eNote 12

Linear Transformations

This eNote investigates an important type of transformation (or map) between vector spaces, viz. linear transformations. It is shown that the kernel and the range for linear transformations are subspaces of the domain and the codomain, respectively. When the domain and the codomain have finite dimensions and a basis has been chosen for each, questions about linear maps can be standardized. In that case a linear transformation can be expressed as a product between a so-called standard matrix for the transformation and the coordinates of the vectors that we want to map. Since standard matrices depend on the chosen bases, we describe how the standard matrices are changed when one of the bases or both are replaced. The prerequisite for the eNote is knowledge about systems of linear equations, see eNote 6, about matrix algebra, see eNote 7 and about vector spaces, see eNote 10.

Updated: 15.11.21 David Brander

12.1 About Maps

A *map* (also known as a *function*) is a rule f that for every element in a set A attaches an element in a set B, and the rule is written $f : A \rightarrow B$. A is called the ***domain*** and B the ***codomain***.

CPR-numbering is a map from the set of citizens in Denmark into $\mathbb{R}^{10}$. Note that there is a 10-times infinity of elements in the codomain $\mathbb{R}^{10}$, so luckily we only need a small subset, about five million! The elements in $\mathbb{R}^{10}$ that in a given instant are in use are the ***range*** for the CPR-map.

Elementary functions of the type $f : \mathbb{R} \rightarrow \mathbb{R}$. are simple maps. The meaning of the arrow is that f to every real number x attaches another real number $y = f(x)$. Consider e.g. the continuous function:

$$y = f(x) = \frac{1}{2}x^2 - 2\,. \tag{12-1}$$

Here the function has the form of a calculation procedure: Square the number, multiply the result by one half and subtract 2. Elementary functions have a great advantage in that their graph $\{ (x,y) \,|\, y = f(x) \}$ can be drawn to give a particular overview of the map (Figure 12.1).

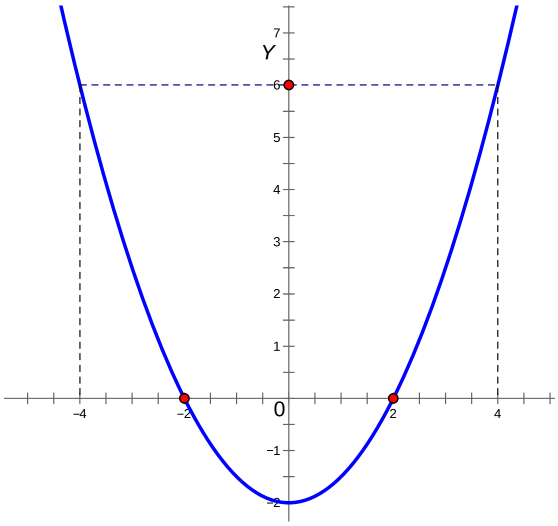


Figure 12.1: Graph of an elementary function

Typical questions in connection with elementary functions reappear in connection with more advanced maps. Therefore let us as an introduction consider some of the most important ones:

1. Determine the zeros of f . This means we must find all x for which $f(x) = 0$. In the example above the answer is $x = -2$ and $x = 2$.

2. Solve for a given b the equation $f(x) = b$. For $b = 6$ there are in the example two solutions: $x = -4$ and $x = 4$.

3. Determine the range for f . We must find all those b for which the equation $f(x) = b$ has a solution. In the example the range is $[\,-2;\infty\,[$.

In this eNote we look at domains, codomains and ranges that are *vector spaces*. A map $f : V \rightarrow W$ attaches to every vector $\mathbf{x}$ in the *domain* V a vector $\mathbf{y} = f(\mathbf{x})$ in the *codomain* W . All the vectors in W that are images of vectors in V together constitute the *range*.

|||| Example 12.1 Mapping from a Vector Space to a Vector Space

A map $g : \mathbb{R}^{2\times 3} \to \mathbb{R}^{2\times 2}$ is given by

$$\mathbf{Y} = g(\mathbf{X}) = \mathbf{X}\mathbf{X}^\top . \tag{12-2}$$

Then e.g.

$$g\left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix} .$$

12.2 Examples of Linear Maps in the Plane

We investigate in the following a map f that has the geometric vectors in the plane as both the domain and codomain. For a given geometric vector $\mathbf{x}$ we will by $\hat{\mathbf{x}}$ understand its *hat vector*, i.e. $\mathbf{x}$ rotated $\pi/2$ counter-clockwise. Consider the map f given by

$$\mathbf{y} = f(\mathbf{x}) = 2\,\hat{\mathbf{x}} . \tag{12-3}$$

To every vector in the plane there is attached its hat vector multiplied (extended) by 2. In Figure 12.2 two vectors $\mathbf{u}$ and $\mathbf{v}$ and their images $f(\mathbf{u})$ and $f(\mathbf{v})$ are drawn.

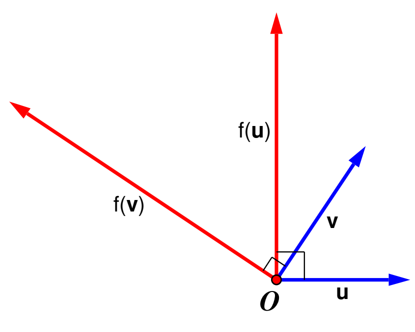


Figure 12.2: Two vectors (blue) and their images (red).

Figure 12.2 gives rise to a couple of interesting questions: How is the vector sum $\mathbf{u} + \mathbf{v}$ mapped? More precisely: How is the image vector $f(\mathbf{u} + \mathbf{v})$ related to the two image vectors $f(\mathbf{u})$ and $f(\mathbf{v})$? And what is the relation between the image vectors $f(k\mathbf{u})$ and $f(\mathbf{u})$, when k is a given real number?

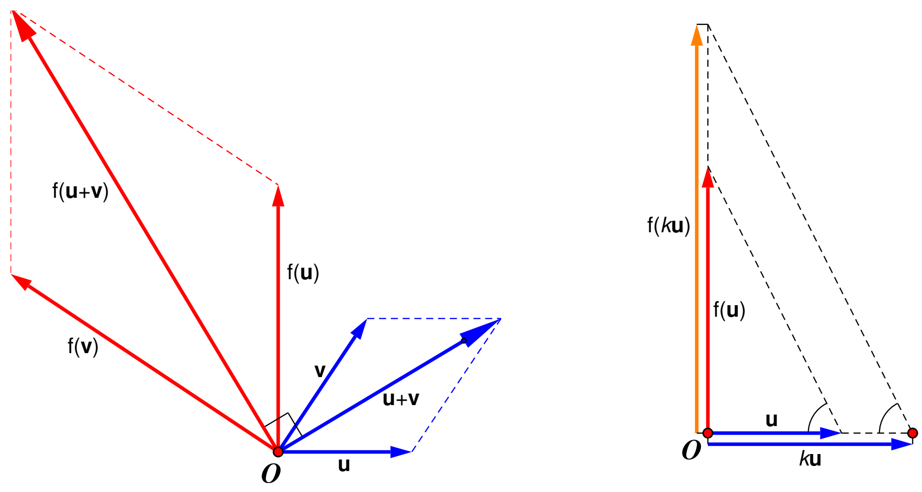


Figure 12.3: Construction of $f(\mathbf{u}+\mathbf{v})$ and $f(k\mathbf{u})$.

As indicated in Figure 12.3, f satisfies two very simple rules:

$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) \text{ and } f(k\mathbf{u}) = k\,f(\mathbf{u})\,. \tag{12-4}$$

Using the well known computational rules for hat vectors

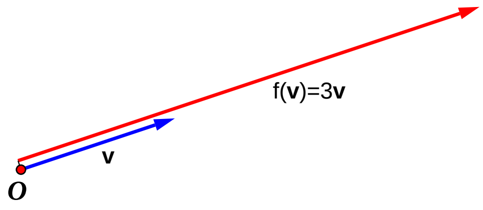
1. $\widehat{\mathbf{u}+\mathbf{v}} = \hat{\mathbf{u}} + \hat{\mathbf{v}}$.
2. $\widehat{k\mathbf{u}} = k\hat{\mathbf{u}}$.

we can now confirm the statement (12-4) :

$$\begin{aligned} f(\mathbf{u}+\mathbf{v}) &= 2\widehat{\mathbf{u}+\mathbf{v}} = 2(\hat{\mathbf{u}}+\hat{\mathbf{v}}) = 2\hat{\mathbf{u}}+2\hat{\mathbf{v}} \\ &= f(\mathbf{u}) + f(\mathbf{v}) \\ f(k\mathbf{u}) &= 2\widehat{k\mathbf{u}} = 2k\hat{\mathbf{u}} = k(2\hat{\mathbf{u}}) \\ &= k\,f(\mathbf{u}) \end{aligned}$$

|||| Exercise 12.2

A map f_1 of plane vectors is given by $f_1(\mathbf{v}) = 3\mathbf{v}$:

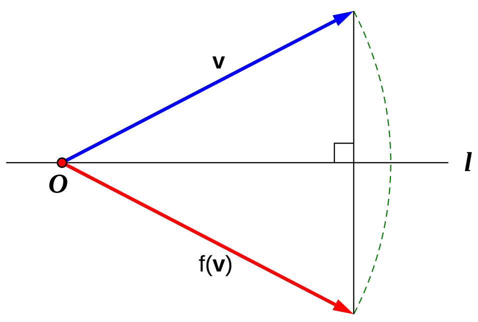


Scaling of vectors

Draw a figure that demonstrates that f_1 satisfies the rules (12-4) .

|||| Exercise 12.3

In the plane a line l through the origin is given. A map f_2 reflects vectors drawn from the origin in l :

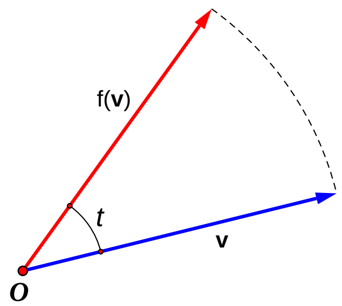


Reflection of a vector

Draw a figure that demonstrates that f_2 satisfies the rules (12-4) .

|||| Exercise 12.4

A map f_3 turns vectors drawn from the origin the angle t about the origin counterclockwise:



Rotation of a vector

Draw a figure that demonstrates that f_3 satisfies the rules (12-4) .

