eNote 13

Eigenvalues and Eigenvectors

This note introduces the concepts of eigenvalues and eigenvectors for linear maps in arbitrary general vector spaces and then delves deeply into eigenvalues and eigenvectors of square matrices. Therefore the note is based on knowledge about general vector spaces, see eNote 11, on knowledge about algebra with matrices, see eNote 7 and eNote 8, and on knowledge about linear maps see eNote 12.

Update: 7.10.21 David Brander.

13.1 The Eigenvalue Problem for Linear Maps

13.1.1 Introduction

In this eNote we consider linear maps of the type

$$f : V \rightarrow V, \tag{13-1}$$

that is, linear maps where the *domain* and the *codomain* are the same vector space. This gives rise to a special phenomenon, that a vector can be equal to its image vector:

$$f(\mathbf{v}) = \mathbf{v}\,. \tag{13-2}$$

Vectors of this type are called *fixed points* of the map f. More generally we are looking for ***eigenvectors***, that is vectors that are proportional to their image vectors. In this connection one talks about the *eigenvalue problem*: to find a scalar λ and a proper (i.e. non-zero) vector **v** satisfying the vector equation:

$$f(\mathbf{v}) = \lambda \mathbf{v} . \tag{13-3}$$

If λ is a scalar and **v** a proper vector satisfying 13-3 the proportionality factor λ is called an ***eigenvalue*** of f and **v** an *eigenvector* corresponding to λ. Let us, for example, take a linear map $f : G_3 \rightarrow G_3$, that is, a linear map of the set of space vectors into itself, mapping three given vectors as shown in Figure 13.1.

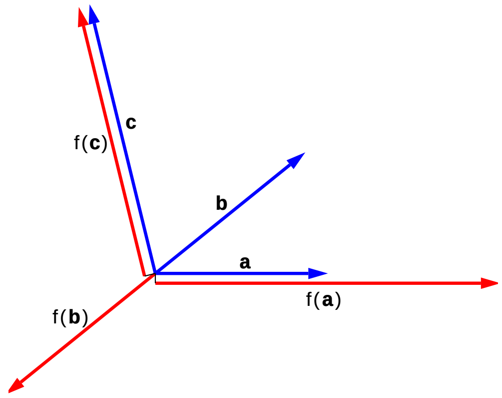

Figure 13.1: Three eigenvectors in space and their image vectors.

As hinted in Figure 13.1 $f(\mathbf{a}) = 2\mathbf{a}$. Therefore 2 is an eigenvalue of f with corresponding eigenvector $\mathbf{a}$. Furthermore $f(\mathbf{b}) = -\mathbf{b}$, so -1 is also an eigenvalue of f with corresponding eigenvector $\mathbf{b}$. And since finally $f(\mathbf{c}) = \mathbf{c}$, 1 is an eigenvalue of f with corresponding eigenvector $\mathbf{c}$. More specifically $\mathbf{c}$ is a fixed point for f.

To solve eigenvalue problems for linear maps is one of the most critical problems in engineering applications of linear algebra. This is closely connected to the fact that a linear map whose mapping matrix with respect to a given basis is a *diagonal matrix* is particularly simple to comprehend and work with. And here the nice rule, that if one chooses a basis consisting of eigenvectors for the map, then the mapping matrix automatically becomes a diagonal matrix.

In the following example we illustrate these points using linear maps in the plane.

|||| Example 13.1 Eigenvalues and Eigenvectors in the Plane

The vector space of vectors in the plane has the symbol $G_2(\mathbb{R})$. We consider a linear map

$$f : G_2(\mathbb{R}) \to G_2(\mathbb{R}) \tag{13-4}$$

of the set of plane vectors into itself, that with respect to a given basis $(\mathbf{a}_1, \mathbf{a}_2)$ has the following diagonal matrix as its mapping matrix:

$${}_{\mathrm{a}}\mathbf{F}_{\mathrm{a}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}. \tag{13-5}$$

Since

$${}_{\mathrm{a}}f(\mathbf{a}_1) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$${}_{\mathrm{a}}f(\mathbf{a}_2) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we have that $f(\mathbf{a}_1) = 2\mathbf{a}_1$ and $f(\mathbf{a}_2) = 3\mathbf{a}_2$. Both basis vectors are thus eigenvectors for f, because $\mathbf{a}_1$ corresponds to the eigenvalue 2 and $\mathbf{a}_2$ corresponds to the eigenvalue 3. The eigenvalues are the diagonal elements in ${}_{\mathrm{a}}\mathbf{F}_{\mathrm{a}}$.

We now consider an arbitrary vector $\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$ and find its image vector:

$${}_{\mathrm{a}}f(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 3x_2 \end{bmatrix}.$$

By the map the x_1-coordinate is multiplied by the eigenvalue 2, while the x_2-coordinate is multiplied by the eigenvalue 3. Geometrically this means that through the map all of the plane "is stretched" first by the factor 2 in the direction $\mathbf{a}_1$ and then by the factor 3 in the direction $\mathbf{a}_2$, see the effect on an arbitrarily chosen vector $\mathbf{x}$ in the figure A:

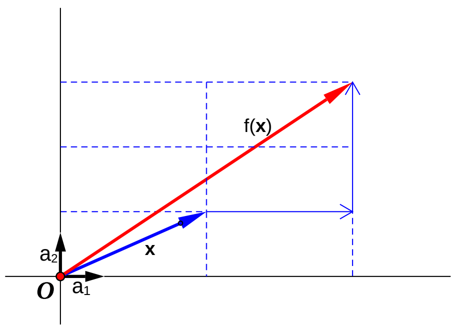

Figure A: The vector $\mathbf{x}$ is stretched horizontally by a factor 2 and vertically by a factor 3.

In Figure B we have chosen the standard basis $(\mathbf{i},\mathbf{j})$ and illustrate how the linear map g that has the mapping matrix

$$_{\mathrm{e}}\mathbf{G}_{\mathrm{e}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

maps the "blue house" into the "red house" by stretching all position vectors in the blue house by the factor 2 in the horizontal direction and by the factor 3 in the vertical direction.

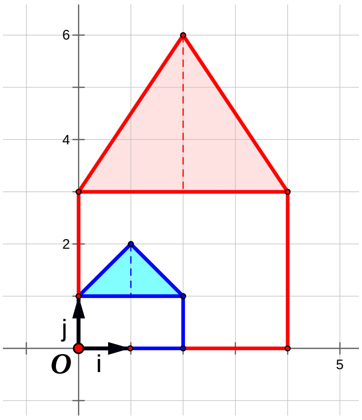

Figure B: The blue house is stretched in the horizontal direction by the factor 2 and vertically by the factor 3.

