eNote 10

Geometric Vectors

The purpose of this note is to give an introduction to geometric vectors in the plane and 3-dimensional space, aiming at the introduction of a series of methods that manifest themselves in the general theory of vector spaces. The key concepts are linear independence and linear dependence, plus basis and coordinates. The note assumes knowledge of elementary geometry in the plane and 3-space, of systems of linear equations as described in eNote 6 and of matrix algebra as described in eNote 7.

Updated 25.09.21 David Brander

By a ***geometric vector*** in the plane $\mathbb{R}^2$ or Euclidean 3-space, $\mathbb{R}^3$, we understand a connected pair consisting of a *length* and a *direction*. Euclidean vectors are written as small bold letters, e.g. $\mathbf{v}$. A vector can be represented by an ***arrow*** with a given initial point and a terminal point. If the vector $\mathbf{v}$ is represented by an arrow with the initial point A and the terminal point B, we use the representation $\mathbf{v} = \overrightarrow{AB}$. All arrows with the same length and direction as the arrow from A to B, also represent $\mathbf{v}$.

Example 10.1 Parallel Displacement Using Vectors

Geometric vectors can be applied in ***parallel displacement*** in the plane and 3-space. In Figure 10.1 the line segment CD is constructed from the line segment AB as follows: all points of AB are displaced by the vector $\mathbf{u}$. In the same way the line segment EF emerges from AB by parallel displacement by the vector $\mathbf{v}$. $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF}$ but notice that e.g. $\overrightarrow{AB} \neq \overrightarrow{FE}$.

In what follows we assume that a *unit line segment* has been chosen, that is a line segment

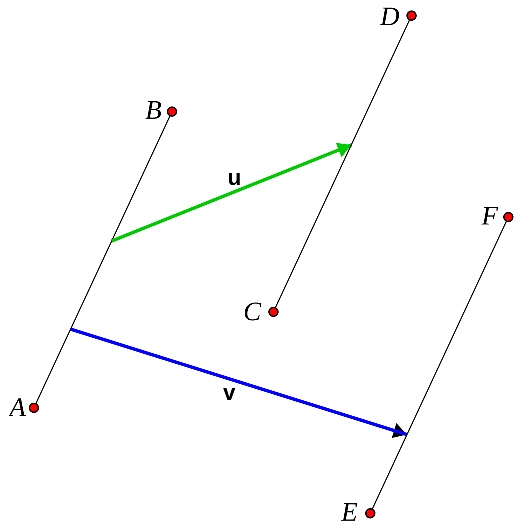


Figure 10.1: Parallel displacement by a vector

that has the length 1. By $|\mathbf{v}|$ we understand the length of the vector $\mathbf{v}$ as the proportionality factor with respect to the unit line segment, that is, a real number. All vectors of the same length as the unit line segment are called ***unit vectors***.

For practical reasons a particular vector that has length 0 and which has no direction is introduced. It is called the ***zero vector*** and is written $\mathbf{0}$. For every point A we put $\overrightarrow{AA} = \mathbf{0}$. Any vector that is not the zero vector is called a *proper* vector.

For every proper vector $\mathbf{v}$ we define the *opposite vector* $-\mathbf{v}$ as the vector that has the same length as $\mathbf{v}$, but the opposite direction. If $\mathbf{v} = \overrightarrow{AB}$, then $\overrightarrow{BA} = -\mathbf{v}$. For the zero vector we put $-\mathbf{0} = \mathbf{0}$.

It is often practical to use a common initial point when different vectors are to be represented by arrows. We choose a fixed point O which we term *the origen*, and consider those representations of the vectors that have O as the initial point. Vectors represented in this way are called ***position vectors***, because every given vector $\mathbf{v}$ has a unique point (position) P that satisifies $\mathbf{v} = \overrightarrow{OP}$. Conversely, every point Q corresponds to a unique vector $\mathbf{u}$ such that $\overrightarrow{OQ} = \mathbf{u}$.

By the *angle between two proper vectors in the plane* we understand the unique angle between their representations radiating from O, in the interval $[0;\pi]$. If a vector $\mathbf{v}$ in the plane is turned the angle $\pi/2$ counter-clockwise, a new vector emerges that is called $\mathbf{v}$'s *hat vector*, it is denoted $\widehat{\mathbf{v}}$.

By the *the angle between two proper vectors in 3-space* we understand the angle between their representations radiating from O in the plane that contains their representations.

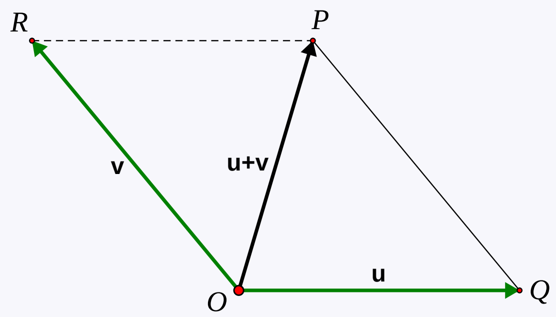
It makes good and useful sense "to add vectors", taking account of the vectors' lengths and directions. Therefore in the following we can introduce some arithmetic operations for geometric vectors. First it concerns two *linear operations*, addition of vectors and multiplication of a vector by a scalar (a real number). Later we will consider three ways of multiplying vectors, viz. the *dot product*, and for vectors in 3-space the *cross product* and the *scalar triple product*.

10.1 Addition and Multiplication by a Scalar

|||| Definition 10.2 Addition

Given two vectors in the plane or 3-space, $\mathbf{u}$ and $\mathbf{v}$. The sum $\mathbf{u}+\mathbf{v}$ is determined in the following way:

- We choose the origin O and mark the position vectors $\mathbf{u} = \overrightarrow{OQ}$ and $\mathbf{v} = \overrightarrow{OR}$.
- By parallel displacement of the line segments OR by $\mathbf{u}$ the line segment QP is constructed.
- $\overrightarrow{OP}$ is then the position vector for the sum of $\mathbf{u}$ and $\mathbf{v}$, in short $\mathbf{u}+\mathbf{v} = \overrightarrow{OP}$.



In physics you talk about the "parallelogram of forces": If the object O is influenced by the forces $\mathbf{u}$ and $\mathbf{v}$, the *resulting force* can be determined as the vector sum $\mathbf{u} + \mathbf{v}$, the direction of which gives the direction of the resulting force, and the length of which gives the magnitude of the resulting force. If in particular $\mathbf{u}$ and $\mathbf{v}$ are of the same length, but have opposite directions, the resulting force is equal to the $\mathbf{0}$-vector.

We then introduce multiplication of a vector by a scalar:

|||| Definition 10.3 Multiplication by a Scalar

Given a vector $\mathbf{v}$ in the plane or 3-space and a scalar k. If $\mathbf{v} = \mathbf{0}$, we have $k\mathbf{v} = \mathbf{v}k = \mathbf{0}$. Otherwise by the product $k\mathbf{v}$ the following is understood:

- If $k > 0$, then $k\mathbf{v} = \mathbf{v}k$ is the vector that has the same direction as $\mathbf{v}$ and which is k times as long as $\mathbf{v}$.
- If $k = 0$, then $k\mathbf{v} = \mathbf{0}$.
- If $k < 0$, then $k\mathbf{v} = \mathbf{v}k$ is the vector that has the *opposite direction* of $\mathbf{v}$ and which is $-k = |\,k\,|$ as long as $\mathbf{v}$.

|||| Example 10.4 Multiplication by a Scalar

A given vector $\mathbf{v}$ is multiplied by -1 and 2, respectively:

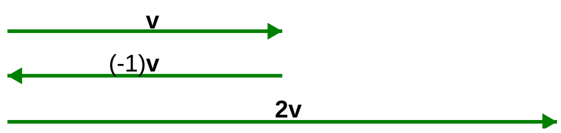


Figure: Multiplication of vector by -1 and 2

It follows immediately from the defintion 10.3 that multiplication of a vector by -1 gives the vector's opposite vector, in short

$$(-1)\mathbf{u} = -\mathbf{u}\,.$$

Thus we use the following way of writing

$$(-5)\mathbf{v} = -(5\mathbf{v}) = -5\mathbf{v}\,.$$

From the definition 10.3 the *zero rule* follows immediately for geometric vectors:

$$k\mathbf{v} = \mathbf{0} \Leftrightarrow k = 0 \;\text{ or }\; \mathbf{v} = \mathbf{0}\,.$$

In the following example it is shown that multiplication of an arbitrary vector by an arbitrary scalar can be performed by a genuine compasses and ruler construction.

|||| Example 10.5 Geometrical Multiplication

Given a vector $\mathbf{a}$ and a line segment of length k, we wish to construct the vector $k\mathbf{a}$.