All maps mentioned in this section are *linear*, because they satisfy (12-4) . We now turn to a general treatment of linear mappings between vector spaces.

12.3 Linear Maps

|||| Definition 12.5 Linear Map

Let V and W be two vector spaces and let $\mathbb{L}$ denote either $\mathbb{R}$ or $\mathbb{C}$. A map $f : V \to W$ is called ***linear*** if for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars $k \in \mathbb{L}$ it satisfies the following two *linearity requirements*:

$$L_1 : \quad f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) \, .$$
$$L_2 : \quad f(k\mathbf{u}) = k\, f(\mathbf{u}) \, .$$

V is called the *domain* and W the *codomain* for f .

By putting $k = 0$ in the linearity requirement L_2 in the definition 12.5 , we see that

$$f(\mathbf{0}) = \mathbf{0}\,. \qquad (12\text{-}5)$$

In other words for every linear map $f : V \to W$ the zero vector in V is mapped to the zero vector in W .

The image of a linear combination becomes in a very simple way a linear combination of the images of the vectors that are part of the given linear combination:

$$f(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_p\mathbf{v}_p) = k_1 f(\mathbf{v}_1) + k_2 f(\mathbf{v}_2) + \ldots + k_p f(\mathbf{v}_p)\,. \qquad (12\text{-}6)$$

This result is obtained by repeated application of L_1 and L_2.

|||| Example 12.6 Linear Map

A map $f : \mathbb{R}^2 \to \mathbb{R}^4$ is given by the rule

$$f(x_1, x_2) = (0, x_1, x_2, x_1 + x_2)\,. \qquad (12\text{-}7)$$

$\mathbb{R}^2$ and $\mathbb{R}^4$ are vector spaces and we investigate whether f is a linear map. First we test the left hand side and the right hand side of L_1 with the vectors $(1, 2)$ and $(3, 4)$:

$$\begin{aligned} f(\,(1,2) + (3,4)\,) &= f(4,6) = (0,4,6,10)\,. \\ f(1,2) + f(3,4) &= (0,1,2,3) + (0,3,4,7) = (0,4,6,10)\,. \end{aligned}$$

Then L_2 is tested with the vector (2,3) and the scalar 5:

$$\begin{aligned} f(\,5\cdot(2,3)\,) &= f(10,15) = (0,10,15,25)\,. \\ 5\cdot f(2,3) &= 5\cdot(0,2,3,5) = (0,10,15,25)\,. \end{aligned}$$

The investigatíon suggests that f is linear. This is now shown generally. First we test L_1:

$$\begin{aligned} f(\,(x_1,x_2) + (y_1,y_2)\,) &= f(x_1+y_1, x_2+y_2) = (0, x_1+y_1, x_2+y_2, x_1+x_2+y_1+y_2)\,. \\ f(x_1,x_2) + f(y_1,y_2) &= (0,x_1,x_2,x_1+x_2) + (0,y_1,y_2,y_1+y_2) \\ &= (0, x_1+y_1, x_2+y_2, x_1+x_2+y_1+y_2)\,. \end{aligned}$$

Then we test L_2:

$$\begin{aligned} f(\,k\cdot(x_1,x_2)\,) &= f(k\cdot x_1, k\cdot x_2) = (0, k\cdot x_1, k\cdot x_2, k\cdot x_1 + k\cdot x_2)\,. \\ k\cdot f(x_1,x_2) &= k\cdot(0,x_1,x_2,x_1+x_2) = (0, k\cdot x_1, k\cdot x_2, k\cdot x_1 + k\cdot x_2)\,. \end{aligned}$$

It is seen that f satisfies both linearity requirements and therefore is linear.

|||| Example 12.7 A Map that is Not Linear

In the example 12.1 we considered the map $g : \mathbb{R}^{2\times 3} \to \mathbb{R}^{2\times 2}$ given by

$$\mathbf{Y} = g(\mathbf{X}) = \mathbf{X}\mathbf{X}^{\top} . \tag{12-8}$$

That this map is *not* linear, can be shown by finding an example where either L_1 or L_2 is not valid. Below we give an example of a matrix $\mathbf{X}$ that does not satisfy $g(2\mathbf{X}) = 2\,g(\mathbf{X})$:

$$g\left(2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = g\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}.$$

But

$$2\,g\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore g does not satisfy the linearity requirements L_2, hence g is not linear.

|||| Example 12.8 Linear Map

A map $f : P_2(\mathbb{R}) \to \mathbb{R}$ is given by the rule

$$f\big(\,P(x)\,\big) = P'(1)\,. \tag{12-9}$$

For every second degree polynomial the slope of the tangent at $x = 1$ is attached. An arbitrary second degree polynomial P can be written as $P(x) = ax^2 + bx + c$, where a, b and c are real constants. Since $P'(x) = 2ax + b$ we have:

$$f\big(\,P(x)\,\big) = 2a + b\,.$$

If we put $P_1(x) = a_1x^2 + b_1x + c_1$ and $P_2(x) = a_2x^2 + b_2x + c_2$, we get

$$\begin{aligned} f\big(\,P_1(x) + P_2(x)\,\big) &= f\big(\,(a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2)\,\big) \\ &= \big(\,2(a_1 + a_2) + (b_1 + b_2)\,\big) \\ &= (2a_1 + b_1) + (2a_2 + b_2) \\ &= f\big(\,P_1(x)\,\big) + f\big(\,P_2(x)\,\big)\,. \end{aligned}$$

Furthermore for every real number k and every second degree polynomial $P(x)$:

$$\begin{aligned} f\big(\,k \cdot P(x)\,\big) &= f\big(\,k \cdot ax^2 + k \cdot bx + k \cdot c\,\big) \\ &= (2k \cdot a + k \cdot b) = k \cdot (2a + b) \\ &= k \cdot f\big(\,P(x)\,\big)\,. \end{aligned}$$

It is hereby shown that f satisfies the linearity conditions L_1 and L_2, and that f thus is a linear map.

|||| Exercise 12.9

By $C^\infty(\mathbb{R})$ we understand the vector space consisting of all functions $f : \mathbb{R} \to \mathbb{R}$ that can be differentiated an arbitrary number of times. One example (among infinitely many) is the sine function. Consider the map $\mathrm{D} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ that to a function $f(x) \in C^\infty(\mathbb{R})$ assigns its derivative:

$$\mathrm{D}(f(x)) = f'(x)\,.$$

Show that D is a linear map.

12.4 Kernel and Range

The zeros of an elementary function $f : \mathbb{R} \to \mathbb{R}$ are all the real numbers x that satisfy $f(x) = 0$. The corresponding concept for linear maps is called the *kernel*. The range of an elementary function $f : \mathbb{R} \to \mathbb{R}$ are all the real numbers b for each of which a real number x exists such that $f(x) = b$. The corresponding concept for linear maps is also called the *range* or *image*. The kernel is a subspace of the domain and the range is a subspace of the codomain. This is now shown.

|||| Definition 12.10 Kernel and Range

By the ***kernel*** of a linear map $f : V \to W$ we understand the set:

$$\ker(f) = \{\, \mathbf{x} \in V \mid f(\mathbf{x}) = \mathbf{0} \in W \,\}\,. \tag{12-10}$$

By the ***range*** or ***image*** of f we understand the set:

$$f(V) = \{\, \mathbf{b} \in W \mid \text{At least one } \mathbf{x} \in V \text{ exists with } f(\mathbf{x}) = \mathbf{b} \,\}\,. \tag{12-11}$$

|||| Theorem 12.11 The Kernel and the Range are Subspaces

Let $f : V \to W$ be a linear map. Then:

1. The kernel of f is a subspace of V.
2. The range $f(V)$ is a subspace of W.

|||| Proof

1) First, the kernel is not empty, as $f(\mathbf{0}) = 0$ by linearity. So we just need to prove that the kernel of f satisfies the stability requirements, see Theorem 11.42. Assume that $\mathbf{x}_1 \in V$ and $\mathbf{x}_2 \in V$, and that k is an arbitrary scalar. Since (using L_1):

$$f(\mathbf{x}_1 + \mathbf{x}_2) = f(\mathbf{x}_1) + f(\mathbf{x}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}\,,$$

the kernel of f is stable with respect to addition. Moreover (using L_2):

$$f(k\mathbf{x}_1) = k\,f(\mathbf{x}_1) = k\,\mathbf{0} = \mathbf{0}\,,$$

the kernel of f is also stable with respect to multiplication by a scalar. In total we had shown that the kernel of f is a subspace of V.