We now investigate another map h, the mapping matrix of which, with respect to the standard basis, is not a diagonal matrix:

$$_{\mathrm{e}}\mathbf{H}_{\mathrm{e}} = \begin{bmatrix} 7/3 & 2/3 \\ 1/3 & 8/3 \end{bmatrix}.$$

Here it is not possible to decide directly whether the map is composed of two stretchings in two given directions. And the mapping of the blue house by h as shown in the figure below does not give a clue directly:

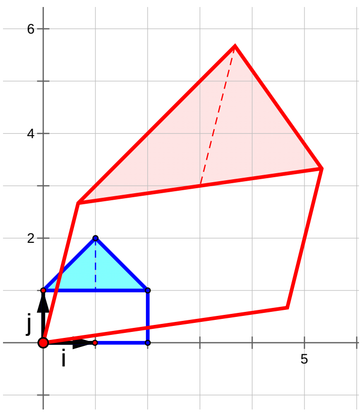

Figure C: House

But it is actually also possible in the case of h to choose a basis consisting of two linearly independent eigenvectors for h. Let $\mathbf{b}_1$ be given by the e-coordinates $(2,-1)$ and $\mathbf{b}_2$ by the e-coordinates $(1,1)$. Then we find that

$$_{\mathrm{e}}h(\mathbf{b}_1) = \begin{bmatrix} 7/3 & 2/3 \\ 1/3 & 8/3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

and

$$_{\mathrm{e}}h(\mathbf{b}_2) = \begin{bmatrix} 7/3 & 2/3 \\ 1/3 & 8/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In other words, $h(\mathbf{b}_1) = 2\mathbf{b}_1$ and $h(\mathbf{b}_2) = 3\mathbf{b}_2$. We see that $\mathbf{b}_1$ and $\mathbf{b}_2$ are eigenvectors for h, and when we choose $(\mathbf{b}_1, \mathbf{b}_2)$ as basis, the mapping matrix for h with respect to this basis takes the form:

$$_{\mathrm{b}}\mathbf{G}_{\mathrm{b}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Surprisingly it thus shows that the mapping matrix for h also can be written in the form (13-5). The map h is also composed of two stretchings with the factors 2 and 3. Only the stretching *directions* are now determined by the eigenvectors $\mathbf{b}_1$ and $\mathbf{b}_2$. This is more evident if we map a new blue house whose principal lines are parallel to the b-basis vectors:

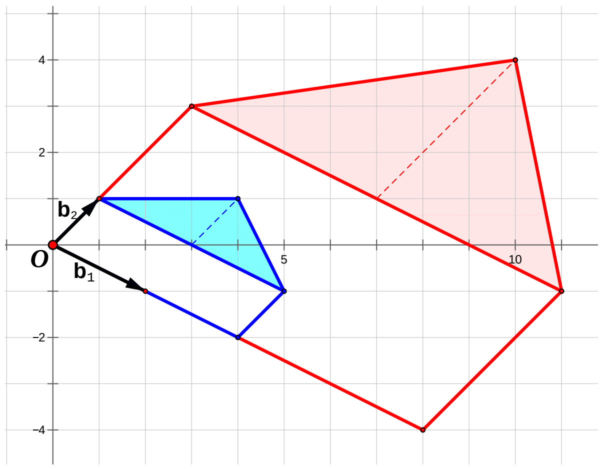

Figure D: The blue house is stretched by the factor 2 and the factor 3, respectively, in the directions of the eigenvectors

Thus we have illustrated: If you can find two linearly independent eigenvectors for a linear map in the plane it is possible:

1. to write its mapping matrix in diagonal form by choosing the eigenvectors as basis
2. to describe the map as stretchings in the directions of the eigenvectors with the corresponding eigenvectors as stretching factors.

13.1.2 Eigenvalues and their Corresponding Eigenvectors

The *eigenvalue problem* for a linear map is briefly about answering the question: do any proper vectors, each with its image vector proportional to the vector itself, exist. The short answer to this is that this cannot be answered in general, it depends on the particular map. In the following we try to pinpoint what can actually be said generally about the eigenvalue problem.

|||| Definition 13.2 Eigenvalue and Eigenvector

Let $f : V \to V$ be a linear map of the vector space V into itself. If a proper vector $\mathbf{v} \in V$ and a scalar λ exist such that

$$f(\mathbf{v}) = \lambda \mathbf{v}\,, \tag{13-6}$$

then the proportionality factor λ is called an ***eigenvalue*** of f, while $\mathbf{v}$ is called an ***eigenvector*** corresponding to λ.

If, in Definition 13.2, it were not required to find a *proper* vector that satisfies $f(\mathbf{v}) = \lambda\mathbf{v}$, then every scalar λ would be an eigenvalue, since for any scalar λ $f(\mathbf{0}) = \lambda\,\mathbf{0}$ is valid. On the other hand, for a given eigenvalue, it is a matter of convention whether or not to say that the zero vector is also a corresponding eigenvector. Most commonly, the zero vector is not considered to be an eigenvector.

The number 0 can be an eigenvalue. This is so if a proper vector $\mathbf{v}$ exists such that $f(\mathbf{v}) = \mathbf{0}$, since we then have $f(\mathbf{v}) = 0\mathbf{v}$.

If a linear map f has one eigenvector $\mathbf{v}$, then it has infinitely many eigenvectors. This is a simple consequence of the following theorem.

|||| Theorem 13.3 Eigenspace

If λ is an eigenvalue of a linear map $f : V \to V$, denote by E_λ the set: $E_\lambda := \{\mathbf{v} \in V \mid f(\mathbf{v}) = \lambda\mathbf{v}\}$. Then E_λ is a vector subspace of V.

|||| Proof

Let $f : V \to V$ be a linear map of the vector space V into itself, and assume that λ is an eigenvalue of f. Obviously E_λ is not empty, since it contains the zero vector. We shall show that the it satisfies the two stability requirements for subspaces, see Theorem 11.42. Let k be an arbitrary scalar, and let $\mathbf{u}$ and $\mathbf{v}$ be two arbitrary elements of E_λ. Then the following is valid :

$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v} = \lambda(\mathbf{u}+\mathbf{v})\,.$$

Thus the vector sum $\mathbf{u}+\mathbf{v} \in E_\lambda$ and thus we have shown that E_λ satisfies the stability requirement with respect to addition. Furthermore the following is valid:

$$f(k\mathbf{u}) = kf(\mathbf{u}) = k(\lambda\mathbf{u}) = \lambda(k\mathbf{u})\,.$$

Thus we have shown stabilit with respect to multiplication by a scalar. Together we have shown that E_λ is a subspace of the domain.

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Theorem 13.3 yields the following definition:

|||| Definition 13.4 Eigenvector Space

Let $f : V \to V$ be a linear map of the vector space V to itself, and let λ be an eigenvalue of f.

By the *eigenvector space* (or in short the *eigenspace*) E_λ corresponding to λ we understand the subspace:

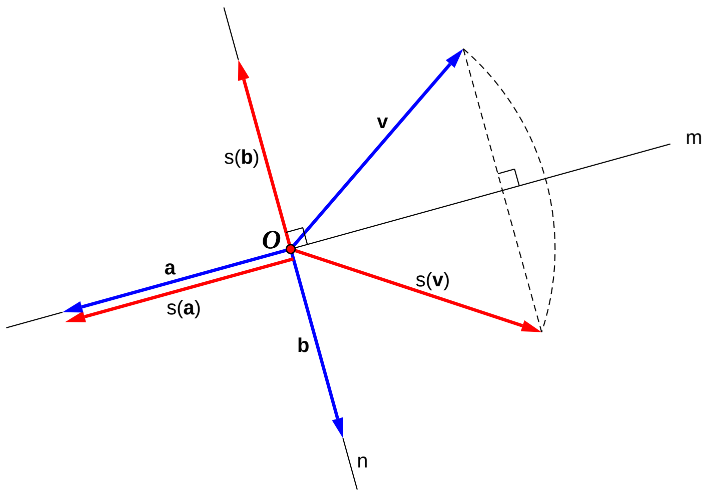
$$E_\lambda = \{\mathbf{v} \in V \mid f(\mathbf{v}) = \lambda\mathbf{v}\}\ .$$

If E_λ is finite-dimensional, $\dim(E_\lambda)$ is called the *geometric multiplicity* of λ, denoted $\mathrm{gm}(\lambda)$.

In the following example we consider a linear map that has two eigenvalues, both with the geometric multiplicity 1.

|||| Example 13.5 Eigenspace for Reflection

In the plane a straight line through the origin is drawn. By s we denote the linear map that maps a vector $\mathbf{v}$, drawn from the origin, in its reflection $s(\mathbf{v})$ in m:



The eigenvalue problem for the reflection in m.