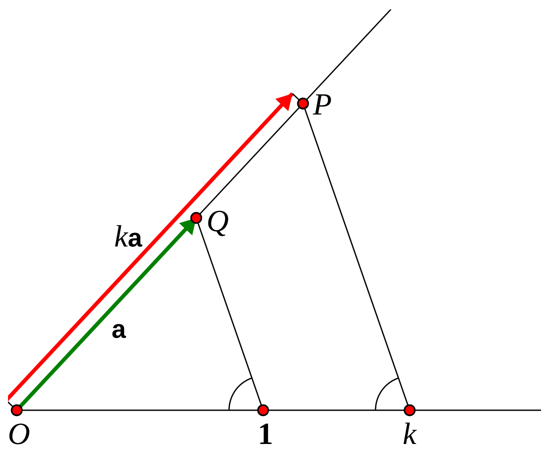


Figure: Multiplication of a vector by an arbitrary scalar

First the position vector $\overrightarrow{OQ} = \mathbf{a}$ is marked. Then with O as the initial point we draw a line which is used as a ruler and which is not parallel to $\mathbf{a}$, and where the numbers 1 and k are marked. The triangle OkP is drawn so it is congruent with the triangle $O1Q$. Since the two triangles are similar it must be true that $k\mathbf{a} = \overrightarrow{OP}$.

|||| Exercise 10.6

Given two parallel vectors **a** and **b** and a ruler line. How can you using a pair of compasses and the ruler line construct a line segment of the length k given that $\mathbf{b} = k\mathbf{a}$.

|||| Exercise 10.7

Given the proper vector **v** and a ruler line. Draw the vector $\frac{1}{|\mathbf{v}|}\,\mathbf{v}$.

Parametric representations for straight lines in the plane or 3-space are written using proper vectors. Below we first give an example of a line through the origin and then an example of a line not passing through the origin.

|||| Example 10.8 Parametric Representation of a Straight Line

Given a straight line l through the origin, we wish to write the points on the line using a ***parametric representation***:

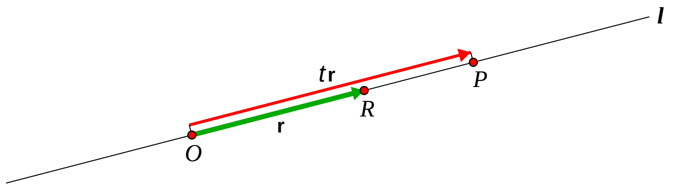


Figure: Parametric representation for a line through the origin

A point R on l different from the origin is chosen. The vector $\mathbf{r} = \overrightarrow{OR}$ is called a ***direction vector*** for l. For every point P on l corresponds exactly one real number t that satisfies $\overrightarrow{OP} = t\mathbf{r}$. Conversely, to every real number t corresponds exactly one point P on l so that $\overrightarrow{OP} = t\mathbf{r}$. As t traverses the real numbers from -∞ to +∞, P will traverse all of l in the direction determined by **r**. Then

$$\{\, P \mid \overrightarrow{OP} = t\mathbf{r} \text{ where } t \in \mathbb{R} \,\}$$

is a parametric representation of l.

|||| Example 10.9 Parametric Representation of a Straight Line

The line m does not go through the origin. We wish to describe the points on m by use of a parametric representation:

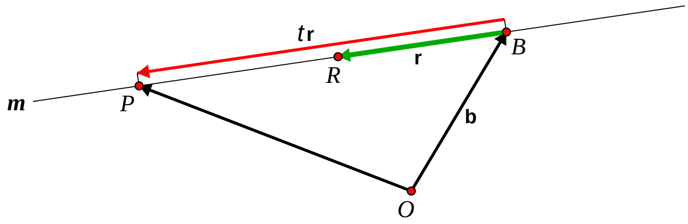


Figure: Parametric representation of a line

First an *initial point* B on m is chosen, and we put $\mathbf{b} = \overrightarrow{OB}$. A point $R \in m$ different from B is chosen. The vector $\mathbf{r} = \overrightarrow{BR}$ is then a directional vector for m. To every point P on m corresponds exactly one real number t that fulfils $\overrightarrow{OP} = \mathbf{b} + t\mathbf{r}$. Conversely, to every number t exactly one point P on m corresponds so that $\overrightarrow{OP} = \mathbf{b} + t\mathbf{r}$. When t traverses the real numbers from -∞ to +∞, P will traverse all of m in the direction determined by $\mathbf{r}$. Then

$$\{\, P \mid \overrightarrow{OP} = \mathbf{b} + t\mathbf{r} \text{ where } t \in \mathbb{R} \,\}$$

is a parametric representation for m.

Parametric representations can also be used for the description of line segments. This is the subject of the following exercise.

|||| Exercise 10.10

Consider the situation in example 10.9. Draw the oriented line segment with the parametric representation

$$\{\, P \mid \overrightarrow{OP} = \mathbf{b} + t\mathbf{r},\ \text{where } t \in [\,-1; 2\,] \,\}.$$

|||| Exercise 10.11

Given two (different) points A and B. Describe with a parametric representation the oriented line segment from A to B.

We will need more advanced arithmetic rules for addition of geometric vectors and multiplication of geometric vectors by scalars than the ones we have given in the examples above. These are sketched in the following theorem and afterwards we will discuss examples of how they can be justified on the basis of already defined arithmetic operations and theorems known from elementary geometry.

|||| Theorem 10.12 Arithmetic Rules

For arbitrary geometric vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and for arbitrary real numbers k_1 and k_2 the following arithmetic rules are valid:

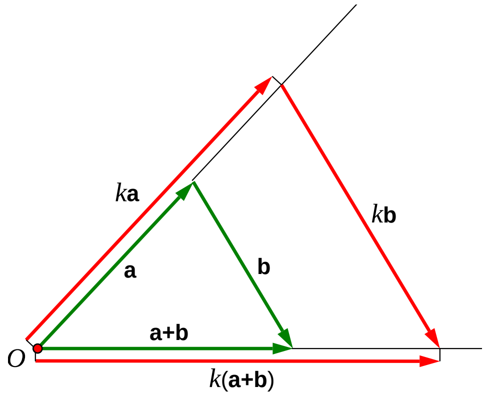
1.	$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	Addition is commutative
2.	$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$	Addition is associative
3.	$\mathbf{u}+\mathbf{0}=\mathbf{u}$	The zero vector is neutral for addition
4.	$\mathbf{u}+(-\mathbf{u})=\mathbf{0}$	The sum of a vector and its opposite is $\mathbf{0}$
5.	$k_1(k_2\mathbf{u})=(k_1k_2)\mathbf{u}$	Scalar multiplication is associative
6.	$(k_1+k_2)\mathbf{u}=k_1\mathbf{u}+k_2\mathbf{u}$	The distributive rules apply
7.	$k_1(\mathbf{u}+\mathbf{v})=k_1\mathbf{u}+k_1\mathbf{v}$	(The distributive rules apply)
8.	$1\mathbf{u}=\mathbf{u}$	The scalar 1 is neutral in the product with vectors

The arithmetic rules in Theorem 10.12 can be illustrated and proven using geometric constructions. Let us as an example take the first rule, the commutative rule. Here we just have to look at the figure in the definition 10.2, where $\mathbf{u}+\mathbf{v}$ is constructed. If we construct $\mathbf{v}+\mathbf{u}$, we will displace the line segment OQ with v and consider the emerging line segment RP_2. It must be true that the parallelogram $OQPR$ is identical to the parallelogram OQP_2R and hence $P_2=P$ and $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

In the following two exercises the reader is asked to explain two of the other arithmetic rules.

|||| Exercise 10.13

Explain using the diagram the arithmetic rule $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$.



|||| Exercise 10.14

Draw a figure that illustrates the rule $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.

For a given vector $\mathbf{u}$ it is obvious that the opposite vector $-\mathbf{u}$ is the only vector that satisfies the equation $\mathbf{u}+\mathbf{x} = \mathbf{0}$. For two arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$ it is also obvious that exactly one vector exists that satisfies the equation $\mathbf{u}+\mathbf{x} = \mathbf{v}$, viz. the vector $\mathbf{x} = \mathbf{v}+(-\mathbf{u})$ which is illustrated in Figure 10.2.