2) The range $f(V)$ is non-empty, as it contains the zero vector. We now show that it satisfies the stability requirements. Suppose that $\mathbf{b}_1 \in f(V)$ and $\mathbf{b}_2 \in f(V)$, and that k is an arbitrary scalar. There exist, according to the definition, see (12.10), vectors $\mathbf{x}_1 \in V$ and $\mathbf{x}_2 \in V$ that satisfy $f(\mathbf{x}_1) = \mathbf{b}_1$ and $f(\mathbf{x}_2) = \mathbf{b}_2$. We need to show that there exists an $\mathbf{x} \in V$ such that $f(\mathbf{x}) = \mathbf{b}_1 + \mathbf{b}_2$. There is, namely $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, since

$$f(\mathbf{x}_1 + \mathbf{x}_2) = f(\mathbf{x}_1) + f(\mathbf{x}_2) = \mathbf{b}_1 + \mathbf{b}_2\,.$$

Hereby it is shown that $f(V)$ is stable with respect to addition. We will, in a similar way, show that there exists an $\mathbf{x} \in V$ such that $f(\mathbf{x}) = k\mathbf{b}_1$. Here we choose $\mathbf{x} = k\mathbf{x}_1$, then

$$f(\mathbf{x}) = f(k\mathbf{x}_1) = kf(\mathbf{x}_1) = k\mathbf{b}_1\,,$$

from which it appears that $f(V)$ is stable with respect to multiplication by a scalar. In total we have shown that $f(V)$is a subspace of W.

■

But why is it so interesting that the kernel and the range of a linear map are subspaces? The answer is that it becomes simpler to describe them when we know that they possess vector space properties and we thereby in advance know their structure. It is particularly elegant when we can determine the kernel and the range by giving a basis for them. This we will try in the next two examples.

|||| Example 12.12 Determination of Kernel and Range

A linear map $f : \mathbb{R}^3 \to R^2$ is given by the rule:

$$f(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3\,, -x_1 - 2x_2 - x_3)\,. \tag{12-12}$$

We wish to determine the kernel of f and the range $f(\mathbb{R}^3)$ (note that it is given that f is linear. So we omit the proof of that).

Determination of the kernel:
We shall solve the equation

$$f(\mathbf{x}) = \mathbf{0} \Leftrightarrow \begin{bmatrix} x_1 + 2x_2 + x_3 \\ -x_1 - 2x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{12-13}$$

This is a system of linear equations consisting of two equations in three unknowns. The corresponding augmented matrix is

$$\mathbf{T} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & -2 & -1 & 0 \end{bmatrix} \to \mathrm{rref}(\mathbf{T}) = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the system of equations has the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The solution set is spanned by two linearly independent vectors. Therefore we can conclude that the kernel of f is a 2-dimensional subspace of $\mathbb{R}^3$ that is precisely characterized by a basis:

$$\text{Basis for the kernel}: \big(\,(-2,1,0), (-1,0,1)\,\big)\,.$$

There is an entire plane of vectors in the space, that by insertion into the expression for f give the image $\mathbf{0}$. This basis yields all of them.

Determination of the range:
We shall find all those $\mathbf{b} = (b_1, b_2)$ for which the following equation has a solution:

$$f(\mathbf{x}) = \mathbf{b} \Leftrightarrow \begin{bmatrix} x_1 + 2x_2 + x_3 \\ -x_1 - 2x_2 - x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{12-14}$$

Note, that it is not x_1, x_2 and x_3, we are looking for, as we usually do in such a system of equations. Rather it is b_1 and b_2 of the right hand side, which we will determine exactly *in those cases, when solutions exist*! Because when the system has solution of a particular right hand side, then this right-hand side *must* be in the image space that we are looking for.

This is a system of linear equations consisting of two equations in three unknowns. The corresponding augmented matrix is

$$\mathbf{T} = \begin{bmatrix} 1 & 2 & 1 & b_1 \\ -1 & -2 & -1 & b_2 \end{bmatrix} \rightarrow \operatorname{rref}(\mathbf{T}) = \begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 0 & 0 & b_1 + b_2 \end{bmatrix}$$

If $b_1 + b_2 = 0$, that is if $b_1 = -b_2$, the system of equations has infinitely many solutions. If on the contrary $b_1 + b_2 \neq 0$ there is no solution. All those $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ that are images of at least one $\mathbf{x} \in R^3$ evidently tcan be written as:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We conclude that $f(V)$ is a 1-dimensional subspace of $\mathbb{R}^2$ that can be characterized precisely by a basis:

$$\text{Basis for the range} : \big((-1, 1) \big) .$$

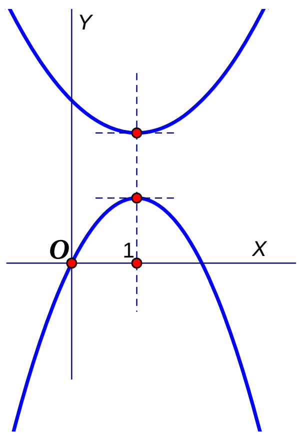


Figure 12.4: Two vectors in the kernel (Exercise 12.13)

|||| Exercise 12.13 Determination of Kernel and Range

In example 12.8 it was shown that the map $f : P_2(\mathbb{R}) \rightarrow \mathbb{R}$ given by the rule

$$f\big(P(x)\big) = P'(1)\,. \tag{12-15}$$

is linear. The kernel of f consists of all second degree polynomials that satisfy $P'(1) = 0$. The graphs for a couple of these are shown in Figure 12.4:

Determine the kernel of f.

In eNote 6 the relation bewteen the solution set for an inhomogeneous system of linear equations and the corresponding homogeneous linear system of equations is presented in Theorem 6.37 (the structural theorem). We now show that a corresponding relation exists for all linear equations.

|||| Theorem 12.14 The Structural Theorem for Linear Equations

Let $f : V \rightarrow W$ be a liner map and $\mathbf{y}$ an arbitrary proper vector in W. Furthermore let $\mathbf{x}_0$ be an arbitrary (so-called particular) solution to the inhomogeneous linear equation

$$f(\mathbf{x}) = \mathbf{y}\,. \tag{12-16}$$

Then the general solution L_{inhom} to the linear equation is given by

$$L_{inhom} = \left\{\, \mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1 \;\middle|\; \mathbf{x}_1 \in \ker(f) \,\right\}\,, \tag{12-17}$$

or in short

$$L_{inhom} = \mathbf{x}_0 + \ker(f)\,. \tag{12-18}$$

|||| Proof

The theorem contains two assertions. The one is that the sum of $\mathbf{x}_0$ and an arbitrary vector from the $\ker(f)$ belongs to L_{inhom}. The other is that an arbitrary vector from L_{inhom} can be written as the sum of $\mathbf{x}_0$ and a vector from $\ker(f)$. We prove the two assertions separately:

1. Assume $\mathbf{x}_1 \in \ker(f)$. Then it applies using the linearity condition L_1 :

$$f(\mathbf{x}_1 + \mathbf{x}_0) = f(\mathbf{x}_1) + f(\mathbf{x}_0) = \mathbf{0} + \mathbf{y} = \mathbf{y} \tag{12-19}$$

by which it is also shown that $\mathbf{x}_1 + \mathbf{x}_0$ is a solution to (12-16).

2. Assume that $\mathbf{x}_2 \in L_{inhom}$. The it applies using the linearity condition L_1 :

$$f(\mathbf{x}_2 - \mathbf{x}_0) = f(\mathbf{x}_2) - f(\mathbf{x}_0) = \mathbf{y} - \mathbf{y} = \mathbf{0} \Leftrightarrow \mathbf{x}_2 - \mathbf{x}_0 \in \ker(f)\,. \tag{12-20}$$

Thus a vector $\mathbf{x}_1 \in \ker(f)$ exists that satisfy

$$\mathbf{x}_2 - \mathbf{x}_0 = \mathbf{x}_1 \Leftrightarrow \mathbf{x}_2 = \mathbf{x}_0 + \mathbf{x}_1 \tag{12-21}$$

whereby we have stated $\mathbf{x}_2$ in the form wanted. The proof is hereby complete.

■

|||| Exercise 12.15

Consider the map $\mathrm{D} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ from Exercise 12.9 that to the function $f \in C^\infty(\mathbb{R})$ relates its derivative:

$$\mathrm{D}(\,f(x)\,) = f'(x)\,.$$

State the complete solution to inhomogeneous linear eequation

$$\mathrm{D}(f(x)) = x^2$$

and interprete this in the light of the structural theorem.

12.5 Mapping Matrix

All linear maps from a finite dimensional domain V to a finite dimensional codomain W can be described by a *mapping matrix*. This is the subject of this subsection. The prerequisite is only that a basis for both V and W is chosen, and that we turn from vector calculation to calculation using the coordinates with respect to the chosen bases. The great advantage by this setup is that we can construct general methods of calculation valid for all linear maps between finite dimensional vector spaces. We return to this subject, see section 12.6. First we turn to mapping matrix construction.

