eNote 14

Similarity and Diagonalization

In this eNote it is explained how certain square matrices can be diagonalized by the use of eigenvectors. Therefore it is presumed that you know how to determine eigenvalues and eigenvectors for a square matrix and furthermore that you know about algebraic and geometric multiplicity.

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If we consider a linear map $f : V \to V$ of an n-dimensional vector space V to itself, then the mapping matrix for f with respect to an arbitrary basis for f is a square, $n \times n$ matrix. If two bases a and b for V are given, then the relation between the corresponding mapping matrices ${}_{\mathrm{a}}\mathbf{F}_{\mathrm{a}}$ and ${}_{\mathrm{b}}\mathbf{F}_{\mathrm{b}}$ are given by

$$ {}_{\mathrm{b}}\mathbf{F}_{\mathrm{b}} = ({}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}})^{-1} \cdot {}_{\mathrm{a}}\mathbf{F}_{\mathrm{a}} \cdot {}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}} \tag{14-1} $$

where ${}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}} = \begin{bmatrix} {}_{\mathrm{a}}\mathbf{b}_1 & {}_{\mathrm{a}}\mathbf{b}_2 & \cdots & {}_{\mathrm{a}}\mathbf{b}_n \end{bmatrix}$ is the change of basis matrix that shifts from b to a coordinates.

It is of special interest if a basis v consisting of eigenvectors for f can be found. Viz. let a be an arbitrary basis for V and ${}_{\mathrm{a}}\mathbf{F}_{\mathrm{a}}$ the corresponding mapping matrix for f. Furthermore let v be an eigenvector basis for V with respect to f. From Theorem 13.14 in eNote 13 it appears that the mapping matrix for f with respect to the v-basis is a diagonal matrix $\mathbf{\Lambda}$ in which the diagonal elements are the eigenvalues of f. If $\mathbf{V}$ denotes the change of basis matrix that shifts from v-coordinates to the a-coordinate vectors, according to (14-1)$\mathbf{\Lambda}$ will appear as

$$ \mathbf{\Lambda} = \mathbf{V}^{-1} \cdot {}_{\mathrm{a}}\mathbf{F}_{\mathrm{a}} \cdot \mathbf{V}\,. \tag{14-2} $$

Naturally formula 14-1 and formula 14-2 inspire questions that take their starting point in square matrices: Which conditions should be satisfied in order for two given square matrices to be interpreted as mapping matrices for the same linear map with respect to two different bases? And which conditions should a square matrix satisfy in order to be a mapping matrix for a linear map that in another basis has a diagonal matrix as a mapping matrix? First we study these questions in a pure matrix algebra context and return in the last subsection to the mapping viewpoint. For this purpose we now introduce the concept similar matrices.

14.1 Similar Matrices

|||| Definition 14.1 Similar Matrices

Given the $n \times n$-matrices $\mathbf{A}$ and $\mathbf{B}$. One says that $\mathbf{A}$ is *similar to* $\mathbf{B}$ if an invertible matrix $\mathbf{M}$ can be found such that

$$\mathbf{B} = \mathbf{M}^{-1}\,\mathbf{A}\,\mathbf{M}\,. \tag{14-3}$$

|||| Example 14.2 Similar Matrices

Given the matrices $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 8 & 21 \\ -3 & -10 \end{bmatrix}$.

The matrix $\mathbf{M} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ is invertible and has the inverse matrix $\mathbf{M}^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

Consider the following calculation:

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 21 \\ -3 & -10 \end{bmatrix}.$$

This shows that $\mathbf{A}$ is similar to $\mathbf{B}$.

If $\mathbf{A}$ is similar to $\mathbf{B}$, then $\mathbf{B}$ is also similar to $\mathbf{A}$. If we put $\mathbf{N} = \mathbf{M}^{-1}$ then $\mathbf{N}$ is invertible and

$$\mathbf{B} = \mathbf{M}^{-1}\,\mathbf{A}\,\mathbf{M} \Leftrightarrow \mathbf{M}\,\mathbf{B}\,\mathbf{M}^{-1} = \mathbf{A} \Leftrightarrow \mathbf{A} = \mathbf{N}^{-1}\,\mathbf{B}\,\mathbf{N}\,,$$

Therefore one uses the phrase: $\mathbf{A}$ and $\mathbf{B}$ are *similar matrices* .

|||| Theorem 14.3 Similarity Is Transitive

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be $n \times n$-matrices. If $\mathbf{A}$ is similar to $\mathbf{B}$ and $\mathbf{B}$ is similar to $\mathbf{C}$ then $\mathbf{A}$ is similar to $\mathbf{C}$.

|||| Exercise 14.4

Prove Theorem 14.3.

