the mapping matrix for f with respect to the monomial bases in $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$, $\mathbf{x}$ is the coordinate matrix for the unknown polynomial with respect to the monomial basis in $P_2(\mathbb{R})$, and $\mathbf{b}$ is the coordinate matrix for the right-hand side of the differential equation with respect to the monomial basis in $P_3(\mathbb{R})$.

Thus, when restricted to $P_2(\mathbb{R})$, the differential equation becomes an inhomogeneous system of linear equations. The first three columns of the augmented matrix $\mathbf{T}$ of the system are given by $\mathbf{F}$, while the fourth column is $\mathbf{b}$:

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -2 & 1 & 2 & 7 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & -4 \end{bmatrix} \rightarrow \operatorname{rref}(\mathbf{T}) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the rank of $\mathbf{T}$ is seen to be 3, the differential equation has only one solution. Since the fourth column in $\operatorname{rref}(\mathbf{T})$ states the coordinate vector of the solution with respect to the monomial basis in $P_2(\mathbb{R})$, the solution can immediately be stated as:

$$x_0(t) = -1 + t + 2t^2.$$

|||| Exercise 16.28

1. Solve the differential equation in Example 16.27 by the guess method or the general solution formula.
2. How does the general solution differ from the one found in the example?

|||| Exercise 16.29

Replace the right-hand side in the differential equation in Example 16.27 by the function $q(t) = 1$.

1. Show, using matrix computation, that the differential equation does not have a solution in the subspace $P_2(\mathbb{R})$ given in the example.
2. Using Maple (or other software), find the solution $x_0(t)$ to the differential equation that satisfies the initial value condition $x_0(t) = 0$ and draw its graph.