Let $\mathbf{A}$ be a real or complex $m \times n-$matrix. We consider a map $f : \mathbb{L}^n \rightarrow \mathbb{L}^m$ that has the form of a f matrix-vector product:

$$\mathbf{y} = f(\mathbf{x}) = \mathbf{A}\,\mathbf{x}\,. \tag{12-22}$$

Using the matrix product computation rules from Theorem 7.13, we obtain for every choice of $\mathbf{x}_1\,, \mathbf{x}_2 \in \mathbb{R}^n$ and every scalar k:

$$f(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{A}\,(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{A}\mathbf{x}_1 + \mathbf{A}\mathbf{x}_2 = f(\mathbf{x}_1) + f(\mathbf{x}_2)\,,$$

$$f(k\mathbf{x}_1) = \mathbf{A}\,(k\mathbf{x}_1) = k(\mathbf{A}\,\mathbf{x}_1) = kf(\mathbf{x}_1)\,.$$

We see that the map satisfies the linearity requirements L_1 and L_2 . Therefore every map of the form (12-22) is linear.

|||| Example 12.16 Matrix-Vector Product as a Linear Map

The formula:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

defines a particular linear map from the vector space $\mathbb{R}^2$ to the vector space $\mathbb{R}^3$.

But also the opposite is true: Every linear map between finite-dimensional vector spaces can be written as a matrix-vector product in the form (12-22) if we replace $\mathbf{x}$ and $\mathbf{y}$ with their coordinates with respect to a chosen basis for the domain and codomain, respectively. This we show in the following.

We consider a linear map $f : V \rightarrow W$ where V is an n-dimensional and W is an m-dimensional vector space, see Figure 12.5

For V a basis a is chosen and for W a basis c. This means that a given vector $\mathbf{x} \in V$ can be written as a unique linear combination of the a-basis vectors and that the image $\mathbf{y} = f(\mathbf{x})$ can be written as a unique linear combination of the c-basis vectors:

$$\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n \;\text{ and }\; \mathbf{y} = y_1\mathbf{c}_1 + y_2\mathbf{c}_2 + \cdots + y_m\mathbf{c}_m\,.$$

This means that $(x_1, x_2, \ldots, x_n)$ is the set of coordinates for $\mathbf{x}$ with respect to the a-basis, and that $(y_1, y_2, \ldots, y_m)$ is the set of coordinates for $\mathbf{y}$ with respect to the c-basis.

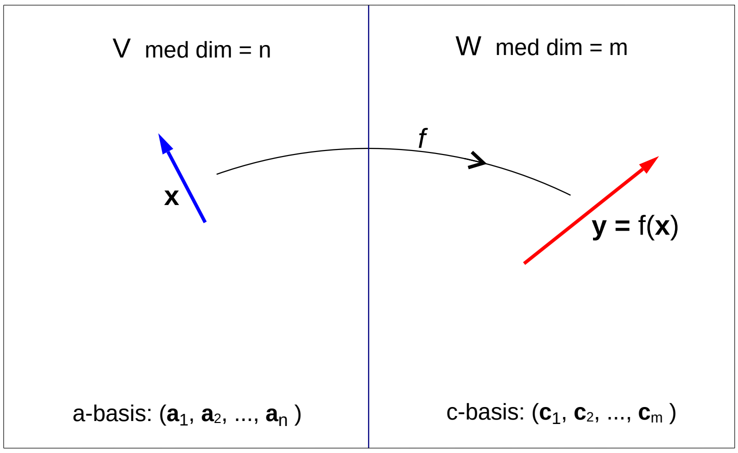


Figure 12.5: Linear map

We now pose the question: How can we describe the relation between the a-coordinate vector for the vector $\mathbf{x} \in V$ and the c-coordinate vector for the image vector $\mathbf{y}$? In other words we are looking for the relation between:

$$
{}_{\mathrm{c}}\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad \text{and} \quad {}_{\mathrm{a}}\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} .
$$

This we develop through the following rewritings where we first, using L_1 and L_2, get $\mathbf{y}$ written as a linear combination of the images of the a-vectos.

$$
\begin{aligned}
\mathbf{y} &= f(\mathbf{x}) \\
&= f(x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n) \\
&= x_1 f(\mathbf{a}_1) + x_2 f(\mathbf{a}_2) + \cdots + x_n f(\mathbf{a}_n) \,.
\end{aligned}
$$

Hereafter we can investigate the coordinate vector for $\mathbf{y}$ with respect to the c-basis, while we first use the cooordinate theorem, see Theorem 11.34, and thereafter the definition on matrix-vector product, see Definition 7.7.

$$
\begin{aligned}
{}_{\mathrm{c}}\mathbf{y} &= {}_{\mathrm{c}}\big(\, x_1 f(\mathbf{a}_1) + x_2 f(\mathbf{a}_2) + \cdots + x_n f(\mathbf{a}_n) \,\big) \\
&= x_1\, {}_{\mathrm{c}}f(\mathbf{a}_1) + x_2\, {}_{\mathrm{c}}f(\mathbf{a}_2) + \cdots + x_n\, {}_{\mathrm{c}}f(\mathbf{a}_n) \\
&= \big[\, {}_{\mathrm{c}}f(\mathbf{a}_1) \quad {}_{\mathrm{c}}f(\mathbf{a}_2) \quad \cdots \quad {}_{\mathrm{c}}f(\mathbf{a}_n) \,\big]\, {}_{\mathrm{a}}\mathbf{x} \,.
\end{aligned}
$$

The matrix $\left[\,{}_{\mathrm{c}}f(\mathbf{a}_1)\quad {}_{\mathrm{c}}f(\mathbf{a}_2)\quad \cdots\quad {}_{\mathrm{c}}f(\mathbf{a}_n)\,\right]$ in the last equation is called the *mapping matrix* for f with respect to the a-basis for V and the c-basis for W.

Thus we have achieved this important result: The coordinate vector ${}_{\mathrm{c}}\mathbf{y}$ can be found by multiplying the coordinate vector ${}_{\mathrm{a}}\mathbf{x}$ on the left by the mapping matrix. We now summarize the results in the following.

|||| Definition 12.17 Mapping Matrix

Let $f : V \to W$ be a linear map from an n-dimensional vector space V to an m-dimensional vector space W. By the ***mapping matrix*** for f with respect to the basis a of V and basis c of W we understand the $m \times n$-matrix:

$$ {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}} = \left[\,{}_{\mathrm{c}}f(\mathbf{a}_1)\quad {}_{\mathrm{c}}f(\mathbf{a}_2)\quad \cdots\quad {}_{\mathrm{c}}f(\mathbf{a}_n)\,\right]. \tag{12-23} $$

The mapping matrix for f thus consists of the coordinate vectors with respect to the basis c of the images of the n basis vectors in basis a.

The main task for a mapping matrix is of course to determine the images in W of the vectors in V, and this is justified in the following theorem which summarizes the investigations above.

|||| Theorem 12.18 Main Theorem of Mapping Matrices

Let V be an n-dimensional vector space with a chosen basis a and W an m-dimensional vector space with a chosen basis c .

1. For a linear map $f : V \to W$ it is valid that if $\mathbf{y} = f(\mathbf{x})$ is the image of an arbitrary vector $\mathbf{x} \in V$, then:

$$ {}_{\mathrm{c}}\mathbf{y} = {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\,{}_{\mathrm{a}}\mathbf{x} \tag{12-24} $$

   where ${}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}$ is the mapping matrix for f with respect to the basis a of V and the basis c of W .

2. Conversely, assume that the images $\mathbf{y} = g(\mathbf{x})$ for a map $g : V \to W$ can be obtained in the coordinate form as

$$ {}_{\mathrm{c}}\mathbf{y} = {}_{\mathrm{c}}\mathbf{G}_{\mathrm{a}}\,{}_{\mathrm{a}}\mathbf{x} \tag{12-25} $$

   where ${}_{\mathrm{c}}\mathbf{G}_{\mathrm{a}} \in \mathbb{L}^{m \times n}$. Then g is linear and ${}_{\mathrm{c}}\mathbf{G}_{\mathrm{a}}$ is the mapping matrix for g with respect to the basis a of V and basis c of W .

Below are three examples of the construction and elementary use of mapping matrices.

|||| Example 12.19 Construction and Use of a Mapping Matrix

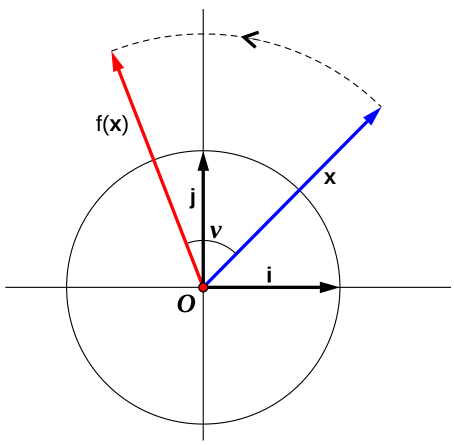


Figure: Linear rotation about the origin

Rotation of plane vectors drawn from the origin is a simple example of a linear map, see Exercise 12.4. Let v be an arbitrary angle, and let f be the linear mapping that rotates an arbitrary vector the angle v about the origin counterclockwise, (see the figure above).