Let $\mathbf{a}$ be an arbitrary proper vector that lies on m. Since

$$s(\mathbf{a}) = \mathbf{a} = 1 \cdot \mathbf{a}$$

1 is an eigenvalue of s. The eigenspace E_1 is the set of vectors that lie on m.

We now draw a straight line n through the origin, perpendicular to m. Let $\mathbf{b}$ be an arbitrary proper vector lying on n. Since

$$s(\mathbf{b}) = -\mathbf{b} = (-1) \cdot \mathbf{b}\,,$$

-1 is an eigenvalue of s. The eigenspace E_{-1} is the set of vectors that lie on n.

That not all linear maps have eigenvalues and thus eigenvectors is evident from the following example.

|||| Example 13.6

Let us investigate the eigenvalue problem for the linear map $f : G_2 \to G_2$ that to every proper vector $\mathbf{v}$ in the plane assigns its hat vector:

$$f(\mathbf{v}) = \widehat{\mathbf{v}}\,.$$

Since a proper vector $\mathbf{v}$ never can be proportional (parallel) to its hat vector, then for any scalar λ we have

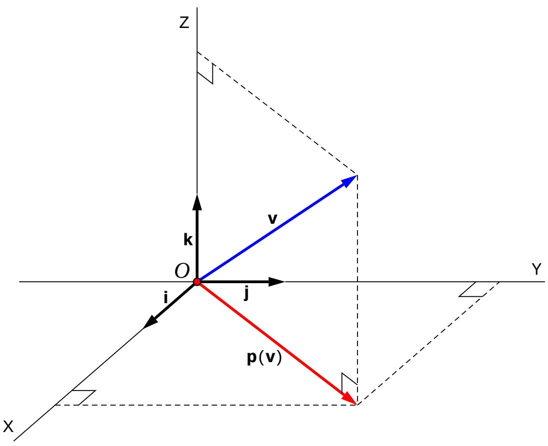
$$\widehat{\mathbf{v}} \neq \lambda \mathbf{v}\,.$$

Therefore eigenvalues and eigenvectors for f do not exist.

From the following exercise we see that the dimension of an eigenspace can be greater than 1.

|||| Exercise 13.7

In space an ordinary $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$-coordinate system is given. All vectors are drawn from the origin. The map p projects vectors down onto the (X, Y)-plane in space:



Eigenvalue problem for the projection down onto the (X, Y)-plane.

It is shown in Exercise 12.28 that p is linear. Determine all eigenvalues and the eigenspaces that correspond to the eigenvalues, solely by mental calculation (ponder).

|||| Example 13.8 The Eigenvalue Problem for Differentiation

We consider the linear map $f : C^\infty(\mathbb{R}) \to C^\infty$ given by

$$f(x(t)) = x'(t)\,.$$

Let λ be an arbitrary scalar. Since

$$f(\mathrm{e}^{\lambda t}) = \lambda\,\mathrm{e}^{\lambda t}\,,$$

λ is an eigenvalue of f and $\mathrm{e}^{\lambda t}$ is an eigenvector that corresponds to λ.

Since all solutions to the differential equation

$$x'(t) = \lambda x(t)$$

is given by $k \cdot \mathrm{e}^{\lambda t}$ where k is an arbitrary real number, the eigenspace corresponding to λ is determined by

$$E_\lambda = \left\{\, k \cdot \mathrm{e}^{\lambda t} \;\middle|\; k \in \mathbb{R} \,\right\}\,.$$

13.1.3 Theoretical Points

The following corollary gives an important result for linear maps of a vector space into itself. It is valid even if the vector space considered is of infinite dimension.

|||| Corollary 13.9

Let $f : V \to V$ be a linear map of a vector space V into itself, and assume

1. that f has a series of eigenvalues with corresponding eigenspaces,
2. that some of the eigenspaces are chosen, and within each of the chosen eigenspaces some linearly independent vectors are chosen,
3. and that all the so chosen vectors are consolidated in a single set of vectors v.

Then v is a linearly independent set of vectors.

|||| Proof

Let $f : V \to V$ be a linear map, and let v be a set of vectors that are put together according to points 1. to 3. in Corollary 13.9. We shall prove that v is linearly independent. The flow of the proof is that we assume the opposite, that is, v is linearly dependent, and show that this leads to a contradiction.

First we delete vectors from v to get a basis for span$\{v\}$. There must be at least one vector in v that does not correspond to the basis. We choose one of these, let us call it $\mathbf{x}$. Now we write $\mathbf{x}$ as a linear combination of the basis vectors, in doing so we leave out the trivial terms, i.e. those with the coefficient 0:

$$\mathbf{x} = k_1\mathbf{v}_1 + \cdots + k_m\mathbf{v}_m \tag{13-7}$$

We term the eigenvalue that corresponds to $\mathbf{x}$ λ, and the eigenvalues corresponding to $\mathbf{v}_i$ λ_i. From (13-7) we can obtain an expression for $\lambda\mathbf{x}$ in two different ways, partly by multiplying (13-7) by λ, partly by finding the image by f of the right and left hand side in (13-7):

$$\lambda\mathbf{x} = \lambda k_1\mathbf{v}_1 + \cdots + \lambda k_m\mathbf{v}_m$$
$$\lambda\mathbf{x} = \lambda_1 k_1\mathbf{v}_1 + \cdots + \lambda_m k_m\mathbf{v}_m$$

Subtracting the lower from the upper equation yields:

$$\mathbf{0} = k_1(\lambda - \lambda_1)\mathbf{v}_1 + \cdots + k_m(\lambda - \lambda_m)\mathbf{v}_m\,. \tag{13-8}$$

If all the coefficients to the vectors on the right hand side of (13-8) are equal to zero, then $\lambda = \lambda_i$ for all $i = 1, 2, \ldots, m$. But then $\mathbf{x}$ and all the basis vectors $\mathbf{v}_i$ that are chosen form the same eigenspace, and therefore they should collectively be linearly independent, this is how they are chosen. This contradicts that $\mathbf{x}$ is a linear combination of the basis vectors.

Therefore at least one of the coefficients in (13-8) must be different from 0. But then the zero vector is written as a proper linear combination of the basis vectors. This contradicts the requirement that a basis is linearly independent.

Conclusion: the assumption that v is a linearly independent set of vectors, necessarily leads to a contradiction. Therefore v is linearly independent.

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|||| Example 13.10 The Linear Independence of Eigenvectors

A linear map $f : V \to V$ has three eigenvalues λ_1, λ_2 and λ_3 that have the geometric multiplicities 2, 1 and 3, respectively. The set of vectors $(\mathbf{a}_1, \mathbf{a}_2)$ is a basis for E_{λ_1}, $(\mathbf{b})$ is a basis for

E_{λ_2} , and $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ is a basis for E_{λ_3} . Then it follows from corollary 13.9 that any selection of the six basis vectors is a linearly independent set of vectors.

Corollary 13.9 is useful because it leads directly to the following important results:

|||| Theorem 13.11 General Properties

Let V be a vector space with $\dim(V) = n$, and let $f : V \to V$ be a linear map of V into itself. Then:

1. Proper eigenvectors that correspond to different eigenvalues for f , are linearly independent.
2. f can at the most have n different eigenvalues.
3. If f has n different eigenvalues, then a basis for V exists consisting of eigenvectors for f .
4. The sum of the geometric multiplicities of eigenvalues for f can at the most be n .
5. If and only if the sum of the geometric multiplicities of the eigenvalues for f is equal to n, a basis for V exists consisting of eigenvectors for f .

|||| Exercise 13.12

The first point in 13.11 is a simple special case of Corollary 13.9 and therefore follows directly from the corollary. The second point can be proved like this:

Assume that a linear map has k different eigenvalues. We choose a proper vector from each of the k eigenspaces. The set of the k chosen vectors is then (in accordance with the corollary 13.9) linearly independent, and k must therefore be less than or equal to the dimension of the vector space (see Corollary 11.21).