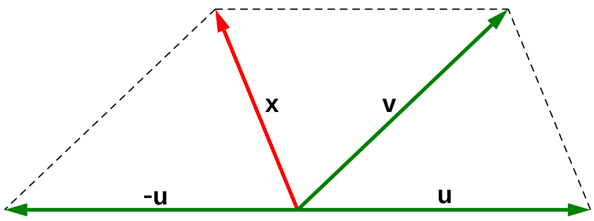


Figure 10.2: Opposite of a vector

Therefore we can introduce *subtraction of vectors* as a variation of addition like this:

|||| Definition 10.15 Subtraction

By the difference of two vectors $\mathbf{v}$ and $\mathbf{u}$ we understand the vector

$$\mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u})\,. \tag{10-1}$$

It is not necessary to introduce a formal definition of *division* of a vector by a scalar, we consider this as a rewriting of multiplication by a scalar:

$$\text{Division by a scalar :} \qquad \frac{\mathbf{v}}{k} = \frac{1}{k} \cdot \mathbf{v}\ ;\ k \neq 0$$

10.2 Linear Combinations

A point about the arithmetic rule $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ from the theorem 10.12 is that parentheses can be left out in the process of adding a series of vectors, since it has no consequences for the resulting vector in what order the vectors are added. This is the background for *linear combinations* where sets of vectors are multiplied by scalars and thereafter written as a sum.

|||| Definition 10.16 Linear Combination

When the real numbers $k_1, k_2, \ldots, k_n$ are given and in the plane or 3-space the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ then the sum

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$$

is called a ***linear combination*** of the n given vectors.

If all the coefficients $k_1, \cdots, k_n$ are equal to 0, the linear combination is called *improper*, or *trivial*, but if at the least one of the coefficients is different from 0, it is *proper*, or *non-trivial*.

|||| Example 10.17 Construction of a Linear Combination

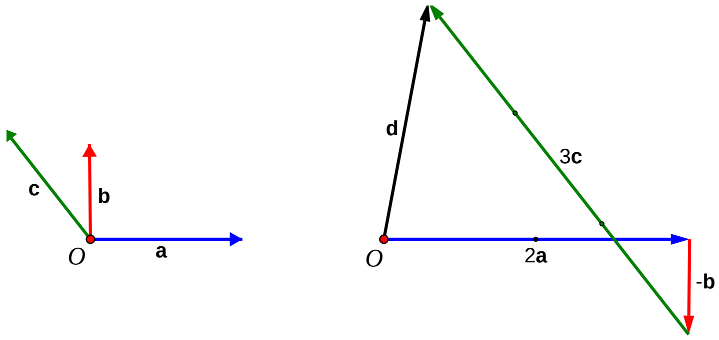


Figure: Construction of a linear combination

In the diagram, to the left the vectors **a**, **b** and **c** are drawn. On the figure to the right we have constructed the linear combination $\mathbf{d} = 2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.

|||| Exercise 10.18

There are given in the plane the vectors **u**, **v**, **s** and **t**, plus the parallelogram A, see diagram.

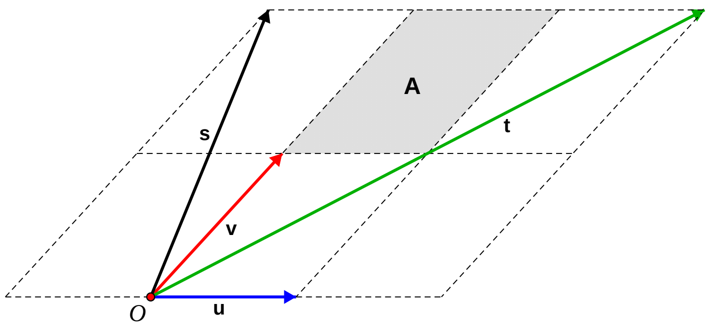


Figure: Linear combinations

1. Write **s** as a linear combination of **u** og **v**.
2. Show that **v** can be expressed by the linear combination $\mathbf{v} = \frac{1}{3}\mathbf{s} + \frac{1}{6}\mathbf{t}$.
3. Draw the linear combination $\mathbf{s} + 3\mathbf{u} - \mathbf{v}$.
4. Determine real numbers a, b, c and d such that A can be described by the *parametric*

representation

$$A = \{ P \mid \overrightarrow{OP} = x\mathbf{u} + y\mathbf{v} \text{ with } x \in [a;b] \text{ and } y \in [c;d] \}.$$

10.3 Linear Dependence and Linear Independence

If two vectors have representations on the same straight line, one says that they are *linearly dependent*. It is evident that two proper vectors are linearly dependent if they are parallel; otherwise they are *linearly independent*. We can formulate it as follows: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if the one can be obtained from the other by multiplication by a scalar different from 0, if e.g. there exists a number $k \neq 0$ such that

$$\mathbf{v} = k\mathbf{u}.$$

We wish to generalize this original meaning of the concepts of linear dependence and independence such that the concepts can be used for an arbitrary set of vectors.

|||| Definition 10.19 Linear Dependence and Independence

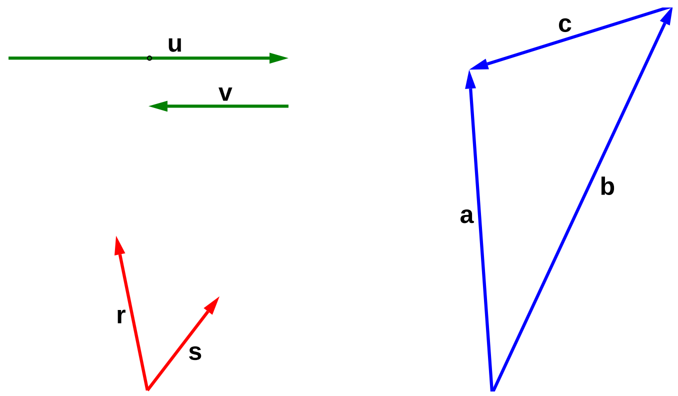
A set of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ where $n \geq 2$,is called ***linearly dependent*** if at least one of the vectors can be written as a linear combination of the others.

If none of the vectors can be written as a linear combination of the others, the set is called ***linearly independent***.

NB: A set that only consists of one vector is called linearly dependent if the vector is the $\mathbf{0}$-vector, otherwise linearly independent.

|||| Example 10.20 Linearly Dependent and Linearly Independent Sets of Vectors

In the plane are given three sets of vectors $(\mathbf{u}, \mathbf{v})$, $(\mathbf{r}, \mathbf{s})$ and $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, as shown.



The set $(\mathbf{u}, \mathbf{v})$ is linearly dependent since for this example we have

$$\mathbf{u} = -2\mathbf{v}\,.$$

Also the set $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is linearly dependent, since e.g.

$$\mathbf{b} = \mathbf{a} - \mathbf{c}\,.$$

Only the set $(\mathbf{r}, \mathbf{s})$ is linearly independent.

|||| Exercise 10.21

Explain that three vectors in 3-space are linearly dependent if and only if they have representations lying in the same plane. What are the conditions three vectors must fulfill in order to be linearly independent?

|||| Exercise 10.22

Consider (intuitively) what is the maximum number of vectors a set of vectors in the plane can comprise, if the set is to be linearly independent. The same question in 3-space.