We wish to determine the mapping matrix for f with respect to the standard basis for vectors in the plane. Therefore we need the images of the basis vectors $\mathbf{i}$ and $\mathbf{j}$:

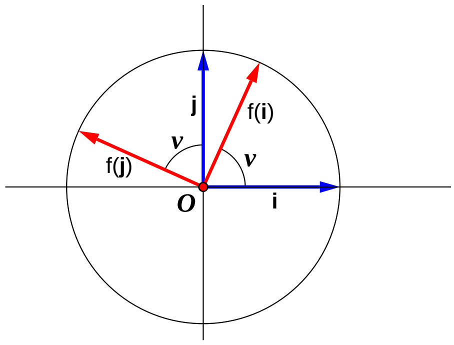


Figure: Determination of mapping matrix

It is seen that $f(\mathbf{i}) = (\cos(v), \sin(v))$ and $f(\mathbf{j}) = (-\sin(v), \cos(v))$. Therefore the mapping matrix we are looking for is

$$ {}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}} = \begin{bmatrix} \cos(v) & -\sin(v) \\ \sin(v) & \cos(v) \end{bmatrix}. $$

The coordinates for the image $\mathbf{y} = f(\mathbf{x})$ of a given vector $\mathbf{x}$ are thus given by the formula:

$$ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(v) & -\sin(v) \\ \sin(v) & \cos(v) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. $$

|||| Example 12.20 Construction and Use of a Mapping Matrix

In a 3-dimensional vector space V, a basis $a = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is chosen, and in 2-dimensional vector space W a basis $c = (\mathbf{c}_1, \mathbf{c}_2)$ is chosen. A linear map $f : V \to W$ satisfies:

$$ f(\mathbf{a}_1) = 3\mathbf{c}_1 + \mathbf{c}_2\,,\; f(\mathbf{a}_2) = 6\mathbf{c}_1 - 2\mathbf{c}_2 \text{ and } f(\mathbf{a}_3) = -3\mathbf{c}_1 + \mathbf{c}_2\,. \tag{12-26} $$

We wish to find the image under f of the vector $\mathbf{v} = \mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3 \in V$ using the mapping matrix ${}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}$. The mapping matrix is easily constructed since we already from (12-26) know the images of the basis vectors in V :

$$ {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}} = \left[\, {}_{\mathrm{c}}f(\mathbf{a}_1) \quad {}_{\mathrm{c}}f(\mathbf{a}_2) \quad {}_{\mathrm{c}}f(\mathbf{a}_3) \,\right] = \begin{bmatrix} 3 & 6 & -3 \\ 1 & -2 & 1 \end{bmatrix} $$

Since $\mathbf{v}$ has the set of coordinates $(1,2,1)$ with respect to basis a, we find the coordinate vector for $f(\mathbf{v})$ like this:

$$ {}_{\mathrm{c}}f(\mathbf{v}) = {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\,{}_{\mathrm{a}}\mathbf{v} = \begin{bmatrix} 3 & 6 & -3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \end{bmatrix}. $$

Hence we have found $f(\mathbf{v}) = 12\mathbf{c}_1 - 2\mathbf{c}_2$.

|||| Example 12.21 Construction and Use of a Mapping Matrix

A linear map $f : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is given by:

$$ f(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 + 2x_2 + x_4 \\ 2x_1 - x_2 + 2x_3 - x_4 \\ x_1 - 3x_2 + 2x_3 - 2x_4 \end{bmatrix}. \tag{12-27} $$

Let us determine the mapping matrix for f with respect to the standard basis e of $\mathbb{R}^4$ and the standard basis e of $\mathbb{R}^3$. First we find the images of the four basis vectors in $\mathbb{R}^4$ using the rule (12-27):

$$ f(1,0,0,0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad f(0,1,0,0) = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}, $$

$$ f(0,0,1,0) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \quad f(0,0,0,1) = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}. $$

We can now construct the mapping matrix for f:

$$ {}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{bmatrix}. \tag{12-28} $$

We wish to find the image $\mathbf{y} = f(\mathbf{x})$ of the vector $\mathbf{x} = (1,1,1,1)$. At our disposal we have of course the rule (12-27), but we choose to find the image using the mapping matrix:

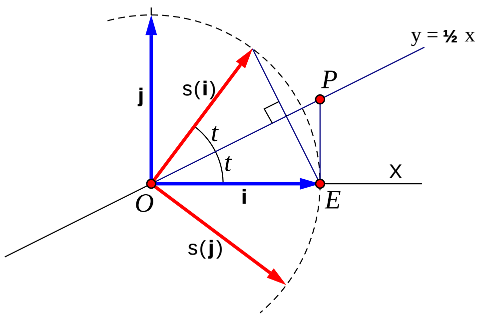
$$ {}_{\mathrm{e}}\mathbf{y} = {}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}}\,{}_{\mathrm{e}}\mathbf{x} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}. $$

Thus we have found that $y = f(1,1,1,1) = (4,2,-2)$.

|||| Exercise 12.22

In the plane is given a customary $(O, \mathbf{i}, \mathbf{j})$-coordinate system. Reflection of position vectors about the line $y = \frac{1}{2}x$ is a linear map, let us call it s.

Determine $s(\mathbf{i})$ and $s(\mathbf{j})$, construct the mapping matrix ${}_e\mathbf{S}_e$ for s and determine an expression k for the reflection of an arbitrary position vector $\mathbf{v}$ with the coordinates (v_1, v_2) with respect to the standard basis. The figure below contains some hints for the determination of $s(\mathbf{i})$. Proceed similarly with $s(\mathbf{j})$.



Reflection of the standard basis vectors.

12.6 On the Use of Mapping Matrices

The mapping matrix tool has a wide range of applications. It allows us to translate questions about linear maps between vector spaces to questions about matrices and coordinate vectors that allow immediate calculations. The methods only require that bases in each of the vector spaces be chosen, and that the mapping matrix that belongs to the two bases has been formed. In this way we can reduce problems as diverse as that of finding polynomials with certain properties, finding the result of a geometrical construction and finding the solution of differential equations, to problems that can be solved through the use of matrix algebra.

As a recurrent example in this section we look at a linear map $f : V \rightarrow W$ where V is a 4-dimensional vector space with chosen basis $a = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$, and where W is a

3-dimensional vector space with chosen basis $c = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$. The mapping matrix for f is:

$$ {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}} = \begin{bmatrix} 1 & 3 & -1 & 8 \\ 2 & 0 & 4 & -2 \\ 1 & -1 & 3 & -4 \end{bmatrix} . \tag{12-29} $$

12.6.1 Finding the Kernel of f

To obtain the kernel of f you must find all the $\mathbf{x} \in V$ that are mapped to $\mathbf{0} \in W$. That is you solve the vector equation

$$ f(\mathbf{x}) = \mathbf{0} . $$

This equation is according to the Theorem 12.18 equivalent to the matrix equation

$$ {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\, {}_{\mathrm{a}}\mathbf{x} = {}_{\mathrm{c}}\mathbf{0} $$

$$ \Leftrightarrow \begin{bmatrix} 1 & 3 & -1 & 8 \\ 2 & 0 & 4 & -2 \\ 1 & -1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

that corresponds to the homogeneous system of linear equations with the augmented matrix:

$$ \mathbf{T} = \left[\begin{array}{cccc|c} 1 & 3 & -1 & 8 & 0 \\ 2 & 0 & 4 & -2 & 0 \\ 1 & -1 & 3 & -4 & 0 \end{array}\right] \rightarrow \mathrm{rref}(\mathbf{T}) = \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] . $$

It is seen that the solution set is spanned by two linear independent vectors: $(-2, 1, 1, 0)$ and $(1, -3, 0, 1)$. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the two vectors in V that are determined by the a-coordinates like this:

$$ {}_{\mathrm{a}}\mathbf{v}_1 = (-2, 1, 1, 0) \;\text{ and }\; {}_{\mathrm{a}}\mathbf{v}_2 = (1, -3, 0, 1) . $$

Since the two coordinate vectors are linearly independent, $(\mathbf{v}_1, \mathbf{v}_2)$ is a basis for the kernel of f, and the kernel of f has the dimension 2.

Point: The number $n = 4$ of unknowns in the solved system of equations is by definition equal to the number of columns in ${}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}$ that again is equal to $\dim(V)$, see definition 12.17. Moreover we notice that the coefficient matrix of the system of equations is equal to ${}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}$. If the rank of the coefficient matrix is k , we know that the solution set, and therefore the kernel, will be spanned by $(n - k)$ linearly independent directional vectors where k is the rank of the coefficient matrix. Therefore we have:

$$ \dim(\ker(f)) = n - \rho\,({}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}) = 4 - 2 = 2 . $$

|||| **Method 12.23 Determination of the Kernel**

In a vector space V a basis a is chosen, and in a vector space W a basis c is chosen. The kernel of a linear map $f : V \to W$, in coordinate form, can be found as the solution set for the homogeneous system of linear equations with the augmented matrix

$$\mathbf{T} = \left[\, {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}} \;\middle|\; {}_{\mathrm{c}}\mathbf{0} \,\right].$$

The kernel is a subspace of V and its dimension is determined by:

$$\dim(\ker(f)) = \dim(V) - \rho\left({}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\right). \tag{12-30}$$

12.6.2 Solving the Vector Equation *f(x)* = *b*

How can you decide whether a vector $\mathbf{b} \in W$ belongs to the image for a given linear map? The question is whether (at least) one $\mathbf{x} \in V$ exists that is mapped to $\mathbf{b}$. And the question can be extended to how to determine all $\mathbf{x} \in V$ with this property that is mapped in $\mathbf{b}$.