Similarly, show how the last three points in Theorem 13.11 follow from Corollary 13.9.

Motivated by Theorem 13.11 we introduce the concept eigenbasis:

|||| Definition 13.13 Eigenvector basis

Let $f : V \to V$ be a linear map of a finite-dimensional vector space V into itself.

By an *eigenvector basis*, or in short *eigenbasis*, for V with respect to f we understand a basis consisting of eigenvectors for f .

Now we can present this subsection's main result:

|||| Theorem 13.14 Main Theorem

Let $f : V \to V$ be a linear map of an n-dimensional vector space V into itself, and let $v = (\mathbf{v}_1, \ldots \mathbf{v}_n)$ be a basis for V . Then:

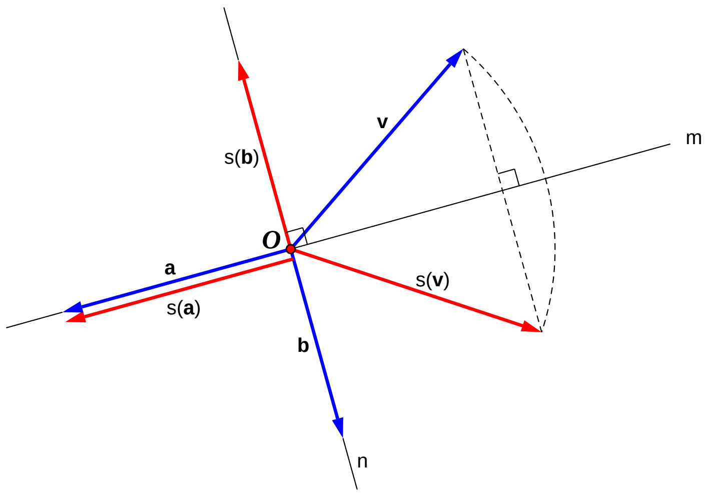
1. The mapping matrix ${}_{\mathrm{v}}\mathbf{F}_{\mathrm{v}}$ for f with respect to v is a diagonal matrix if and only if v is an eigenbasis for V with respect to f .

2. Assume that v is an eigenbasis for V with respect to f . Let $\mathbf{\Lambda}$ denote the diagonal matrix that is the mapping matrix for f with respect to v . The order of the diagonal elements in $\mathbf{\Lambda}$ is then determined from the basis like this: The basis vector $\mathbf{v}_i$ corresponds to the eigenvalue λ_i that is in the i'th column in $\mathbf{\Lambda}$.

The proof of this theorem can be found in eNote 14 (see Theorem 14.7).

|||| Example 13.15 Diagonal Matrix for Reflection

Let us again consider the situation in example 13.5, where we considered the map s that reflects vectors drawn from the origin in the line m:



Reflection about m.

We found that $\mathbf{a}$ is an eigenvector that corresponds to the eigenvalue 1, and that $\mathbf{b}$ is an eigenvector that corresponds to the eigenvalue -1. Since the plane has the dimension 2 it follows from Theorem 13.14 that if we choose the basis $(\mathbf{a}, \mathbf{b})$, then f has the following mapping matrix with respect to this basis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

|||| Example 13.16 Linear Maps without Eigenvalues

In the example 13.6 we found that the map, which maps a vector in the plane onto its hat vector, has no eigenvalues. Therefore there is no eigenbasis for the map, and therefore it cannot be described by a diagonal matrix for this map.

|||| Example 13.17 Diagonalisation of a Complex Map

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a linear map that satisfies

$$f(z_1, z_2) = (-z_2, z_1).$$

Since:

$$f(i,1) = (-1,i) = i\,(i,1) \quad \text{and} \quad f(-i,1) = (-1,-i) = (-i)(-i,1)\,,$$

it is seen that i is an eigenvalue of f with a corresponding eigenvector $(i,1)$, and that $-i$ is an eigenvalue of f with a corresponding eigenvector $(-i,1)$.

Since $(i,1)$ and $(-i,1)$ are linearly independent, $\big((i,1),(-i,1)\big)$ is an eigenbasis for $\mathbb{C}^2$ with respect to f. The mapping matrix for f with respect to this basis is in accordance with Theorem 13.14

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

|||| Exercise 13.18

Consider once more the situation in Example 13.7. Choose two different eigenbases (bases consisting of eigenvectors for p), and determine in each of the two cases the diagonal matrix that will become the mapping matrix for p with respect to the chosen basis.

13.2 The Eigenvalue Problem for Square Matrices

When a linear map $f : V \to V$ maps an n-dimensional vector space V into the vector space itself the mapping matrix for f with respect to the arbitrarily chosen basis a becomes a *square* matrix. The eigenvalue problem $f(\mathbf{v}) = \lambda\mathbf{v}$ is the equivalent of the matrix equation:

$$_{\mathrm{a}}\mathbf{F}_{\mathrm{a}} \cdot {}_{\mathrm{a}}\mathbf{v} = \lambda \cdot {}_{\mathrm{a}}\mathbf{v}\,. \tag{13-9}$$

Thus we can formulate an eigenvalue problem for square matrices generally, that is without necessarily having to think about a square matrix as a mapping matrix. We will standardize the method, when eigenvalues and eigenvectors for square matrices are to be determined. At the same time, due to (13-9), we get methods for finding eigenvalues and eigenvectors for all linear maps of a vector space into itself, that can be described by mapping matrices.

First we define what is to be understood by the eigenvalue problem for a square matrix.

|||| Definition 13.19 The Eigenvalue Problem for Matrices

Solving the eigenvalue problem for a square real $n \times n$-matrix $\mathbf{A}$ means to find a scalar λ and a proper vector $\mathbf{v} = (v_1, \ldots, v_n)$ satisfying the equation:

$$\mathbf{A}\,\mathbf{v} = \lambda\mathbf{v}\,. \tag{13-10}$$

If this equation is satisfied for a pair af λ and $\mathbf{v} \neq \mathbf{0}$, λ is termed an *eigenvalue* of $\mathbf{A}$ and $\mathbf{v}$ an *eigenvector* of $\mathbf{A}$ corresponding to λ.

|||| Example 13.20 The Eigenvalue Problem for a Square Matrix

We wish to investigate whether $\mathbf{v}_1 = (2,3)$, $\mathbf{v}_2 = (4,4)$ and $\mathbf{v}_3 = (2,-1)$ are eigenvectors for $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \tag{13-11}$$

For this we write the eigenvalue problem, as stated in Definition 13.19.

$$\begin{aligned} \mathbf{A}\mathbf{v}_1 &= \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 1 \cdot \mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 &= \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix} = 2 \cdot \mathbf{v}_2 \\ \mathbf{A}\mathbf{v}_3 &= \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix} \neq \lambda \cdot \mathbf{v}_3\,. \end{aligned} \tag{13-12}$$

From this we see that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for $\mathbf{A}$. $\mathbf{v}_1$ corresponding to the eigenvalue 1, and $\mathbf{v}_2$ corresponding to the eigenvalue 2.

Furthermore we see that $\mathbf{v}_3$ is not an eigenvector for $\mathbf{A}$.

|||| Example 13.21 The Eigenvalue Problem for a Square Matrix

Given the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

0 is an eigenvalue of $\mathbf{A}$ and $(1,1)$ an eigenvector for $\mathbf{A}$ corresponding to the eigenvalue 0.

|||| Example 13.22 The Eigenvalue Problem for a Square Matrix

Given the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} = i \begin{bmatrix} -i \\ 1 \end{bmatrix},$$

i is a complex eigenvalue of $\mathbf{A}$ and $(-i, 1)$ is a complex eigenvector for $\mathbf{A}$ corresponding to the eigenvalue i.