When investigate whether or not a given set of vectors is linearly independent or linearly dependent, the definition 10.19 does not give a practical procedure. It might be easier to use the theorem that follows below. This theorem is based on the fact that a set of vectors is linearly dependent if and only if the **0**-vector can be written as a proper linear combination of the vectors. Assume – as a prerequisite to the theorem – that the set $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is linearly dependent because

$$\mathbf{c} = 2\mathbf{a} - 3\mathbf{b}.$$

Then the **0**-vector can be written as the proper linear combination

$$2\mathbf{a} - 3\mathbf{b} - \mathbf{c} = \mathbf{0}\,.$$

Conversely assume that the **0**-vector is a proper linear combination of the vectors $\mathbf{u}, \mathbf{v}$ og $\mathbf{w}$ like this:

$$2\mathbf{u} - 2\mathbf{v} + 3\mathbf{w} = \mathbf{0}\,.$$

Then we have (e.g.) that

$$\mathbf{w} = -\frac{2}{3}\mathbf{u} + \frac{2}{3}\mathbf{v}$$

and hence the vectors are linearly dependent.

|||| Theorem 10.23 Linear Independence

Let $k_1, k_2, \ldots, k_n$ be real numbers. That the set of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is linearly independent implies that the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{0} \qquad (10\text{-}2)$$

is only satisfied when all the coefficients $k_1, k_2, \ldots, k_n$ are equal to 0.

|||| Proof

Assume that the set $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is linearly dependent, and let v_i be a vector that can be written as a linear combination of the other vectors. We reorder (if necessary) the set, such that $i = 1$, following which $\mathbf{v}_1$ can be written in the form

$$\mathbf{v}_1 = k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n \;\Leftrightarrow\; \mathbf{v}_1 - k_2\mathbf{v}_2 - \cdots - k_n\mathbf{v}_n = \mathbf{0}\,. \qquad (10\text{-}3)$$

The **0**-vector is hereby written in the form (10-2), in which not all the coefficients are 0, because the coefficient to $\mathbf{v}_1$ is 1.

Conversely, assume that the set is written in the form (10-2), and let $k_i \neq 0$. We reorder (if necessary) the set such that $i = 1$ following which we have

$$k_1\mathbf{v}_1 = -k_2\mathbf{v}_2 - \cdots - k_n\mathbf{v}_n \;\Leftrightarrow\; \mathbf{v}_1 = -\frac{k_2}{k_1}\mathbf{v}_2 - \cdots - \frac{k_n}{k_1}\mathbf{v}_n\,. \tag{10-4}$$

From this we see that the set is linearly independent.

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|||| Example 10.24 Linearly Independent Set

Every set of vectors containing the zero vector is linearly dependent. Consider e.g. the set $(\mathbf{u}, \mathbf{v}, \mathbf{0}, \mathbf{w})$. It is obvious that the zero-vector can be written as the other three vectors:

$$\mathbf{0} = 0\mathbf{u} + 0\mathbf{v} + 0\mathbf{w}\,,$$

where the zerovector is written as a linear combination of the other vectors in the set.

Parametric representations for planes in 3-space is written using two linearly independent vectors. Below we first give an example of a plane through the origin, then an example of a plane that does not contain the origin.

|||| Example 10.25 Parametic Representation for a Plane

Given a plane in 3-space through the origin as shown. We wish to describe the points in the plane by a ***parametric representation***.

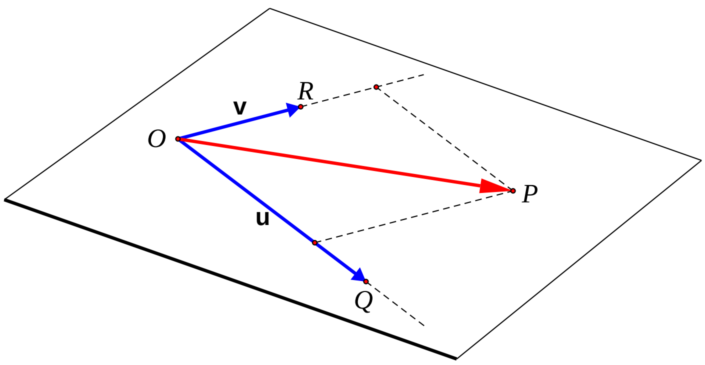


Figure: A plane in 3-space through the origin

In the given plane we choose two points Q and R, both not the origin, and that do not lie on a common line through the origin. The vectors $\mathbf{u} = \overrightarrow{OQ}$ and $\mathbf{v} = \overrightarrow{OR}$ will then be linearly independent, and are called ***direction vectors*** of the plane. For every point P in the plane we have exactly one pair of numbers (s,t) such that $\overrightarrow{OP} = s\mathbf{u} + t\mathbf{v}$. Conversely, for every pair of real numbers (s,t) exists exactly one point P in the plane that satisfies $\overrightarrow{OP} = s\mathbf{u} + t\mathbf{v}$. Then

$$\{P \mid \overrightarrow{OP} = s\mathbf{u} + t\mathbf{v}\,;\ (s,t) \in \mathbb{R}^2\}$$

is a parametric representation of the given plane.

|||| Example 10.26 Parametric Representation for a Plane

A plane in 3-space does not contain the origin. We wish to describe the plane using a parametric representation.

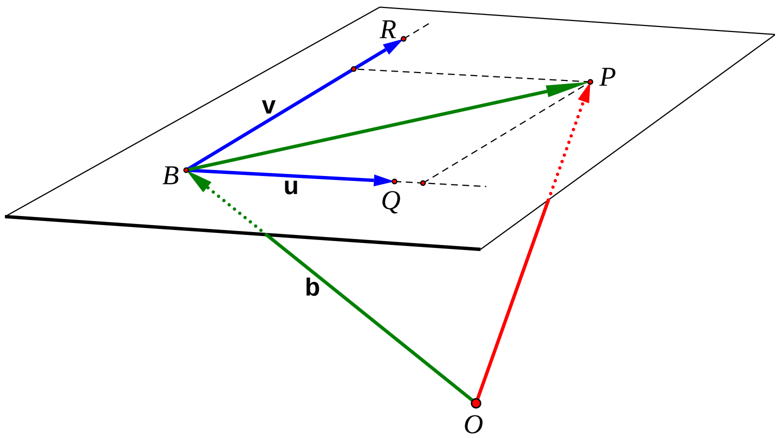


Figure: A plane in 3-space

First we choose an initial point B in the plane, and we put $\mathbf{b} = \overrightarrow{OB}$. Then we choose two linearly independent direction vectors $\mathbf{u} = \overrightarrow{BQ}$ and $\mathbf{v} = \overrightarrow{BR}$ where Q and R belong to the plane. To every point P in the plane corresponds exactly one pair of real numbers (s,t), such that

$$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP} = \mathbf{b} + s\mathbf{u} + t\mathbf{v}\,.$$

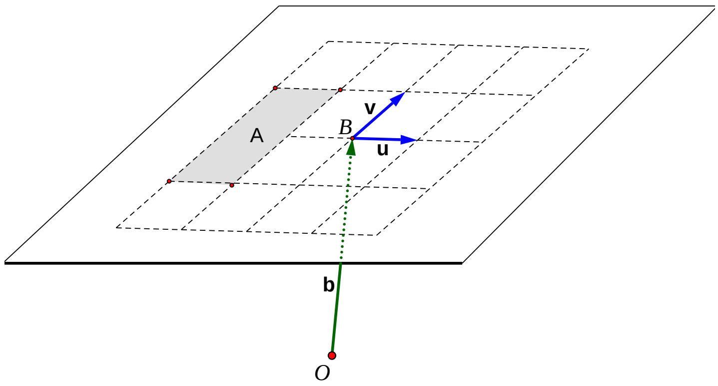
Conversely, to every pair of real numbers (s,t) corresponds exactly one point P in the plane as given by this vector equation. Then

$$\{P \mid \overrightarrow{OP} = \mathbf{b} + s\mathbf{u} + t\mathbf{v}\,;\ (s,t) \in \mathbb{R}^2\}$$

is a parametric representation for the given plane.

|||| Exercise 10.27

Give a parametric representation for the parallelogram A lying in the plane shown:



10.4 The Standard Bases in the Plane and Space

In *analytic geometry* one shows how numbers and equations can describe geometric objects and phenomena including vectors. Here the concept of coordinates is decisive. It is about how we determine the position of the geometric objects in 3-space and relative to one another using numbers and tuples of numbers. To do so we need to choose a number of vectors which we appoint as ***basis vectors***. The basis vectors ***are ordered***, that is they are given a distinct order, and thus they constitute a ***basis***. When a basis is given all the vectors can be described using coordinates, which we assemble in so called coordinate vectors. How this whole procedure takes place we first explain for the standard bases in the plane and 3-space. Later we show that often it is useful to use other bases than the standard bases and how the coordinates of a vector in different bases are related.

|||| Definition 10.28 Standard Basis in the Plane

By a ***standard basis*** or an ***ordinary basis*** for the geometric vectors in the plane we understand an ordered set of two vectors $(\mathbf{i},\mathbf{j})$ that satisfies:

- $\mathbf{i}$ has the length 1 .
- $\mathbf{j} = \widehat{\mathbf{i}}$ (that is $\mathbf{j}$ is the hat vector of $\mathbf{i}$).