We consider the linear map $f : V \to W$ that is represented by the mapping matrix (12-29) and choose as our example the vector $\mathbf{b} \in W$ that has c-coordinates $(1, 2, 3)$. We will solve the vector equation

$$f(\mathbf{x}) = \mathbf{b}.$$

If we calculate with coordinates the vector equation corresponds to the following matrix equation

$${}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\, {}_{\mathrm{a}}\mathbf{x} = {}_{\mathrm{c}}\mathbf{b}$$

that is the matrix equation

$$\begin{bmatrix} 1 & 3 & -1 & 8 \\ 2 & 0 & 4 & -2 \\ 1 & -1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

that corresponds to an inhomogeneous system of linear equations with the augmented matrix:

$$\mathbf{T} = \left[\begin{array}{cccc|c} 1 & 3 & -1 & 8 & 1 \\ 2 & 0 & 4 & -2 & 2 \\ 1 & -1 & 3 & -4 & 3 \end{array}\right]$$

that by Gauss-Jordan elimination is reduced to

$$\operatorname{rref}(\mathbf{T}) = \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Since the rank of the augmented matrix is larger than the rank of the coefficient matrix, the inhomogeneous system of equations has no solutions. We have thus found a vector in W that has no “original vector” in V.

|||| Method 12.24 Solution of the Vector Equation *f(x) = b*

In a vector space V a basis a is chosen, and in a vector space W a basis c is chosen. For a linear map $f : V \to W$, and a proper vector $\mathbf{b} \in W$, the equation

$$f(\mathbf{x}) = \mathbf{b}$$

can be solved using the inhomogeneous system of linear equations that has the augmented matrix

$$\mathbf{T} = \left[\,{}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}} \;\middle|\; {}_{\mathrm{c}}\mathbf{b}\,\right]$$

If solutions exist and $\mathbf{x}_0$ is one of these solutions the whole solution set can be written as:

$$\mathbf{x}_0 + \ker(f)\,.$$

An inhomogeneous system of linear equation consisting of m equations in n unknowns, with the coefficient matrix $\mathbf{A}$ and the right-hand side $\mathbf{b}$ can in matrix form be written as

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,.$$

The map $f : \mathbb{L}^n \to \mathbb{L}^m$ given by

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

is linear. The linear equation $f(\mathbf{x}) = \mathbf{b}$ is thus equivalent to the considered system of linear equations. Thus we can see that the structural theorem for systems of linear equations (see eNote 6 Theorem 6.37) is nothing but a particular case of the general structural theorem for linear maps (Theorem 12.14).

12.6.3 Determining the Image Space

Above we have found that the image space for a linear map is a subspace of the codomain, see theorem 12.11. How can this subspace be delimited and characterized?

Again we consider the linear map $f : V \to W$ that is represented by the mapping matrix (12-29). Since the basis $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ for V is chosen we can write all the vectors in V at once:

$$\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4\,,$$

where we imagine that x_1, x_2, x_3 og x_4 run through all conceivable combinations of real values. But then all images in W of vectors in V can be written as:

$$\begin{aligned} f(\mathbf{x}) &= f(x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4) \\ &= x_1 f(\mathbf{a}_1) + x_2 f(\mathbf{a}_2) + x_3 f(\mathbf{a}_3) + x_4 f(\mathbf{a}_4)\,, \end{aligned}$$

where we have used L_1 og L_2, and where we continue to imagine that x_1, x_2, x_3 and x_4 run through all conceivable combinations of real values. But then:

$$f(V) = \operatorname{span}\{\, f(\mathbf{a}_1), f(\mathbf{a}_2), f(\mathbf{a}_3), f(\mathbf{a}_4)\,\}\,.$$

The image space is thus spanned by the images of the a-basis vectors! But then we can (according to Method 11.47Method in eNote 11) determine a basis for the image space by finding the leading 1's in the reduced row echelon form of

$$\begin{bmatrix} {}_c\mathbf{f}(\mathbf{a}_1) & {}_c\mathbf{f}(\mathbf{a}_2) & {}_c\mathbf{f}(\mathbf{a}_3) & {}_c\mathbf{f}(\mathbf{a}_4) \end{bmatrix}.$$

This is the mapping matrix for f with respect to the chosen bases

$${}_c\mathbf{F}_a = \begin{bmatrix} 1 & 3 & -1 & 8 \\ 2 & 0 & 4 & -2 \\ 1 & -1 & 3 & -4 \end{bmatrix}$$

that by Gauss-Jordan elimination is reduced to

$$\operatorname{rref}({}_c\mathbf{F}_a) = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To the two leading 1's in $\operatorname{rref}({}_c\mathbf{F}_a)$ correspond the first two columns in ${}_c\mathbf{F}_a$. We thus conclude:

Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be the two vectors in W determined by c-coordinates as:

$${}_c\mathbf{w}_1 = (1,2,1) \text{ and } {}_c\mathbf{w}_2 = (3,0,-1)\,.$$

Then $(\mathbf{w}_1, \mathbf{w}_2)$ is a basis for the image space $f(V)$.

|||| Method 12.25 Determination of the Image Space

In a vector space V a basis a is chosen, and in a vector space W a basis c is chosen. The image space $f(V)$ for a linear mapping $f : V \to W$ can be found from

$$\mathrm{rref}({}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}) \tag{12-31}$$

in the following way: If there is no leading 1 in the i'th column in (12-31) then $f(\mathbf{a}_i)$ is removed from the vector set $\big(f(\mathbf{a}_1), f(\mathbf{a}_2), \dots, f(\mathbf{a}_n)\big)$. After this thinning the vector set constitutes a basis for $f(V)$.

Since the number of leading 1's in (12-31) is equal to the number of basis vectors in the chosen basis for $f(V)$ it follows that

$$\dim(f(V)) = \rho\left({}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\right). \tag{12-32}$$

12.7 The Dimension Theorem

In the method of the preceding section 12.23 we found the following expression for the dimension of the kernel of a linear map $f : V \to W$:

$$\dim(\ker(f)) = \dim(V) - \rho\left({}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\right). \tag{12-33}$$

And in method 12.25 a corresponding expression for the image space $f(V)$:

$$\dim(f(V)) = \rho\left({}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\right). \tag{12-34}$$

By combining (12-33) and (12-34) a remarkably simple relationship between the domain, the kernel and the image space for a linear map is achieved:

|||| Theorem 12.26 The Dimension Theorem (or Rank-Nullity Theorem)

Let V and W be two finite dimensional vector spaces. For a linear map $f : V \to W$ we have:

$$\dim(V) = \dim(\ker(f)) + \dim(f(V)).$$

Here are some direct consequences of Theorem 12.26:

The image space for a linear map can never have a higher dimension than the domain.

If the kernel only consists of the **0**-vector, the image space *keeps* the dimension of the domain.

If the kernel has the dimension $p > 0$, then p dimensions *disappear* through the map.

|||| Exercise 12.27

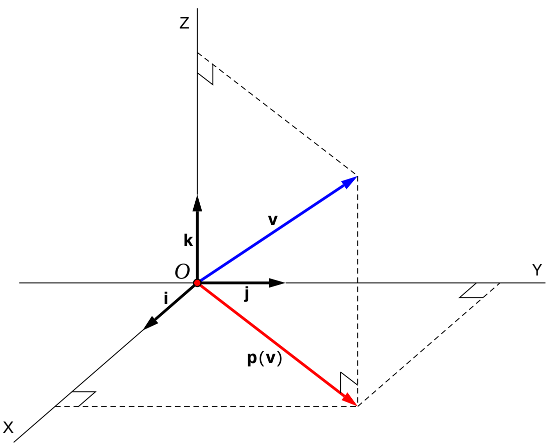
A linear map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ has, with respect to the standard basis for $\mathbb{R}^3$, the mapping matrix

$$ {}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix} . $$

It is stated that the kernel of f has the dimension 1. Find by mental calculation, a basis for $f(V)$.

|||| Exercise 12.28

In 3-space a standard $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$-coordinate system is given. The map p projects position vectors down into (x, y)-plane in space:



Projection down into the (X, Y)-plane

Show that p is linear and construct the mapping matrix ${}_{\mathrm{e}}\mathbf{P}_{\mathrm{e}}$ for p. Determine a basis for the kernel and the image space of the projection. Check that the Dimension Theorem is fulfilled.

12.8 Change in the Mapping Matrix when the Basis is Changed

In eNote 11 it is shown how the coordinates of a vector change when the basis for the vector space is changed, see method 11.40. We begin this section by repeating the most important points and showing two examples.