For the use in the following investigations we make some important comments to Definition 13.19 .

First we note that even if the square matrix $\mathbf{A}$ in Definition 13.19 is real, one is often interested not only in the real solutions to (13-10), but more generally complex solutions. In other words we seek a scalar $\lambda \in \mathbb{C}$ and a vector $\mathbf{v} \in \mathbb{C}^n$, satisfying (13-10).

Therefore it can be convenient to regard the left-hand side of (13-10) as a map $f : \mathbb{C}^n \to \mathbb{C}^n$ given by:

$$f(\mathbf{v}) = \mathbf{A}\,\mathbf{v}\,.$$

This map is linear, viz. let $\mathbf{u} \in \mathbb{C}^n$, $\mathbf{v} \in \mathbb{C}^n$ and $k \in \mathbb{C}$., then according to the usual arithmetic rules for matrices

$$1. \quad f(\mathbf{u} + \mathbf{v}) = \mathbf{A}\,(\mathbf{u} + \mathbf{v}) = \mathbf{A}\,\mathbf{u} + \mathbf{A}\,\mathbf{v}$$
$$2. \quad f(k\,\mathbf{u}) = \mathbf{A}(k\,\mathbf{u}) = k(\mathbf{A}\,\mathbf{u})$$

By this the linearity is established. Since the eigenvalue problem $f(\mathbf{v}) = \lambda\mathbf{v}$ in this case is *identical* to the eigenvalue problem $\mathbf{Av} = \lambda\mathbf{v}$, we can conclude that results obtained in subsection 9.1 for the eigenvalue problem in general, can be transferred directly to the eigenvalue problem for matrices. Thus let us immediately characterize the set of eigenvectors that correspond to a given eigenvalue of a square, real matrix, compare with Theorem 13.3.

|||| Theorem 13.23 Subspaces of Eigenvectors

Let λ be a real or complex eigenvalue of a real $n \times n$-matrix $\mathbf{A}$. Then the set of complex eigenvectors for $\mathbf{A}$ corresponding to λ, is a subspace in $\mathbb{C}^n$.

If one is only interested in real solutions to the eigenvalue problem for real square matrices, one can alternatively see the left hand side of (13-10) as a real map $f : \mathbb{R}^n \to \mathbb{R}^n$ given by:

$$f(\mathbf{v}) = \mathbf{A}\,\mathbf{v}\,.$$

Of course, this map is linear, too. We get the following version of Theorem 13.23:

|||| Theorem 13.24 Subspaces of Eigenvectors

Let λ be a real eigenvalue of a real $n \times n$-matrix $\mathbf{A}$. Then the set of real eigenvectors for $\mathbf{A}$ corresponding to λ, is a subspace in $\mathbb{R}^n$.

In the light of Theorem 13.23 and Theorem 13.24 we now introduce the concept eigenvector space, compare with Definition 13.4.

|||| Definition 13.25 The Eigenvector Space

Let $\mathbf{A}$ be a square, real matrix, and let λ be an eigenvalue of $\mathbf{A}$.

The subspace of all the eigenvectors that correspond to λ is termed the ***eigenvector space*** (or in short the ***eigenspace***) corresponding to λ and is termed E_λ.

Now we have sketched the structural framework for the eigenvalue problem for square matrices, and we continue in the following two subsections by investigating in an elementary way, how one can begin to find eigenvalues and eigenvectors for square matrices.

13.2.1 To Find the Eigenvalues for a Square Matrix

We wish to determine the eigenvalues that correspond to a real $n \times n$ matrix $\mathbf{A}$. The starting point is the equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}\,, \tag{13-13}$$

First we put $\lambda\mathbf{v}$ onto the left hand of the equality sign, and then $\mathbf{v}$ "is placed outside a pair of brackets". This is possible because $\mathbf{v} = \mathbf{E}\,\mathbf{v}$ where $\mathbf{E}$ is the identity matrix:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \;\Leftrightarrow\; \mathbf{A}\mathbf{v} - \lambda(\mathbf{E}\mathbf{v}) = \mathbf{A}\mathbf{v} - (\lambda\mathbf{E})\mathbf{v} = \mathbf{0} \;\Leftrightarrow\; (\mathbf{A} - \lambda\mathbf{E})\mathbf{v} = \mathbf{0}\,. \tag{13-14}$$

The last equation in (13-14) corresponds to a homogeneous system of linear equations consisting of n equations in the n unknowns $v_1, ..., v_n$, that are the elements in $\mathbf{v} = (v_1, ..., v_n)$. However, it is not possible to solve the system of equations directly, precisely because we do not know λ. We have to continue the work with the coefficient matrix of the system of equations. We give this matrix a special symbol:

$$\mathbf{K}_{\mathbf{A}}(\lambda) = (\mathbf{A} - \lambda\mathbf{E})$$

and is called ***the characteristic matrix*** of $\mathbf{A}$.

Since it is a homogeneous system of linear equations that we have to solve we have two possibilities for the structure of the solution. Either the characteristic matrix is *invertible*, and the the only solution is $\mathbf{v} = \mathbf{0}$. Or the matrix is *singular*, and then infinitely many solutions $\mathbf{v}$ exist. But since Definition 13.19 requires that $\mathbf{v}$ must be a proper vector, that is a vector different from the zero vector, the characteristic matrix must be singular. To investigate whether this is true, we take the determinant of the square matrix. This is zero exactly when the matrix is singular:

$$\det(\mathbf{A} - \lambda\mathbf{E}) = 0\,. \tag{13-15}$$

Note that the left hand side in (13-15) is a polynomial in the variable λ. The polynomial is given a special symbol:

$$K_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{E}) = \det(\mathbf{K}_{\mathbf{A}}(\lambda))$$

and is termed ***the characteristic polynomial*** of $\mathbf{A}$.

The equation that results when the characteristic polynomial is set equal to zero

$$K_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{E}) = \det(\mathbf{K}_{\mathbf{A}}(\lambda)) = 0$$

is termed the ***characteristic equation*** of $\mathbf{A}$.

By the use of the method for calculating the determinant we see that the characteristic polynomial is always an n'th degree polynomial. See also the following examples. The main point is that the roots in the characteristic polynomial (solutions to the character equation) are the eigenvalues of the matrix, because the eigenvalues precisely satisfy that the characteristic matrix is singular.

It is also common to define the characteristic matrix as $\lambda\mathbf{E} - \mathbf{A}$, since the homogeneous equation for this matrix has the same solutions, and the zeros of the corresponding characteristic polynomial $\det(\lambda\mathbf{E} - \mathbf{A}) = 0$ are also the same. But note that $\det(\lambda\mathbf{E} - \mathbf{A}) = (-1)^n \det(\mathbf{A} - \lambda\mathbf{E})$.

|||| Example 13.26 Eigenvalues for 2×2 Matrices

Given two matrices **A** and **B**:

$$\mathbf{A} = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}. \tag{13-16}$$

We wish to determine the eigenvalues for **A** and **B**.

First we consider **A**. Its characteristic matrix reads:

$$\mathbf{K_A}(\lambda) = \mathbf{A} - \lambda\mathbf{E} = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{bmatrix}. \tag{13-17}$$

Now we determine the characteristic polynomial:

$$\begin{aligned} K_{\mathbf{A}}(\lambda) = \det(\mathbf{K_A}(\lambda)) &= \det\left(\begin{bmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{bmatrix}\right) \\ &= (4-\lambda)(-1-\lambda) - (-2)\cdot 3 = \lambda^2 - 3\lambda + 2\,. \end{aligned} \tag{13-18}$$

The polynomial as expected has the degree 2. The characteristic equation can be written and the solutions determined:

$$K_{\mathbf{A}}(\lambda) = 0 \;\Leftrightarrow\; \lambda^2 - 3\lambda + 2 = 0 \;\Leftrightarrow\; \lambda = 1 \;\text{ or }\; \lambda = 2\,. \tag{13-19}$$

Thus **A** has two eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 2$.