By a ***standard coordinate system in the plane*** we understand a standard basis $(\mathbf{i},\mathbf{j})$ together with a chosen the origin O . The coordinate system is written $(O,\mathbf{i},\mathbf{j})$. By the x-axis and the y-axis we understand oriented number axes through O that are parallel to $\mathbf{i}$ and $\mathbf{j}$,respectively.

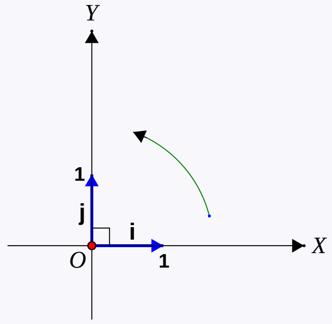


Figure: Standard coordinate system in the plane

|||| Theorem 10.29 Coordinates of a Vector

If $e = (\mathbf{i},\mathbf{j})$ is a standard basis, then any vector $\mathbf{v}$ in the plane can be written in exactly one way as a linear combination of $\mathbf{i}$ and $\mathbf{j}$:

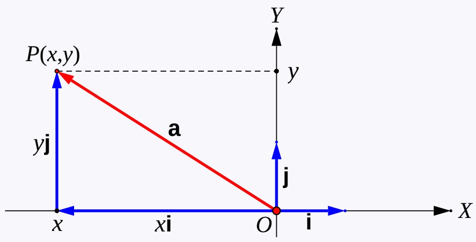
$$\mathbf{v} = x\mathbf{i} + y\mathbf{j}.$$

The coefficients x and y in the linear combination are called $\mathbf{v}$'s *coordinates with respect to the basis e*, or for short $\mathbf{v}$'s e-coordinates, and they are assembled in a ***coordinate vector*** as follows:

$${}_{e}\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

|||| Definition 10.30 The Coordinates of a Point

Let P be any point in the plane, and let $(O, \mathbf{i}, \mathbf{j})$ be a standard coordinate system in the plane. By the coordinates of P with respect to the coordinate system we understand the coordinates of the position vector $\overrightarrow{OP}$ with respect to the standard basis $(\mathbf{i}, \mathbf{j})$.



The introduction of a standard basis and the coordinates of a vector in 3-space is a simple extension of the corresponding coordinates in the plane.

|||| Definition 10.31 Standard Basis in Space

By a ***standard basis*** or an ***ordinary basis*** for the geometric vectors in 3-space we understand an ordered set of three vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ that satisfies:

- $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ all have the length 1.
- $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are pairwise orthogonal.
- When $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are drawn from a chosen point, and we view $\mathbf{i}$ and $\mathbf{j}$ from the endpoint of $\mathbf{k}$, then $\mathbf{i}$ turns into $\mathbf{j}$, when $\mathbf{i}$ is turned by the angle $\frac{\pi}{2}$ counter-clockwise.

By n ***standard coordinate system in 3-space*** we understand a standard basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ together with a chosen the origin O . The coordinate system is written $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$. By the x-axis, the y-axis and the z-axis we understand oriented number axes through the origin that are parallel to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively.

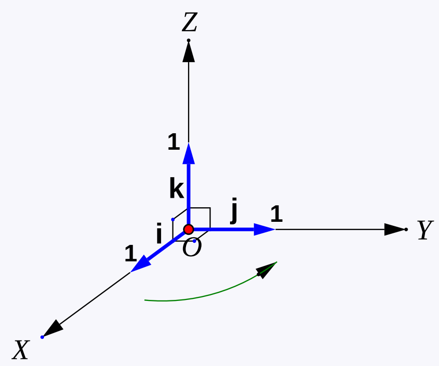


Figure: A standard coordinate system in 3-space.

|||| Theorem 10.32 The Coordinates of a Vector

When $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is a basis, every vector $\mathbf{v}$ in 3-space can be written in exactly one way as a linear combination of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

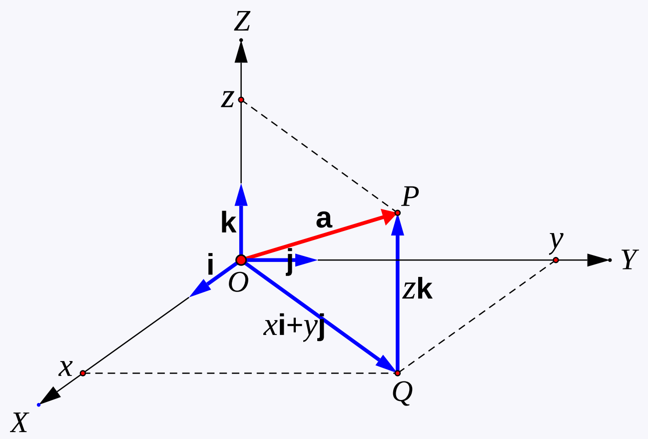
$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

The coefficients x, y and z in the linear combination are called $\mathbf{v}$'s *coordinates with respect to the basis*, or in short $\mathbf{v}$'s e-coordinates, and they are assembled in a ***coordinate vectodr*** as follows:

$$_{\mathrm{e}}\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

|||| Definition 10.33 The Coordinates of a Point

Let P be an arbitrary point in 3-space, and let $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ be a standard coordinate system in 3-space. By the coordinates of P with respect to the coordinate system we understand the coordinates of the position vector $\overrightarrow{OP}$ with respect to the standard basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$.



10.5 Arbitrary Bases for the Plane and Space

If two linearly independent vectors in the plane are given, it is possible to write every other vector as a linear combination of the two given vectors. In Figure 10.3 we consider e.g. the two linearly independent vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ plus two other vectors $\mathbf{u}$ and $\mathbf{v}$: in the plane

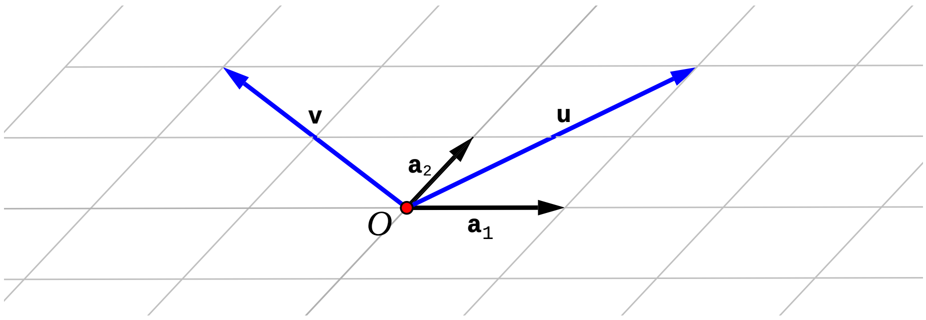


Figure 10.3: Coordinate system in the plane with basis $(\mathbf{a}_1, \mathbf{a}_2)$

We see that $\mathbf{u} = 1\mathbf{a}_1 + 2\mathbf{a}_2$ and $\mathbf{v} = -2\mathbf{a}_1 + 2\mathbf{a}_2$. These linear combinations are unique because $\mathbf{u}$ and $\mathbf{v}$ cannot be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$ using any other coefficients than those written. Similarly, any other vector in the plane can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, and our term for this is that the two vectors ***span*** the whole plane.