Regarding the eigenvalues of similar matrices the following theorem applies.

|||| Theorem 14.5 Similarity and Eigenvalues

If $\mathbf{A}$ is similar to $\mathbf{B}$ then the two matrices have identical eigenvalues with the same corresponding algebraic and geometric multiplicities.

|||| Proof

Let $\mathbf{M}$ be an invertible matrix that satisfies $\mathbf{B} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ and let, as usual, $\mathbf{E}$ denote the identity matrix of the same size as the three given matrices. Then:

$$\begin{aligned} \det(\mathbf{B} - \lambda\mathbf{E}) &= \det(\mathbf{M}^{-1}\mathbf{A}\mathbf{M} - \lambda\mathbf{M}^{-1}\mathbf{E}\mathbf{M}) \\ &= \det(\mathbf{M}^{-1}(\mathbf{A} - \lambda\mathbf{E})\mathbf{M}) \\ &= \det(\mathbf{A} - \lambda\mathbf{E})\,. \end{aligned} \tag{14-4}$$

Thus it is shown that the two matrices have the same characteristic polynomial and thus the same eigenvalues with the same corresponding algebraic multiplicities. Moreover, that they have the same eigenvalues appears from Theorem 14.13 which is given below: When $\mathbf{A}$ and

B can represent the same linear map f with respect to different bases they have identical eigenvalues, viz. the eigenvalues of f.

But the eigenvalues also do have the same geometric multiplicities. This follows from the fact that the eigenspaces for **A** and **B** with respect to any of the eigenvalues can be interpreted as two different coordinate representations of the same eigenspace, viz. the eigenspace for f with respect to the said eigenvalue.

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Note that Theorem 14.5 says that two similar matrices have the same eigenvalues, but not vice versa: that two matrices, which have the same eigenvalues, are similar. There is a difference and only the first statement is true.

Two similar matrices **A** and **B** have the same eigenvalues, but an eigenvector for the one is not generally and eigenvector for the other. But if **v** is an eigenvector for **A** corresponding to the eigenvalue λ then $\mathbf{M}^{-1}\mathbf{v}$ is an eigenvector for **B** corresponding to the eigenvalue λ, where **M** is the invertible matrix that satisfies $\mathbf{B} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$. Viz.:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \quad \Leftrightarrow \quad \mathbf{M}^{-1}\mathbf{A}\mathbf{v} = \mathbf{M}^{-1}\lambda\mathbf{v} \quad \Leftrightarrow \quad \mathbf{B}(\mathbf{M}^{-1}\mathbf{v}) = \lambda(\mathbf{M}^{-1}\mathbf{v})\,. \tag{14-5}$$

|||| Exercise 14.6

Explain that two square $n \times n$-matrices are similar, if they have identical eigenvalues with the same corresponding geometric multiplicities and that the sum of the geometric multiplicities is n.

14.2 Matrix Diagonalization

Consider a matrix **A** and an invertible matrix **V** given by

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \text{ and } \mathbf{V} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}. \tag{14-6}$$

Since

$$\mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{V} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix},$$

$\mathbf{A}$ possesses a special property: it is similar to a diagonal matrix, viz. the diagonal matrix

$$\mathbf{\Lambda} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

In this context one says that $\mathbf{A}$ has been ***diagonallzed by similarity transformation***.

Now we will ask the question whether or not an arbitrary square matrix $\mathbf{A}$ can be diagonalized by a similarity transformation. Therefore we form the equation

$$\mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{V} = \mathbf{\Lambda},$$

where $\mathbf{V}$ is an invertible matrix and $\mathbf{\Lambda}$ is a diagonal matrix. Below we prove that the equation has exactly one solution if the columns of $\mathbf{V}$ are linearly independent eigenvectors for $\mathbf{A}$, and the diagonal elements in $\mathbf{\Lambda}$ are the eigenvalues of $\mathbf{A}$ written such that the i-th column of $\mathbf{V}$ is an eigenvector corresponding to the eigenvalue for the i-th column in $\mathbf{\Lambda}$.