The same technique is used for the determination of possible eigenvalues of **B**.

$$\begin{aligned} \mathbf{K_B}(\lambda) = \mathbf{B} - \lambda\mathbf{E} &= \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -1-\lambda & 4 \\ -2 & 3-\lambda \end{bmatrix} \\ K_{\mathbf{B}}(\lambda) = \det(\mathbf{K_B}(\lambda)) &= \det\left(\begin{bmatrix} -1-\lambda & 4 \\ -2 & 3-\lambda \end{bmatrix}\right) \\ &= (-1-\lambda)(3-\lambda) - 4\cdot(-2) = \lambda^2 - 2\lambda + 5\,. \end{aligned} \tag{13-20}$$

In this case there are no real solutions to $K_{\mathbf{B}}(\lambda) = 0$, because the discriminant $d = (-2)^2 - 4 \cdot 1 \cdot 5 = -16 < 0$, and therefore **B** has no real eigenvalues. But it has two complex eigenvalues. We use the complex "toolbox": The discriminant can be rewritten as $d = (4i)^2$, which gives the two complex solutions

$$\lambda = \frac{2 \pm 4i}{2} \;\Leftrightarrow\; \lambda = 1 + 2i \;\text{ and }\; \bar{\lambda} = 1 - 2i \tag{13-21}$$

Thus **B** has two complex eigenvalues: $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$.

In the following theorem the conclusions of this subsection are summarized.

|||| Theorem 13.27 The Characteristic Polynomial

For the square real $n \times n$-matrix $\mathbf{A}$ consider

1. *The characteristic matrix* $\mathbf{K_A}(\lambda) = \mathbf{A} - \lambda\mathbf{E}$.
2. *The characteristic polynomial* $K_{\mathbf{A}}(\lambda) = \det(\mathbf{K_A}(\lambda)) = \det(\mathbf{A} - \lambda\mathbf{E})$.
3. *The characteristic equation* $K_{\mathbf{A}}(\lambda) = 0$.

Then:

1. The characteristic polynomial is an n'th degree polynomial with the variable λ, and similarly the characteristic equation is an n'th degree equation with the unknown λ.

2. The roots of the characteristic polynomial (the solutions to the characteristic equation) are all the eigenvalues of $\mathbf{A}$.

13.2.2 To Find the Eigenvectors of a Square Matrix

After the eigenvalues of a real $n \times n$ matrix $\mathbf{A}$ are determined, it is possible to determine the corresponding eigenvectors. The procedure starts with the equation

$$(\mathbf{A} - \lambda\mathbf{E})\mathbf{v} = \mathbf{0}, \tag{13-22}$$

that was achieved in (13-14). Since the eigenvalues are now known, the homogeneous system of linear equations corresponding to (13-22) can be solved with respect to the n unknowns $v_1, ..., v_n$ that are the elements in $\mathbf{v} = (v_1, ..., v_n)$. We just have to substitute the eigenvalues one after one. As mentioned above, the characteristic matrix is singular when the substituted λ is an eigenvalue. Therefore infinitely many solutions to the system of equations exist. Finding these corresponds to finding all eigenvectors $\mathbf{v}$ that correspond to λ.

In the following method we summarize the problem of determining eigenvalues and the corresponding eigenvectors of a square matrix.

|||| Method 13.28 Determination of Eigenvectors

All (real or complex) eigenvalues λ for the square matrix $\mathbf{A}$ are found as the solutions to the ***characteristic equation*** of $\mathbf{A}$:

$$K_{\mathbf{A}}(\lambda) = 0 \;\Leftrightarrow\; \det(\mathbf{A} - \lambda\mathbf{E}) = 0\,. \tag{13-23}$$

Then the eigenvectors $\mathbf{v}$ corresponding to each of the eigenvalues λ can be determined. They are the solutions to the following system of linear equations

$$(\mathbf{A} - \lambda\mathbf{E})\mathbf{v} = \mathbf{0}\,, \tag{13-24}$$

when the eigenvalue λ is substituted. $\mathbf{E}$ is the identity matrix.

Method 13.28 is unfolded in the following three examples that also show a way of characterizing the set of eigenvectors corresponding to a given eigenvalue, in the light of Theorem 13.23 and Theorem 13.24.

|||| Example 13.29 Eigenvectors Belonging to Given Eigenvalues

Given the square matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{13-25}$$

We wish to determine eigenvalues and eigenvectors to $\mathbf{A}$ and use method 13.28. First the characteristic matrix is found:

$$\mathbf{K}_{\mathbf{A}}(\lambda) = \mathbf{A} - \lambda\mathbf{E} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \tag{13-26}$$

Then the characteristic polynomial is formed:

$$\begin{gathered} K_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{E}) \\ = \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)(2-\lambda) - 1\cdot 1 = \lambda^2 - 4\lambda + 3\,. \end{gathered} \tag{13-27}$$

The characteristic equation that is $\lambda^2 - 4\lambda + 3 = 0$, has the solutions $\lambda_1 = 1$ and $\lambda_2 = 3$, which are all the real eigenvalues of $\mathbf{A}$.

In order to determine the eigenvectors corresponding to λ_1, it is substituted into $(\mathbf{A} - \lambda\mathbf{E})\mathbf{v} = \mathbf{0}$, and then we solve the system of linear equations with the augmented matrix:

$$\mathbf{T} = [\mathbf{A} - \lambda_1\mathbf{E} \,|\, \mathbf{0}\,] = \left[\begin{array}{cc|c} 2-1 & 1 & 0 \\ 1 & 2-1 & 0 \end{array}\right]. \tag{13-28}$$

By Gauss-Jordan elimination we get

$$\mathrm{rref}(\mathbf{T}) = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{13-29}$$

Thus there are infinitely many solutions $\mathbf{v} = (v_1, v_2)$, since there is only one non-trivial equation: $v_1 + v_2 = 0$. If we are only looking for one proper eigenvector corresponding to the eigenvalue λ_1, we can put v_2 equal to 1, and we get the eigenvector $\mathbf{v}_1 = (-1, 1)$. All real eigenvectors corresponding to λ_1 can then be written as

$$\mathbf{v} = t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; t \in \mathbb{R}. \tag{13-30}$$

This is a one-dimensional subspace in $\mathbb{R}^2$, viz. the eigenspace that corresponds to 1 that can also be written like this:

$$E_1 = \mathrm{span}\{(-1, 1)\}. \tag{13-31}$$

Now λ_2 is substituted in $(\mathbf{A} - \lambda\mathbf{E})\mathbf{v} = \mathbf{0}$, and we then solve the corresponding system of linear equations that has the augmented matrix

$$\mathbf{T} = [\mathbf{A} - \lambda_2\mathbf{E} \,|\, \mathbf{0}\,] = \left[\begin{array}{cc|c} 2-3 & 1 & 0 \\ 1 & 2-3 & 0 \end{array}\right]. \tag{13-32}$$

By Gauss-Jordan elimination we get

$$\mathrm{rref}(\mathbf{T}) = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]. \tag{13-33}$$

From this we see that $\mathbf{v}_2 = (1, 1)$ is an eigenvector corresponding to the eigenvalue λ_2. All real eigenvectors corresponding to λ_2 can be written as

$$\mathbf{v} = t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; t \in \mathbb{R}. \tag{13-34}$$

This is a one-dimensional subspace in $\mathbb{R}^2$ that can also be written as:

$$E_3 = \mathrm{span}\{(1, 1)\}. \tag{13-35}$$

We will now check our understanding: When $\mathbf{v}_1 = (-1, 1)$ is mapped by $\mathbf{A}$, will the image vector only be a scaling (change of length) of $\mathbf{v}_1$?

$$\mathbf{A}\mathbf{v}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \mathbf{v}_1. \tag{13-36}$$

It is true! It is also obvious that the eigenvalue is 1.