This makes it possible to generalise the concept of a basis. Instead of a standard basis we can choose to use the set of vectors $(\mathbf{a}_1, \mathbf{a}_2)$ as a basis for the vectors in the plane. If we call the basis a , we say that the coefficients in the linear combinations above are *coordinates* for $\mathbf{u}$ and $\mathbf{v}$, respectively, *with respect to a basis a*, which is written like this:

$$ {}_{\mathrm{a}}\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } {}_{\mathrm{a}}\mathbf{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} . \tag{10-5} $$

For the set of geometric vectors in 3-space we proceed in a similar way. Given three linearly independent vectors, then every vector in 3-space can be written as a unique-linear combination of the three given vectors. They *span* all of 3-space. Therefore we can choose three vectors as a basis for the vectors in 3-space and express an arbitrary vector in 3-space by coordinates with respect to this basis. A method for determination of the coordinates is shown in Figure 10.4, where we are given an a-basis $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ plus

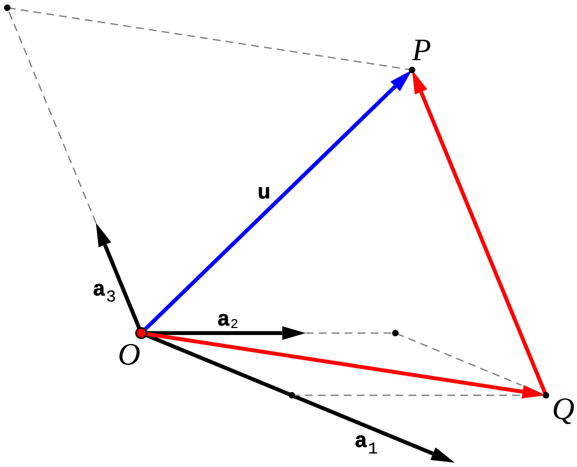


Figure 10.4: Coordinate system with basis $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$

an arbitrary vector $\mathbf{u}$. Through the endpoint P for $\mathbf{u}$ a line parallel to $\mathbf{a}_3$ is drawn, and the point of intersection of this line and the plane that contains $\mathbf{a}_1$ and $\mathbf{a}_2$, is denoted Q. Two numbers k_1 and k_2 exist such that $\overrightarrow{OQ} = k_1\mathbf{a}_1 + k_2\mathbf{a}_2$ because $(\mathbf{a}_1, \mathbf{a}_2)$ constitutes a basis in the plane that contains $\mathbf{a}_1$ and $\mathbf{a}_2$. Furthermore there exists a number k_3 such that $\overrightarrow{QP} = k_3\mathbf{a}_3$ since $\overrightarrow{QP}$ and $\mathbf{a}_3$ are parallel. But then we have

$$\mathbf{u} = \overrightarrow{OQ} + \overrightarrow{QP} = k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + k_3\mathbf{a}_3.$$

$\mathbf{u}$ thereby has the coordinate set (k_1, k_2, k_3) with respect to basis a.

|||| Example 10.34 Coordinates with Respect to an Arbitrary Basis

In 3-space three linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ are given as shown in the Figure.

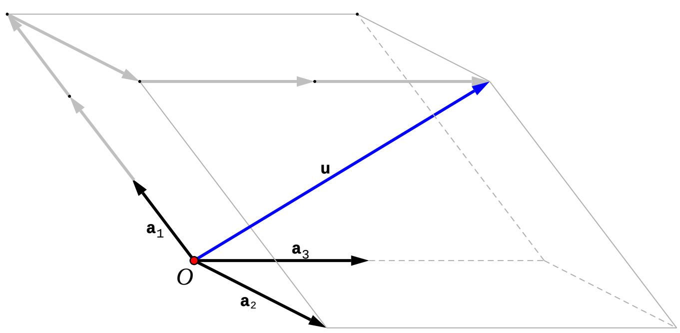


Figure: Coordinate system with basis $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$

Since $\mathbf{u}$ can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ in the following way

$$\mathbf{u} = 3\mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3\,, \tag{10-6}$$

then $\mathbf{u}$ has the coordinates $(3, 1, 2)$ with respect to the basis a given by $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ which we write in short as

$$_{\mathrm{a}}\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}. \tag{10-7}$$

We gather the above considerations about arbitrary bases in the following more formal definition:

|||| Definition 10.35 The Coordinates of a Vector with Respect to a Basis

- By a basis a for the geometric vectors in the plane we will understand an arbitrary ordered set of two linear independent vectors $(\mathbf{a}_1, \mathbf{a}_2)$. Let an arbitrary vector $\mathbf{u}$ be determined by the linear combination $\mathbf{u} = x\mathbf{a}_1 + y\mathbf{a}_2$. The coefficients x and y are called $\mathbf{u}$'s *coordinates with respect to the basis* a, or shorter $\mathbf{u}$'s a-coordinates, and they are assembled in a *coordinate vector* as follows:

$$_{\mathrm{a}}\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{10-8}$$

- By a basis b for the geometric vectors in 3-space we understand an arbitrary ordered set of three linear independent vectors $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$. Let an arbitrary vector $\mathbf{v}$ be determined by the linearly combinationen $\mathbf{v} = x\mathbf{b}_1 + y\mathbf{b}_2 + z\mathbf{b}_3$. The coefficients x, y and z are called $\mathbf{v}$'s *coordinates with respect to the basis* b, or shorter $\mathbf{v}$'s b-coordinates, and they are assembled in a *coordinate vector* as follows:

$$_{\mathrm{b}}\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{10-9}$$

The coordinate set of a given vector will change when we change the basis. This crucial point is the subject of the following exercise.

|||| Exercise 10.36

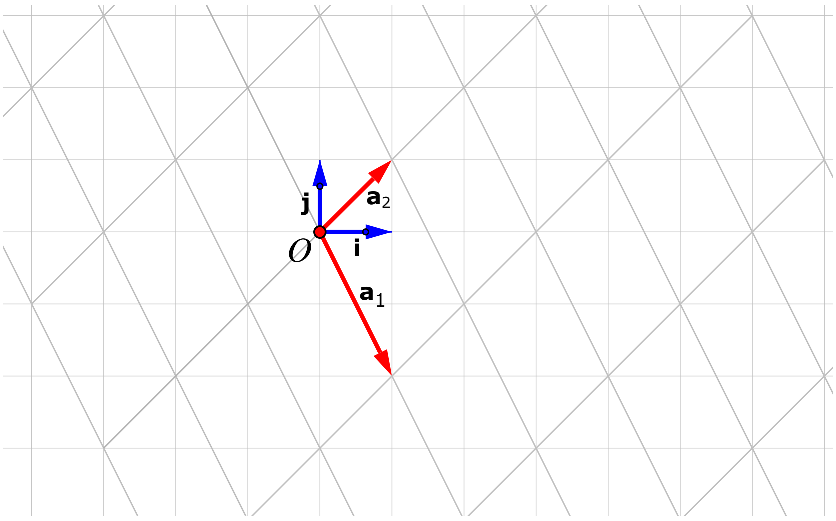


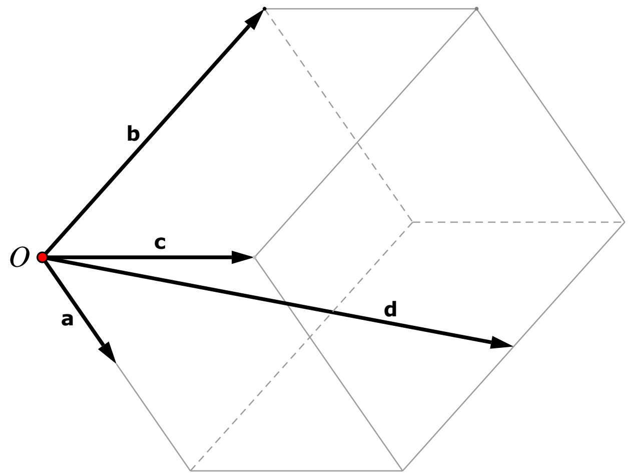
Figure: Change of basis

In the diagram, we are given the standard basis $e = (\mathbf{i}, \mathbf{j})$ in the plane plus another basis $a = (\mathbf{a}_1, \mathbf{a}_2)$.

1. A vector $\mathbf{u}$ has the coordinates $(5, -1)$ with respect to basis e. Determine $\mathbf{u}$'s a-coordinates.

2. A vector $\mathbf{v}$ has the coordinates $(-1, -2)$ with respect to basis a. Determine $\mathbf{v}$'s e-coordinates.

|||| **Exercise 10.37**



1. In the diagram, it is evident that **a**, **b** and **c** are linearly independent. A basis m is therefore given by $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Determine the coordinate vector ${}_{\mathrm{m}}\mathbf{d}$.

2. It is also evident from the figure that $(\mathbf{a}, \mathbf{b}, \mathbf{d})$ is a basis, let us call it n. Determine the coordinate vector ${}_{\mathrm{n}}\mathbf{c}$.

3. Draw, with the origin as the initial point, the vector **u** that has the m-coordinates

$$ {}_{\mathrm{m}}\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}. $$

10.6 Vector Calculations Using Coordinates

When you have chosen a basis for geometric vectors in the plane (or in 3-space), then all vectors can be described and determined using their coordinates with respect to the chosen basis. For the two arithmetic operations, addition and multiplication by a scalar, that were introduced previously in this eNote by geometrical construction, we get a particularly practical alternative. Instead of geometrical constructions we can carry out calculations with the coordinates that correspond to the chosen basis.