Assume that in V an a-basis $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ is given, and that a new b-basis $(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ is chosen in V. If a vector $\mathbf{x}$ has the b-coordinate vector ${}_{\mathrm{b}}\mathbf{x}$, then its a-coordinate vector can be calculated as the matrix vector-product

$$ {}_{\mathrm{a}}\mathbf{v} = {}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}}\, {}_{\mathrm{b}}\mathbf{v} \tag{12-35} $$

where the *change of basis matrix* ${}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}}$ is given by

$$ {}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}} = \left[\, {}_{\mathrm{a}}\mathbf{b}_1 \quad {}_{\mathrm{a}}\mathbf{b}_2 \quad \ldots \quad {}_{\mathrm{a}}\mathbf{b}_n \,\right]. \tag{12-36} $$

We now show two examples of the use of (12-36). In the first example the "new" coordinates are given following which the "old" are calculated. In the second example it is vice versa: the "old" are known, and the "new" are determined.

|||| Example 12.29 From New Coordinates to Old

In a 3-dimensional vector space V a basis $a = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is given, following which a new basis b is chosen consisting of the vectors

$$\mathbf{b}_1 = \mathbf{a}_1 - \mathbf{a}_3,\ \mathbf{b}_2 = 2\mathbf{a}_1 - 2\mathbf{a}_2 + \mathbf{a}_3 \ \text{ and } \ \mathbf{b}_3 = -3\mathbf{a}_1 + 3\mathbf{a}_2 - \mathbf{a}_3\,.$$

Problem: Determine the coordinate vector ${}_{\mathrm{a}}\mathbf{x}$ for $\mathbf{x} = \mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3\,.$

Solution: First we see that

$$ {}_{\mathrm{b}}\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \ \text{ and } \ {}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 3 \\ -1 & 1 & -1 \end{bmatrix}. \tag{12-37}$$

Then we get

$$ {}_{\mathrm{a}}\mathbf{x} = {}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}}\, {}_{\mathrm{b}}\mathbf{x} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 3 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix}. \tag{12-38}$$

|||| Example 12.30 From Old Coordinates to New

In a 2-dimensional vector space W a basis $c = (\mathbf{c}_1, \mathbf{c}_2)$ is given, following which a new basis d is chosen consisting of the vectors

$$\mathbf{d}_1 = 2\mathbf{c}_1 + \mathbf{c}_2 \ \text{ and } \ \mathbf{d}_2 = \mathbf{c}_1 + \mathbf{c}_2\,.$$

Problem: Determine the coordinate vector ${}_{\mathrm{d}}\mathbf{y}$ for $\mathbf{y} = 10\mathbf{c}_1 + 6\mathbf{c}_2\,.$

Solution: First we see that

$$ {}_{\mathrm{c}}\mathbf{y} = \begin{bmatrix} 10 \\ 6 \end{bmatrix} \ \text{ and } \ {}_{\mathrm{c}}\mathbf{M}_{\mathrm{d}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow {}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}} = ({}_{\mathrm{c}}\mathbf{M}_{\mathrm{d}})^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}. \tag{12-39}$$

Then we get

$$ {}_{\mathrm{d}}\mathbf{y} = {}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\, {}_{\mathrm{c}}\mathbf{y} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}. \tag{12-40}$$

We now continue to consider how a mapping matrix is changed when the basis for the domain or the codomain is changed.

For two vector spaces V and W with finite dimension the mapping matrix for a linear map $f : V \rightarrow W$ can only be constructed when a basis for V and a basis for W are chosen. By using the mapping matrix symbol ${}_c\mathbf{F}_a$ we show the foundation to be the pair of given bases a of V and c of W.

Often one wishes to change the basis of V or the basis of W. In the *first* case the coordinates for those vectors $\mathbf{x} \in V$ will change while the coordinates for their images $\mathbf{y} = f(\mathbf{x})$ are unchanged; in the *second* case it is the other way round with the $\mathbf{x}$ coordinates remaining unchanged while the image coordinates change. If the bases of both $V$and W are changed then the coordinates for both $\mathbf{x}$ and $\mathbf{y} = f(\mathbf{x})$ change.

In this section we construct methods for finding the new mapping matrix for f, when we change the basis for either the domain, the codomain or both. First we show how a vector's coordinates change when the basis for the domain is changed (as in detail in Method 11.40 in eNote 11.)

12.8.1 Change of Basis in the Domain

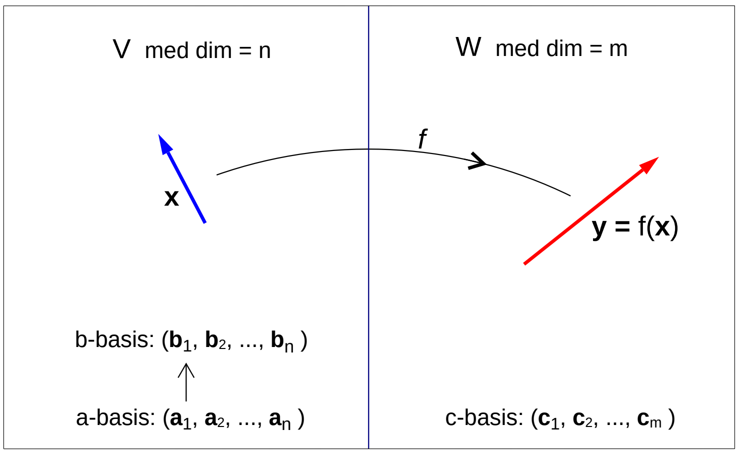


Figure 12.6: Linear map

In Figure 12.6 a linear map $f : V \to W$ is given that, with respect to basis a of V and basis c of W, has the mapping matrix ${}_c\mathbf{F}_a$. We change the basis for V from basis a to basis b. The mapping matrix for f now has the symbol ${}_c\mathbf{F}_b$. Let us find it. The equation

$$\mathbf{y} = f(\mathbf{x})$$

is translated into coordinates and rewritten as:

$${}_c\mathbf{y} = {}_c\mathbf{F}_a\, {}_a\mathbf{x} = {}_c\mathbf{F}_a \left({}_a\mathbf{M}_b\, {}_b\mathbf{x}\right) = \left({}_c\mathbf{F}_a\, {}_a\mathbf{M}_b\right) {}_b\mathbf{x}\,.$$

From this we deduce that the mapping matrix for f with respect to the basis b of V and basis c of W is formed by a matrix product:

$${}_c\mathbf{F}_b = {}_c\mathbf{F}_a\, {}_a\mathbf{M}_b\,. \tag{12-41}$$

|||| Example 12.31 Change of a Mapping Matrix

We consider the 3-dimensional vector space V that is treated in Example 12.29 and the 2-dimensional vector space W that is treated in Example 12.30. A linear map $f : V \to W$ is given by the mapping matrix:

$${}_c\mathbf{F}_a = \begin{bmatrix} 9 & 12 & 7 \\ 6 & 8 & 5 \end{bmatrix}.$$

Problem: Determine $\mathbf{y} = f(\mathbf{x})$ where $\mathbf{x} = \mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3$.

Solution: We try two different ways. 1) We use a-coordinates for $\mathbf{x}$ as found in (12-37):

$${}_c\mathbf{y} = {}_c\mathbf{F}_a\, {}_a\mathbf{x} = \begin{bmatrix} 9 & 12 & 7 \\ 6 & 8 & 5 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}.$$

2) We change the mapping matrix for f:

$${}_c\mathbf{F}_b = {}_c\mathbf{F}_a\, {}_a\mathbf{M}_b = \begin{bmatrix} 9 & 12 & 7 \\ 6 & 8 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 3 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then we can directly use the given b-coordinates for $\mathbf{x}$:

$${}_c\mathbf{y} = {}_c\mathbf{F}_b\, {}_b\mathbf{x} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}.$$

In either case we get $\mathbf{y} = 10\mathbf{c}_1 + 6\mathbf{c}_2$.