Now we check $\mathbf{v}_2$:

$$\mathbf{A}\mathbf{v}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \cdot \mathbf{v}_2. \tag{13-37}$$

$\mathbf{v}_2$ is also as expected an eigenvector and the eigenvalue is 3.

|||| Example 13.30 Complex Eigenvalues and Eigenvectors

In Example 13.26 a matrix **B** is given

$$\mathbf{B} = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix} \tag{13-38}$$

that has no real eigenvalues. But we found two complex eigenvalues, $\lambda_1 = 1+2i$ and $\lambda_2 = 1-2i$.

We substitute λ_1 in $(\mathbf{B}-\lambda\mathbf{E})\mathbf{v} = \mathbf{0}$ and then we solve the corresponding system of linear equations that has the augmented matrix

$$\mathbf{T} = [\mathbf{B}-\lambda_1\mathbf{E}\,|\,\mathbf{0}\,] = \left[\begin{array}{cc|c} -1-(1+2i) & 4 & 0 \\ -2 & 3-(1+2i) & 0 \end{array}\right] \tag{13-39}$$

By Gauss-Jordan elimination we get

$$\operatorname{rref}(\mathbf{T}) = \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{13-40}$$

This corresponds to one non-trivial equation $v_1 + (-1+i)v_2 = 0$, and if we put $v_2 = s$, we see that all the complex eigenvectors corresponding to λ_1 are given by

$$\mathbf{v} = s\cdot\begin{bmatrix} 1-i \\ 1 \end{bmatrix},\ s\in\mathbb{C}\,. \tag{13-41}$$

This is a one-dimensional subspace in $\mathbb{C}^2$, viz. the eigenspace corresponding to the eigenvalue $1+2i$ which we also can state like this:

$$E_{1+2i} = \operatorname{span}\{(1-i,1)\}\,. \tag{13-42}$$

Similarly all complex solutions corresponding to λ_2 are given by

$$\mathbf{v} = s\cdot\begin{bmatrix} 1+i \\ 1 \end{bmatrix},\ s\in\mathbb{C}\,. \tag{13-43}$$

This is a one-dimensional subspace in $\mathbb{C}^2$ which we also can state like this:

$$E_{1-2i} = \operatorname{span}\{(1+i,1)\}\,. \tag{13-44}$$

In the following example we find eigenvalues and corresponding eigenspaces for a 3×3-matrix. It turns out that in this case to one of the eigenvalues corresponds a two-dimensional eigenspace.

|||| Example 13.31 Eigenvalue with Multiplicity 2

Given the matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 6 & 3 & 12 \\ 4 & -5 & 4 \\ -4 & -1 & -10 \end{bmatrix} \qquad (13\text{-}45)$$

First we wish to determine the eigenvalues of $\mathbf{A}$ and use Method 13.28.

$$\det\left(\begin{bmatrix} 6-\lambda & 3 & 12 \\ 4 & -5-\lambda & 4 \\ -4 & -1 & -10-\lambda \end{bmatrix}\right) = -\lambda^3 - 9\lambda^2 + 108 = -(\lambda-3)(\lambda+6)^2 = 0 \qquad (13\text{-}46)$$

From the last factorization it is seen that $\mathbf{A}$ has two different eigenvalues. The eigenvalue $\lambda_1 = -6$ is a double root in the characteristic equation, while the eigenvalue $\lambda_2 = 3$ is a single root.

Now we determine the eigenspace corresponding to $\lambda_1 = -6$, see Theorem 13.23:

$$\left[\begin{array}{ccc|c} 6-(-6) & 3 & 12 & 0 \\ 4 & -5-(-6) & 4 & 0 \\ -4 & -1 & -10-(-6) & 0 \end{array}\right] \rightarrow$$
$$\left[\begin{array}{ccc|c} 12 & 3 & 12 & 0 \\ 4 & 1 & 4 & 0 \\ -4 & -1 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 4 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad (13\text{-}47)$$

Here is only one nontrivial equation: $4x_1 + x_2 + 4x_3 = 0$. If we put x_1 and x_3 equal to the two free parameters s and t all real eigenvectors corresponding to the eigenvalue -6 are given by:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}, \; s, t \in \mathbb{R}. \qquad (13\text{-}48)$$

This is a two-dimensional subspace in $\mathbb{R}^3$ which can also be stated like this:

$$E_{-6} = \operatorname{span}\{(1,-4,0), (0,-4,1)\}. \qquad (13\text{-}49)$$

It is thus possible to find two linearly independent eigenvectors corresponding to λ_1. What about the number of linearly independent eigenvectors for $\lambda_2 = 3$?

$$\left[\begin{array}{ccc|c} 6-3 & 3 & 12 & 0 \\ 4 & -5-3 & 4 & 0 \\ -4 & -1 & -10-3 & 0 \end{array}\right] \rightarrow$$
$$\left[\begin{array}{ccc|c} 3 & 3 & 12 & 0 \\ 4 & -8 & 4 & 0 \\ -4 & -1 & -13 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad (13\text{-}50)$$

Here are two non-trivial equations: $x_1 + x_2 + 4x_3 = 0$ and $x_2 + x_3 = 0$. If we put $x_3 = s$ equal to the free parameter, then all real eigenvectors corresponding to the eigenvalue 3 are given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \cdot \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, \ s \in \mathbb{R}\,. \tag{13-51}$$

This is a one-dimensional subspace in $\mathbb{R}^3$ that can also be stated like this:

$$E_3 = \operatorname{span}\{(-3, -1, 1)\}\,. \tag{13-52}$$

Thus it is only possible to find one linearly independent eigenvector corresponding to λ_2.

|||| Exercise 13.32

Given the square matrix

$$\mathbf{A} = \begin{bmatrix} 5 & -4 & 4 \\ 0 & -1 & 6 \\ 0 & 1 & 4 \end{bmatrix}. \tag{13-53}$$

1. Determine all eigenvalues of $\mathbf{A}$.
2. Determine for each of the eigenvalues the corresponding eigenspace.
3. State at least 3 eigenvectors (not necessarily linearly independent) corresponding to each eigenvalue.

13.2.3 Algebraic and Geometric Multiplicity

As is evident from Example 13.31 it is important to pay attention to whether an eigenvalue is a single root or a multiple root of the characteristic equation of a square real matrix and to the dimension of the corresponding eigenspace. In this subsection we investigate the relation between the two phenomena. This gives rise to the following definitions.

|||| Definition 13.33 Algebraic and Geometric Multiplicity

Let **A** be a square, real matrix, and let λ be an eigenvalue of **A** .

1. λ is said to have ***the algebraic multiplicity*** n when λ is a n-double root in the characteristic equation of the square matrix **A**. This is is termed $\text{am}(\lambda) = n$.

2. λ is said to have ***the geometric multiplicity*** m when the dimension of the eigenvector space corresponding to λ is m. This is termed $\text{gm}(\lambda) = m$. In other words: $\dim(E_\lambda) = \text{gm}(\lambda)$.

We do not always have $\text{am}(\lambda) = \text{gm}(\lambda)$. This is dealt with in Theorem 13.34.

The following theorem has some important properties concerning algebraic and geometric multiplicity of eigenvalues of square matrices, cf. Theorem 13.11.

|||| Theorem 13.34 Properties of Multiplicities

Given a real $n \times n$-matrix **A** .