We illustrate this with an example in the plane with a basis a given by $(\mathbf{a}_1, \mathbf{a}_2)$ plus two vectors **u** and **v** drawn from O, see Figure 10.5. The exercise is to determine the vector $\mathbf{b} = 2\mathbf{u} - \mathbf{v}$, and we will do this in two different ways.

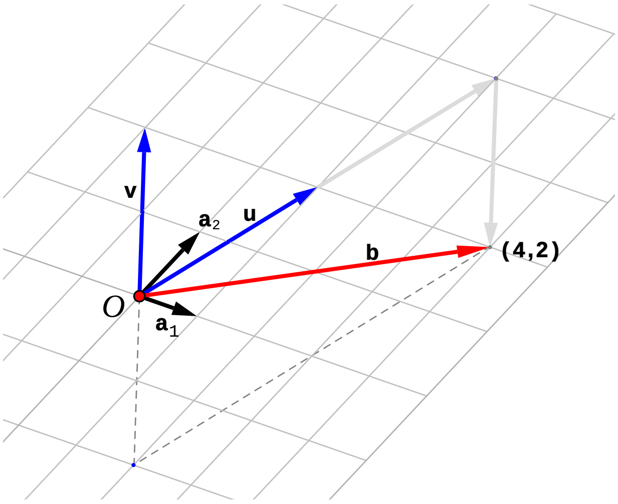


Figure 10.5: Linear combination determined using coordinates

Method 1 (geometric): First we carry through the arithmetic operations as defined in 10.2 and 10.3, cf. the grey construction vectors in Figure 10.5.

Method 2 (algebraic): We read the coordinates for **u** and **v** and carry out the arithmetic operations directly on the coordinates:

$$ {}_{\mathrm{a}}\mathbf{b} = 2\,{}_{\mathrm{a}}\mathbf{u} - {}_{\mathrm{a}}\,\mathbf{v} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}. \tag{10-10} $$

Now **b** can be drawn directly from its coordinates $(4, 2)$ with respect to basis a.

That it is allowed to use this method is stated in the following theorem.

|||| Theorem 10.38 Basic Rules for Coordinate Calculations

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in the plane or in 3-space plus a real number k are given. Moreover, an arbitrary basis a has been chosen.The two arithmetic operations $\mathbf{u}+\mathbf{v}$ and $k\,\mathbf{u}$ can then be carried out using coordinates as follows:

1. ${}_{\mathrm{a}}(\mathbf{u}+\mathbf{v}) = {}_{\mathrm{a}}\mathbf{u} + {}_{\mathrm{a}}\mathbf{v}$
2. ${}_{\mathrm{a}}(k\mathbf{u}) = k\,{}_{\mathrm{a}}\mathbf{u}$

In other words: the coordinates for a vector sum are obtained by adding the coordinates for the summands. And the coordinates for a vector multiplied by a number are the coordinates of the vector multiplied by that number.

|||| Proof

We carry through the proof for the set of geometric vectors in 3-space. Suppose the coordinates for $\mathbf{u}$ and $\mathbf{v}$ with respect to the chosen basis a are given by

$$ {}_{\mathrm{a}}\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } {}_{\mathrm{a}}\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}. \tag{10-11} $$

We then have

$$ \mathbf{u} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3 \text{ og } \mathbf{v} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3 \tag{10-12} $$

and accordingly, through the application of the commutative, associative and distributive arithmetic rules, see Theorem10.12,

$$ \begin{aligned} \mathbf{u}+\mathbf{v} &= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3) \\ &= (u_1+v_1)\mathbf{a}_1 + (u_2+v_2)\mathbf{a}_2 + (u_3+v_3)\mathbf{a}_3 \end{aligned} \tag{10-13} $$

which yields

$$ {}_{\mathrm{a}}(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = {}_{\mathrm{a}}\mathbf{u} + {}_{\mathrm{a}}\mathbf{v} \tag{10-14} $$

so that now the first part of the proof is complete. In the second part of the proof we again use a distributive arithmetic rule, see Theorem 10.12:

$$ k\mathbf{u} = k(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) = (k\cdot u_1)\mathbf{a}_1 + (k\cdot u_2)\mathbf{a}_2 + (k\cdot u_3)\mathbf{a}_3 \tag{10-15} $$

which yields

$$ {}_{\mathrm{a}}(k\mathbf{u}) = \begin{bmatrix} k\cdot u_1 \\ k\cdot u_2 \\ k\cdot u_3 \end{bmatrix} = k\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = k\,{}_{\mathrm{a}}\mathbf{u} \tag{10-16} $$

so that now the second part of the proof is complete.

■

Theorem 10.38 makes it possible to perform more complicated arithmetic operations using coordinates, as shown in the following example.

|||| Example 10.39 Coordinate Vectors for a Linear Combination

The three plane vectors **a**, **b** and **c** have the following coordinate vectors with respect to a chosen basis v:

$$ {}_{\mathrm{v}}\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ {}_{\mathrm{v}}\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } {}_{\mathrm{v}}\mathbf{c} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}. \tag{10-17}$$

Problem: Determine the coordinate vector $\mathbf{d} = \mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$ with respect to basis v.
Solution:

$$\begin{aligned} {}_{\mathrm{v}}\mathbf{d} &= {}_{\mathrm{v}}(\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}) \\ &= {}_{\mathrm{v}}(\mathbf{a} + (-2)\mathbf{b} + 3\mathbf{c}) \\ &= {}_{\mathrm{v}}\mathbf{a} + {}_{\mathrm{v}}(-2\mathbf{b}) + {}_{\mathrm{v}}(3\mathbf{c}) \\ &= {}_{\mathrm{v}}\mathbf{a} - 2\,{}_{\mathrm{v}}\mathbf{b} + 3\,{}_{\mathrm{v}}\mathbf{c} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 16 \\ -3 \end{bmatrix}. \end{aligned}$$

Here the third equality sign is obtained using the first part of Theorem 10.38 and the fourth equality sign from the second part of that theorem.

|||| Example 10.40 The Parametric Representation of a Plane in Coordinates

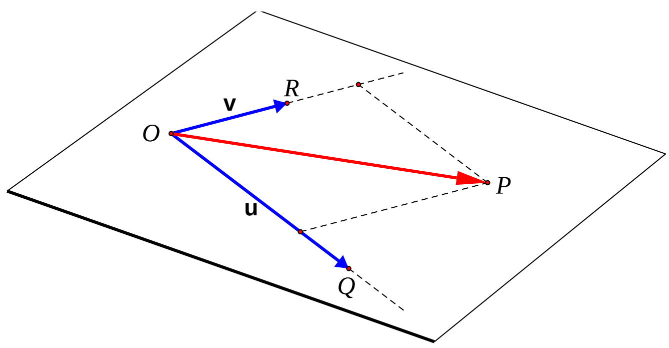


Figure: A plane in 3-space

In accordance with Example 10.25, the plane through the origin shown in the diagram has the parametric representation

$$\{P \mid \overrightarrow{OP} = s\mathbf{u} + t\mathbf{v}\,;\; (s,t) \in \mathbb{R}^2\}. \tag{10-18}$$

Suppose that in 3-space we are given a basis a and that

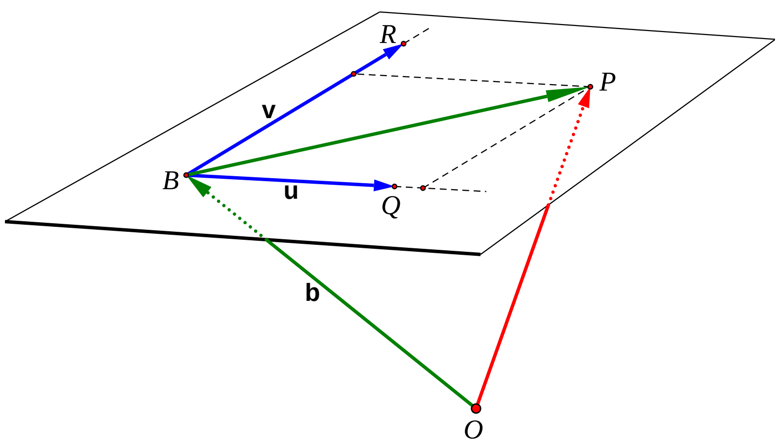
$$_{\mathrm{a}}\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } {_{\mathrm{a}}\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