We note that this is in agreement with the example-matrices in (14-6) above:

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{14-7}$$

and

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -2 \\ 1 \end{bmatrix}. \tag{14-8}$$

We see from (14-7) that the first column of $\mathbf{V}$ as expected is an eigenvector for $\mathbf{A}$ corresponding to the first diagonal element in $\mathbf{\Lambda}$, and we see in (14-8) that the second column of $\mathbf{V}$ is an eigenvector corresponding to the second diagonal element in $\mathbf{\Lambda}$.

|||| Theorem 14.7 Diagonalization by Similarity Transformation

If a square $n \times n$-matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ corresponding to the n (not necessarily different) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, it can be diagonalized by the similarity transformation

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}, \tag{14-9}$$

where

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \quad \text{and} \quad \mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n). \tag{14-10}$$

If $\mathbf{A}$ does not have n linearly independent eigenvectors, it cannot be diagonalized by a similarity transformation.

|||| Proof

Suppose that $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and $\mathbf{v}_i$ corresponds to the eigenvalue λ_i, for $i = 1 \ldots n$. Then the following equations are valid:

$$\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \quad , \quad \mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \quad , \quad \ldots \quad , \quad \mathbf{A}\mathbf{v}_n = \lambda_n \mathbf{v}_n \tag{14-11}$$

The n equations can be gathered in a system of equations:

$$\begin{aligned} &\begin{bmatrix} \mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \cdots & \mathbf{A}\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1\lambda_1 & \mathbf{v}_2\lambda_2 & \cdots & \mathbf{v}_n\lambda_n \end{bmatrix} \\ \Leftrightarrow\ &\mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\ \Leftrightarrow\ &\mathbf{A}\mathbf{V} = \mathbf{V}\mathbf{\Lambda} \end{aligned} \tag{14-12}$$

Now all the eigenvectors are inserted (vertically one after the other) in the matrix $\mathbf{V}$ in the same order as that of the eigenvalues in the diagonal of the matrix $\mathbf{\Lambda}$ that outside the diagonal contains only zeroes. Since the eigenvectors are linearly independent the matrix $\mathbf{V}$ is invertible. Therefore the inverse $\mathbf{V}^{-1}$ exists, and we multiply by this from the left on both sides of the equality sign:

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{V}^{-1}\mathbf{V}\mathbf{\Lambda} \ \Leftrightarrow\ \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}\,. \tag{14-13}$$

Thus the first part of the theorem is proved. Suppose on the contrary that $\mathbf{A}$ can be diagonalized by a similarity transformation. Then an invertible matrix $\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$ and a diagonal matrix $\mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$ exist such that

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}\,. \tag{14-14}$$

If we now repeat the transformations in the first part of the proof only now in the opposite order, it is seen that (14-14) is the equivalent of the following n equations:

$$\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \quad , \quad \mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \quad , \quad \ldots \quad , \quad \mathbf{A}\mathbf{v}_n = \lambda_n \mathbf{v}_n \tag{14-15}$$

from which it appears that $\mathbf{v}_i$ for $i = 1 \ldots n$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_i.

Therefore diagonalization by similarity transformation can only be obtained by the method described in the first part of the theorem.

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The following theorem can be of great help when one investigates whether matrices can be diagonalized by similarity in different contexts . The main result is already given in Theorem 14.7, but here we refine the conditions by drawing upon previously proven theorems about the eigenvalue problem for linear maps and matrices.

|||| Theorem 14.8 Matrix Diagonalizability

For a given $n \times n$-matrix $\mathbf{A}$ we have:

$\mathbf{A}$ can be diagonalized by a similarity transformation

1. if n different eigenvalues for $\mathbf{A}$ exist.
2. if the sum of the geometric multiplicities of the eigenvalues is n .

$\mathbf{A}$ *cannot* be diagonalized by similarity transformation

3. if the sum of the geometric multiplicities of the eigenvalues is less than n .
4. if an eigenvalue λ with $\operatorname{gm}(\lambda) < \operatorname{am}(\lambda)$ exists.

|||| Proof

Ad. 1. If a proper eigenvector from each of the n eigenspaces is chosen, it follows from Corollary 13.9 that the collected set of n eigenvectors is linearly independent. Therefore, according to Theorem 14.7, $\mathbf{A}$ can be diagonalized by similarity transformation.

Ad. 2: If a basis from each of the eigenspaces is chosen, then the collected set of the chosen n eigenvectors according to Corollary 13.9 is linearly independent. Therefore, according to Theorem 14.7 $\mathbf{A}$ can be diagonalized by similarity transformation .