1. **A** has at the most n different real eigenvalues, and also the sum of algebraic multiplicities of the real eigenvalues is at the most n .

2. **A** has at the most n different complex eigenvalues, but the sum of the algebraic multiplicities of the complex eigenvalues is equal to n .

3. If λ is a real or complex eigenvalue of **A**, then:

$$1 \leq \text{gm}(\lambda) \leq \text{am}(\lambda) \leq n \tag{13-54}$$

That is, the geometric multiplicity of an eigenvalue will at the least be equal to 1, it will be less than or equal to the algebraic multiplicity of the eigenvalue, which in turn will be less than or equal to the number of rows and columns in **A**.

|||| Exercise 13.35

Check that all three points in Theorem 13.34 are valid for the eigenvalues and eigenvectors in example 13.31.

Let us comment upon 13.34:

Points 1 and 2 follow directly from the theory of polynomials. The characteristic polynomial for a real $n \times n$-matrix $\mathbf{A}$ is an n'th degree polynomial, and it has at the most n different roots, counting both real and complex ones. Furthermore the sum of the multiplicities of the real roots is at the most n, whereas the sum of the multiplicities to the complex roots is equal to n.

We have previously shown that for every linear map of an n-dimensional vector space into itself the sum of the geometric multiplicities of the eigenvalues for f can at the most be n, see Theorem 13.11. Note that this can be deduced directly from the statements about multiplicities in Theorem 13.34.

As something new and interesting it is postulated in point 3 that the geometric multiplicity of a single eigenvalue can be less than the algebraic multiplicity. This is demonstrated in the following summarizing Example 13.36. Furthermore the geometric multiplicity of a single eigenvalue cannot be greater than the algebraic one. The proof of point 3 in Theorem 13.34 is left out.

|||| Example 13.36 Geometric Multiplicity Less than Algebraic Multiplicity

Given the matrix

$$\mathbf{A} = \begin{bmatrix} -9 & 10 & 0 \\ -3 & 1 & 5 \\ 1 & -4 & 6 \end{bmatrix} \tag{13-55}$$

The eigenvalues of $\mathbf{A}$ are determined:

$$\det\left(\begin{bmatrix} -9-\lambda & 10 & 0 \\ -3 & 1-\lambda & 5 \\ 1 & -4 & 6-\lambda \end{bmatrix}\right) = -\lambda^3 - 2\lambda^2 + 7\lambda - 4 = -(\lambda+4)(\lambda-1)^2 = 0\,. \tag{13-56}$$

From the factorization in front of the last equality sign we get that $\mathbf{A}$ has two different eigenvalues: $\lambda_1 = -4$ and $\lambda_2 = 1$. Moreover $\operatorname{am}(-4) = 1$ and $\operatorname{am}(1) = 2$, as can be seen from the factorization.

The eigenspace corresponding to $\lambda_1 = -4$ is determined by solving $(\mathbf{A} - \lambda_1 \mathbf{E})\mathbf{v} = \mathbf{0}$:

$$\begin{array}{c}\left[\begin{array}{ccc|c} -9-(-4) & 10 & 0 & 0 \\ -3 & 1-(-4) & 5 & 0 \\ 1 & -4 & 6-(-4) & 0 \end{array}\right] \rightarrow \\ \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & -2 & 10 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\end{array} \tag{13-57}$$

There are two non-trivial equations: $v_1 - 2v_2 = 0$ and $v_2 - 5v_3 = 0$. If we put v_3 equal to the free parameter we see that all real eigenvectors corresponding to λ_1 can be stated as

$$E_{-4} = \left\{ s \cdot (10,5,1) \;\middle|\; s \in \mathbb{R} \right\} = \text{span}\{(10,5,1)\}\,. \tag{13-58}$$

We have that $\text{gm}(-4) = \dim(E_{-4}) = 1$, and that an eigenvector to λ_1 is $\mathbf{v}_1 = (10,5,1)$. It is seen that $\text{gm}(-4) = \text{am}(-4)$.

Similarly for $\lambda_2 = 1$:

$$\begin{array}{c}\left[\begin{array}{ccc|c} -9-1 & 10 & 0 & 0 \\ -3 & 1-1 & 5 & 0 \\ 1 & -4 & 6-1 & 0 \end{array}\right] \rightarrow \\ \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -3 & 5 & 0 \\ 0 & -3 & 5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\end{array} \tag{13-59}$$

Again we have two non-trivial equations: $v_1 - v_2 = 0$ and $3v_2 - 5v_3 = 0$. If we put $v_3 = 3s$ we see that all to λ_2 corresponding real eigenvectors can be stated as

$$E_1 = \left\{ s \cdot (5,5,3) \;\middle|\; s \in \mathbb{R} \right\} = \text{span}\{(5,5,3)\}\,. \tag{13-60}$$

This gives the following results: $\text{gm}(1) = \dim(E_1) = 1$ and that an eigenvector to $\lambda_2 = \lambda_3$ is $\mathbf{v}_2 = (5,5,3)$. Furthermore it is seen that $\text{gm}(1) < \text{am}(1)$.

13.2.4 More About the Complex Problem

We will use the matrix

$$\mathbf{B} = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}. \tag{13-61}$$

From Example 13.30 in order to make more precise some special phenomena for square, real matrices when their eigenvalue problems are studied in a complex framework.

We found that $\mathbf{B}$ has the eigenvalues, $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. We see that the eigenvalues are conjugate numbers. Another remarkable thing in Example 13.30 is that where

$$\mathbf{v} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to $\lambda_1 = 1 + 2i$, then the conjugate vector

$$\overline{\mathbf{v}} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

is an eigenvector for $\lambda_2 = 1 - 2i$. Both are examples of general rules:

|||| Theorem 13.37 Conjugate Eigenvalues and Eigenvectors

For a square, real matrix $\mathbf{A}$ we have:

1. If λ is a complex eigenvalue of $\mathbf{A}$ in rectangular form $\lambda = a + ib$, then $\overline{\lambda} = a - ib$ is also an eigenvalue of $\mathbf{A}$.

2. If $\mathbf{v}$ is an eigenvector for $\mathbf{A}$ coresponding to the complex eigenvalue λ, then the conjugate vector $\overline{\mathbf{v}}$ is an eigenvector for $\mathbf{A}$ corresponding to the conjugate eigenvalue $\overline{\lambda}$.

|||| Proof

The first part of Theorem 13.37 follows from the theory of polynomials. The characteristic polynomial of a square, real matrix is a polynomial with real coefficients. The roots of such a polynomial come in conjugate pairs.

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By the *trace* of a square matrix we understand the sum of the diagonal elements. The trace of $\mathbf{B}$ is thus $-1 + 3 = 2$. Now notice that the sum of the eigenvalues of $\mathbf{B}$ is $(1-i) + (1+i) = 2$, that is equal to the trace of $\mathbf{B}$. This is also a general phenomenon, which we state without proof:

|||| Theorem 13.38 The Trace

For a square, real matrix $\mathbf{A}$, the trace $\mathbf{A}$, i.e. the sum of the diagonal elements in $\mathbf{A}$, is equal to the sum of all (real and/or complex) eigenvalues of $\mathbf{A}$, where every eigenvalue is counted in the sum the number of times corresponding to the algebraic multiplicity of the eigenvalue.

|||| Exercise 13.39

In Example 13.31 we found that the characteristic polynomial for the matrix

$$\mathbf{A} = \begin{bmatrix} 6 & 3 & 12 \\ 4 & -5 & 4 \\ -4 & -1 & -10 \end{bmatrix}$$

has the double root -6 and the single root 3. Prove that Theorem 13.38 is valid in this case.