12.8.2 Change of Basis in the Codomain

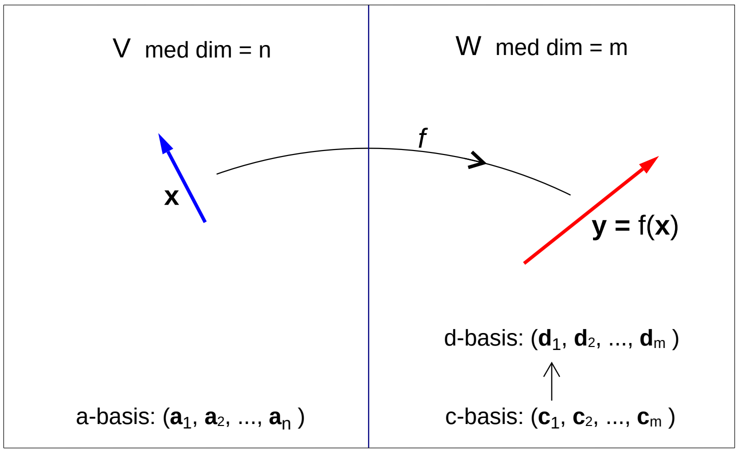


Figure 12.7: Linear map

In Figure 12.7 a linear map $f : V \rightarrow W$ is given that, with respect to the basis a of V and basis c of W has a mapping matrix ${}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}$. We change the basis for W from basis c to basis d . The mapping matrix for f now has the symbol ${}_{\mathrm{d}}\mathbf{F}_{\mathrm{a}}$. Let us find it. The equation

$$\mathbf{y} = f(\mathbf{x})$$

is translated into the matrix equation

$${}_{\mathrm{c}}\mathbf{y} = {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\,{}_{\mathrm{a}}\mathbf{x}$$

that is equivalent to

$${}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\,{}_{\mathrm{c}}\mathbf{y} = {}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\,({}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\,{}_{\mathrm{a}}\mathbf{x})$$

from which we get that

$${}_{\mathrm{d}}\mathbf{y} = ({}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\,{}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}})\,{}_{\mathrm{a}}\mathbf{x}\,.$$

From this we deduce that the mapping matrix for f with respect to the a-basis for V and the d-basis for W is formed by a matrix product:

$${}_{\mathrm{d}}\mathbf{F}_{\mathrm{a}} = {}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\,{}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\,. \tag{12-42}$$

|||| Example 12.32 Change of Mapping Matrix

We consider the 3-dimensional vector space V that is treated in Example 12.29 and the 2-dimensional vector space W that is treated in Example 12.30. A linear map $f : V \to W$ is given by the mapping matrix:

$$ {}_{c}\mathbf{F}_{a} = \begin{bmatrix} 9 & 12 & 7 \\ 6 & 8 & 5 \end{bmatrix}. $$

Problem: Given the vector $\mathbf{x} = -4\mathbf{a}_1 + 5\mathbf{a}_2 - 2\mathbf{a}_3$. Determine the image $\mathbf{y} = f(\mathbf{x})$ as a linear combination of $\mathbf{d}_1$ and $\mathbf{d}_2$.

Solution: We try two different ways.
1) We use the given mapping matrix:

$$ {}_{c}\mathbf{y} = {}_{c}\mathbf{F}_{a}\, {}_{a}\mathbf{x} = \begin{bmatrix} 9 & 12 & 7 \\ 6 & 8 & 5 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}. $$

And translate the result to d-coordinates using (12-40):

$$ {}_{d}\mathbf{y} = {}_{d}\mathbf{M}_{c}\, {}_{c}\mathbf{y} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} $$

2) We change the mapping matrix for f using (12-39):

$$ {}_{d}\mathbf{F}_{a} = {}_{d}\mathbf{M}_{c}\, {}_{c}\mathbf{F}_{a} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 9 & 12 & 7 \\ 6 & 8 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 \\ 3 & 4 & 3 \end{bmatrix}. $$

Then we can directly read the d-coordinates:

$$ {}_{d}\mathbf{y} = {}_{d}\mathbf{F}_{a}\, {}_{a}\mathbf{x} = \begin{bmatrix} 3 & 4 & 2 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}. $$

In both cases we get $\mathbf{y} = 4\mathbf{d}_1 + 2\mathbf{d}_2$.

12.8.3 Change of Basis in both the Domain and Codomain

In Figure 12.8 a linear map $f : V \to W$ is given that, with respect to the basis a for V and basis c for W, has the mapping matrix ${}_{c}\mathbf{F}_{a}$. We change the basis for V from basis a to basis b, and for W from basis c to basis d. The mapping matrix for f now has the

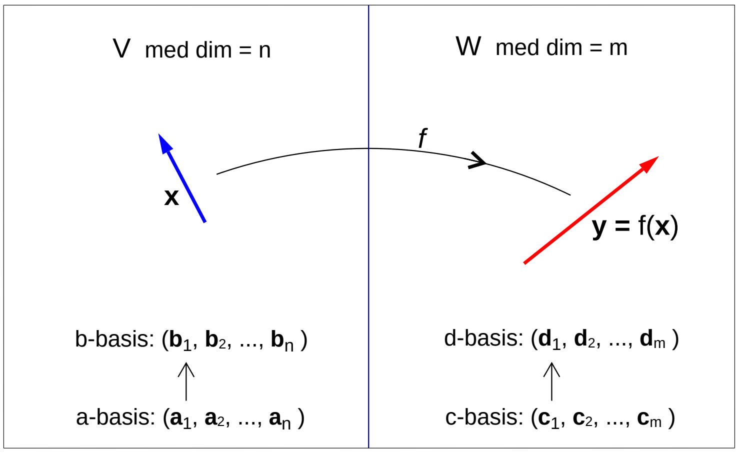


Figure 12.8: Linear map

symbol ${}_{\mathrm{d}}\mathbf{F}_{\mathrm{b}}$. Let us find it. The equation

$$y = f(\mathbf{x})$$

corresponds in coordinates to

$${}_{\mathrm{c}}\mathbf{y} = {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\,{}_{\mathrm{a}}\mathbf{x}$$

that is equivalent to

$${}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\,{}_{\mathrm{c}}\mathbf{y} = {}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\left({}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\left({}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}}\,{}_{\mathrm{b}}\mathbf{x}\right)\right)$$

from which we obtain

$${}_{\mathrm{d}}\mathbf{y} = \left({}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\,{}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\,{}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}}\right){}_{\mathrm{b}}\mathbf{x}\,.$$

From here we deduce that the mapping matrix for f with respect to b-basis of V and d-basis of W is formed by a matrix product:

$${}_{\mathrm{d}}\mathbf{F}_{\mathrm{b}} = {}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\,{}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\,{}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}}\,. \tag{12-43}$$

|||| Example 12.33 Change of Mapping Matrix

We consider the 3-dimensional vector space V that is treated in example 12.29, and the 2-dimensional vector space W that is treated in example 12.30. A linear map $f : V \to W$ is given by the mapping matrix:

$${}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}} = \begin{bmatrix} 9 & 12 & 7 \\ 6 & 8 & 5 \end{bmatrix}.$$

Problem: Given the vector $\mathbf{x} = \mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3$. Determine $\mathbf{y} = f(\mathbf{x})$ as a linear combination of $\mathbf{d}_1$ and $\mathbf{d}_2$.

Solution: We change the mapping matrix using (12-39) and (12-37):

$$ {}_{\mathrm{d}}\mathbf{F}_{\mathrm{b}} = {}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\,{}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\,{}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 9 & 12 & 7 \\ 6 & 8 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 3 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. $$

Then we can directly use the given b-coordinates and directly read the d-coordinates:

$$ {}_{\mathrm{d}}\mathbf{y} = {}_{\mathrm{d}}\mathbf{F}_{\mathrm{b}}\,{}_{\mathrm{b}}\mathbf{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}. $$

Conclusion: $\mathbf{y} = 4\mathbf{d}_1 + 2\mathbf{d}_2$.

The change of basis in this example turns out to be rather practical. With the new mapping matrix ${}_{\mathrm{d}}\mathbf{F}_{\mathrm{b}}$ it is much easier to calculate the image vector: You just add the first and the third coordinates of the given vector and keep the second coordinate!

12.8.4 Summary Concerning Change of Basis

We gather the results concerning change of basis in the subsections above in the following method:

|||| Method 12.34 Change of Mapping Matrix when Changing the Basis

For the vector space V are given a basis $a = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ and a new basis $b = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$. For the vector space W are given a basis $c = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m)$ and a new basis $d = (\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_m)$.

If f is a linear map $f : V \to W$ that, with respect to basis a of V and basis c of W, has the mapping matrix ${}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}$, then:

1. The mapping matrix for f with respect to basis b of $V$and basis c of W is

$$ {}_{\mathrm{c}}\mathbf{F}_{\mathrm{b}} = {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\, {}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}} \,. \tag{12-44} $$

2. The mapping matrix for f with respect to basis a of V and basis d of W is

$$ {}_{\mathrm{d}}\mathbf{F}_{\mathrm{a}} = \left({}_{\mathrm{c}}\mathbf{M}_{\mathrm{d}}\right)^{-1} {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}} = {}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\, {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}} \,. \tag{12-45} $$

3. The mapping matrix for f with respect to basis b of V and basis d of W is

$$ {}_{\mathrm{d}}\mathbf{F}_{\mathrm{b}} = \left({}_{\mathrm{c}}\mathbf{M}_{\mathrm{d}}\right)^{-1} {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\, {}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}} = {}_{\mathrm{d}}\mathbf{M}_{\mathrm{c}}\, {}_{\mathrm{c}}\mathbf{F}_{\mathrm{a}}\, {}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}} \,. \tag{12-46} $$

In the three formulas we have used the change of basis matrices:

$$ {}_{\mathrm{a}}\mathbf{M}_{\mathrm{b}} = \begin{bmatrix} {}_{\mathrm{a}}\mathbf{b}_1 & {}_{\mathrm{a}}\mathbf{b}_2 & \cdots & {}_{\mathrm{a}}\mathbf{b}_n \end{bmatrix} \text{ and } {}_{\mathrm{c}}\mathbf{M}_{\mathrm{d}} = \begin{bmatrix} {}_{\mathrm{c}}\mathbf{d}_1 & {}_{\mathrm{c}}\mathbf{d}_2 & \cdots & {}_{\mathrm{c}}\mathbf{d}_m \end{bmatrix} . $$