Ad. 3: If the sum of the geometric multiplicities is less than n, n linearly independent eigenvectors for $\mathbf{A}$ do not exist. Therefore, according to Theorem 14.7 $\mathbf{A}$ cannot be diagonalized by similarity transformation.

Ad. 4: Since according to Theorem 13.34 point 1, the sum of the algebraic multiplicity is less than or equal to n, and since according to the same theorem point 2 for every eigenvalue λ $\operatorname{gm}(\lambda) \leq \operatorname{am}(\lambda)$, the sum of the geometric multiplicities cannot become n, if one of the geometric multiplicities is less than its algebraic one. Therefore, according to what has just been proved, $\mathbf{A}$ cannot be diagonalized by similarity transformation.

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A typical special case is that of a square $n \times n$-matrix with n different eigenvalues. Theorem 14.8 point 1 guarantees that all matrices of this type can be diagonalized by similarity transformation.

In the following examples we will see how to investigate in practice whether diagonalization by similarity transformation is possible and, if so, carry it through.

|||| Example 14.9

The square matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 3 & 2 \\ 2 & 10 & 4 \\ 2 & 6 & 8 \end{bmatrix} \tag{14-16}$$

has the eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 15$. The vectors $\mathbf{v}_1 = (-2, 0, 1)$ and $\mathbf{v}_2 = (-3, 1, 0)$ are linearly independent vectors correponding to λ_1, and the vector $\mathbf{v}_3 = (1, 2, 2)$ is a proper eigenvector corresponding to λ_2. The collected set of the three eigenvectors is linearly independent according to Corollary 13.9. Therefore, according to Theorem 14.7, it is possible to diagonalize $\mathbf{A}$, because $n = 3$ linearly independent eigenvectors exist. Therefore we can write $\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$, where

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 15 \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}. \tag{14-17}$$

14.3 Complex Diagonalization

What we so far have said about similar matrices is generally valid for square, *complex* matrices. Therefore the basic equation for diagonalization by similarity transformation:

$$\mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{V} = \mathbf{\Lambda},$$

will be understood in the broadest sense, where the matrices $\mathbf{A}$, $\mathbf{V}$ and $\mathbf{\Lambda}$ are complex $n \times n$-matrices. Until now we have limited ourselves to real examples, that is examples where it has been possible to satisfy the basic equation (14.3) with real matrices. We will in the following look upon a special situation that is typical in technical applications of diagonalization: For a given *real* $n \times n$ matrix $\mathbf{A}$ we seek an invertible matrix $\mathbf{M}$ and a diagonal matrix $\mathbf{\Lambda}$ satisfying the basic equation in a broad context where $\mathbf{M}$ and $\mathbf{\Lambda}$ possibly are complex (not real) $n \times n$ matrices.

The following example shows a real 3×3 matrix that cannot be diagonalized (with only non-complex entries in the diagonal) because its characteristic polynomial only has one real root. On the other hand it can be diagonalized in a complex sense.

|||| Example 14.10 Complex Diagonalization of a Real Matrix

The square matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{14-18}$$

has the eigenvalues $\lambda_1 = 2$, $\lambda_2 = -i$ and $\lambda_3 = i$. $\mathbf{v}_1 = (1, 0, 0)$ is a proper eigenvector corresponding to λ_1, $\mathbf{v}_2 = (-2+i, i, 1)$ is a proper eigenvector corresponding to λ_2, and $\mathbf{v}_3 = (-2-i, -i, 1)$ is a proper eigenvector belonging to λ_3. The collected set of the three said eigenvectors is linearly independent according to Corallary 13.9. Therefore, according to Theorem 14.7, it is possible to diagonalize $\mathbf{A}$. Therefore we can write $\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$, where

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -2+i & -2-i \\ 0 & i & -i \\ 0 & 1 & 1 \end{bmatrix}. \tag{14-19}$$

The next example shows a real, square matrix that cannot be diagonalized either in a real or in a complex way.

|||| Example 14.11 Non-Diagonalizable Square Matrix

Given the square matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \tag{14-20}$$

and $\mathbf{A}$ has the eigenvalues $\lambda_1 = 3$ and $\lambda_3 = 5$. The eigenvalue 3 has the algebraic multiplicity 2, but only one linearly independent eigenvector can be chosen, e.g. $\mathbf{v}_1 = (1, -1, 0)$. Thus the eigenvalue has the geometric multiplicity 1. Therefore, according to Theorem 14.7, it is not possible to diagonalize $\mathbf{A}$ by similarity transformation.

|||| Exercise 14.12

For the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 9 \\ 1 & -6 \end{bmatrix} \tag{14-21}$$

the following should be determined:

1. All eigenvalues and their algebraic multiplicities.
2. All corresponding linearly independent eigenvectors and thus the geometric multiplicities of the eigenvectors.
3. If possible, $\mathbf{A}$ is to be diagonalized: Determine a diagonal matrix $\mathbf{\Lambda}$ and an invertible matrix $\mathbf{V}$ for which $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}$. What are the requirements for the diagonalization to be carried through? Which numbers and vectors are used in $\mathbf{\Lambda}$ and $\mathbf{V}$?

14.4 Diagonalization of Linear Maps

In the introduction to this eNote we asked the question: What conditions should be satisfied so that two given square matrices can be interpreted as mapping matrices for the same linear map with respect to two different bases? The answer is simple:

|||| Theorem 14.13 Similar Matrices as Mapping Matrices

An n-dimensional vector space V is given. Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are mapping matrices for the same linear map $f : V \to V$ with respect to two different bases for V if and only if $\mathbf{A}$ and $\mathbf{B}$ are similar.

|||| Exercise 14.14

Prove Theorem 14.13

In the introduction we also asked the question: Which conditions should a square matrix satisfy in order to be a mapping matrix for a linear map that in another basis has a diagonal matrix as a mapping matrix? The answer appears from Theorem 14.7 combined with Theorem 14.13: the matrix must have n linearly independent eigenvectors.

We end the eNote by an example on diagonalization of a linear map, that is finding a suitable basis in which the mapping matrix is diagonal.

|||| Example 14.15 Diagonalization of a Linear Map

A linear map $f : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ is given by the following mapping matrix with respect to the standard monomial basis m:

$$ {}_{\mathrm{m}}\mathbf{F}_{\mathrm{m}} = \begin{bmatrix} -17 & -21 \\ 14 & 18 \end{bmatrix} \tag{14-22} $$

This means that $f(1) = -17 + 14x$ and $f(x) = -21 + 18x$. We wish to investigate whether a (real) eigenbasis for f can be found and if so, how the mapping matrix looks with respect to this basis, and what the basis vectors are.

The eigenvalues of ${}_{\mathrm{m}}\mathbf{F}_{\mathrm{m}}$ are determined:

$$ \det\left(\begin{bmatrix} -17-\lambda & -21 \\ 14 & 18-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 12 = (\lambda+3)(\lambda-4) = 0\,. \tag{14-23} $$

It is already now possible to confirm that a real eigenbasis for f exists since $2 = \dim(P_2(\mathbb{R}))$, viz. $\lambda_1 = -3$ and $\lambda_2 = 4$ each with the algebraic multiplicity 1. Eigenvectors corresponding to λ_1 are determined:

$$ \left[\begin{array}{cc|c} -17+3 & -21 & 0 \\ 14 & 18+3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]. \tag{14-24} $$

This yields an eigenvector ${}_{\mathrm{m}}\mathbf{v}_1 = (-3, 2)$, if the free parameter is put equal to 2. Similarly we get the other eigenvector:

$$ \left[\begin{array}{cc|c} -17-4 & -21 & 0 \\ 14 & 18-4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]. \tag{14-25} $$

This yields an eigenvector ${}_{\mathrm{m}}\mathbf{v}_2 = (-1, 1)$, if the free parameter is put equal to 1.

Thus a real eigenbasis v for f, given by the basis vectors ${}_{\mathrm{m}}\mathbf{v}_1$ and ${}_{\mathrm{m}}\mathbf{v}_2$, exists. We then get

$$ {}_{\mathrm{m}}\mathbf{M}_{\mathrm{v}} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad {}_{\mathrm{v}}\mathbf{F}_{\mathrm{v}} = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix} \tag{14-26} $$

The basis consists of the vectors $\mathbf{v}_1 = -3 + 2x$ and $\mathbf{v}_2 = -1 + x$ and the map is “simple” with respect to this basis.

One can check with the map of $\mathbf{v}_1$:

$$\begin{aligned} f(\mathbf{v}_1) &= f(-3+2x) = -3\cdot f(1) + 2\cdot f(x) \\ &= -3\cdot(-17+14x) + 2\cdot(-21+18x) \\ &= 9-6x = -3(-3+2x) = -3\mathbf{v}_1 \end{aligned} \tag{14-27}$$

It is true!