The parametric representation (10-18) can then be written in coordinate form like this:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{10-19}$$

where $_{\mathrm{a}}\overrightarrow{OP} = (x, y, z)$ and $(s,t) \in \mathbb{R}^2$.

|||| Example 10.41 The Parametric Representation of a Plane in Coordinates



In accordance with Example 10.26 the plane through the origin shown in the diagram has the parametric representation

$$\{P \mid \overrightarrow{OP} = \mathbf{b} + s\mathbf{u} + t\mathbf{v}\,;\; (s,t) \in \mathbb{R}^2\}. \tag{10-20}$$

Suppose that in 3-space we are given a basis a and that

$$_{\mathrm{a}}\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},\; {_{\mathrm{a}}\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } {_{\mathrm{a}}\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

The parametric representation (10-18) can then be written in coordinate form like this:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + s \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{10-21}$$

where ${}_{\mathrm{a}}\overrightarrow{OP} = (x, y, z)$ and $(s, t) \in \mathbb{R}^2$

10.7 Vector Equations and Matrix Algebra

A large number of vector-related problems are best solved by resorting to vector equations. If we wish to solve these equations using the vector coordinates in a given basis, we get systems of linear equations. The problems can then be solved using matrix methods that follow in eNote 6. This subsection gives examples of this and sums up this approach by introducing the *coordinate matrix* concept in the final Exercise 10.45.

|||| Example 10.42 Whether a Vector is a Linear Combination of Other Vectors

In 3-space are given a basis a and three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{p}$ which have the coordinates with respect to the basis a given by:

$${}_{\mathrm{a}}\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \ {}_{\mathrm{a}}\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \text{ and } {}_{\mathrm{a}}\mathbf{p} = \begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}.$$

Problem: Investigate whether $\mathbf{p}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
Solution: We will investigate whether we can find coefficients k_1, k_2, such that

$$k_1\mathbf{u} + k_2\mathbf{v} = \mathbf{p}\,.$$

We arrange the corresponding coordinate vector equation

$$k_1 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$$

which is equivalent to the following system of equations

$$\begin{aligned} 2k_1 + k_2 &= 0 \\ k_1 + 4k_2 &= 7 \\ 5k_1 + 3k_2 &= 1 \end{aligned} \tag{10-22}$$

We consider the augmented matrix $\mathbf{T}$ for the system of equations and give (without details) the reduced row echelon form of the matrix:

$$\mathbf{T} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix} \rightarrow \operatorname{rref}(\mathbf{T}) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \tag{10-23}$$

We see that the system of equations has exactly one solution, $k_1 = -1$ and $k_2 = 2$, meaning that

$$-1\mathbf{u} + 2\mathbf{v} = \mathbf{p}\,.$$

|||| Example 10.43 Whether a Set of Vectors is Linearly Dependent

In 3-space are given a basis v and three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ which with respect to this basis have the coordinates

$${}_{\mathrm{v}}\mathbf{a} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix},\ {}_{\mathrm{v}}\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \text{ and } {}_{\mathrm{v}}\mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Problem: Investigate whether the set of vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is linearly dependent.

Solution: Following theorem 10.23 we can investigate whether there exists a proper linear combination

$$k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c} = \mathbf{0}\,.$$

We look at the corresponding coordinate vector equation

$$k_1 \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + k_3 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

that is equivalent to the following homogeneous system of linear equations

$$\begin{aligned} 5k_1 + k_2 + 2k_3 &= 0 \\ k_1 + 3k_3 &= 0 \\ 3k_1 + 4k_2 + k_3 &= 0 \end{aligned} \tag{10-24}$$

We arrange the augmented matrix $\mathbf{T}$ of the system of equations and give (without details) the reduced row echelon form of the matrix:

$$\mathbf{T} = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 1 & 0 & 3 & 0 \\ 3 & 4 & 1 & 0 \end{bmatrix} \rightarrow \operatorname{rref}(\mathbf{T}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{10-25}$$

We see that the system of equations only have the zero solution $k_1 = 0, k_2 = 0$ and $k_3 = 0$. The set of vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is therefore linearly independent. Therefore you may choose $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ as a new basis for the set of vectors in 3-space.

In the following example we continue the discussion of the relation between coordinates and change of basis from exercise 10.36

|||| Example 10.44 The New Coordinates after Change of Basis

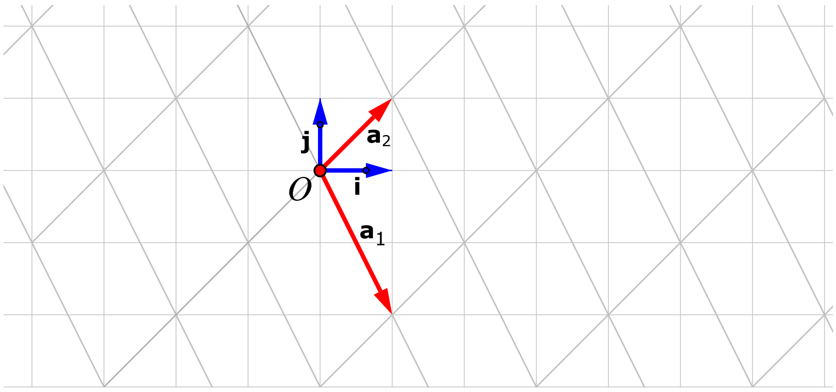


Figure: Change of basis

In the diagram we are given a standard basis e= $(\mathbf{i}, \mathbf{j})$ and another basis a= $(\mathbf{a}_1, \mathbf{a}_2)$. When the basis is changed, the coordinates of any given vector are changed. Here we give a systematic method for expressing the change in coordinates using a matrix-vector product. First we read the e-coordinates of the vectors in basis a:

$$ {}_{\mathrm{e}}\mathbf{a_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ and } {}_{\mathrm{e}}\mathbf{a_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{10-26} $$

1. *Problem*: Suppose a vector $\mathbf{v}$ has the set of coordinates ${}_{\mathrm{a}}\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Determine the e-coordinates of $\mathbf{v}$.

   Solution: We have that $\mathbf{v} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2$ and therefore following Theorem 10.38:

   $$ {}_{\mathrm{e}}\mathbf{v} = v_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} $$

   If we put $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$, we express $\mathbf{v}$'s e-coordinates by the matrix-vector product

   $$ {}_{\mathrm{e}}\mathbf{v} = \mathbf{M} \cdot {}_{\mathrm{a}}\mathbf{v} \tag{10-27} $$

2. *Problem*: Suppose a vector $\mathbf{v}$ has the set of coordinates ${}_{\mathrm{e}}\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Determine the a-coordinates of $\mathbf{v}$.

Solution: We multiply from the left on both sides of 10-27 with the inverse matrix to **M** and get a-coordinates of **v** expressed by the matrix-vector product:

$$_{\mathrm{a}}\mathbf{v} = \mathbf{M}^{-1} \cdot {_{\mathrm{e}}\mathbf{v}} \tag{10-28}$$

|||| Exercise 10.45

By a ***coordinate matrix*** with respect to a given basis a for a set of vectors me mean the matrix that is formed by combining the vector's a-coordinate columns to form a matrix.
Describe the matrix **T** in example 10.42 and 10.43 and the matrix **M** in 10.44 as coordinate matrices.

10.8 Theorems about Vectors in a Standard Basis

In this subsection we work with standard coordinate systems, both in the plane and in 3-space. We introduce two different multiplications between vectors, the *dot product* which is defined both in the plane and in 3-space, and the *cross product* that is only defined in 3-space. We look at geometric applications of these types of multiplication and at geometrical interpretations of determinants.

10.8.1 The Dot Product of two Vectors

|||| Definition 10.46 The Dot Product in the Plane

In the plane are given two vectors ${}_e\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and ${}_e\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. By the *dot product* (or the *scalar product*) of **a** and **b** we refer to the number

$$\mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 \,. \tag{10-29}$$

|||| Definition 10.47 The Dot Product in Space

In 3-space are given two vectors ${}_e\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and ${}_e\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. By the *dot product* (or the *scalar product*) of **a** and **b** we understand the number

$$\mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 \,. \tag{10-30}$$

For the dot product between two vectors the following rules of calculation apply.

|||| Theorem 10.48 Arithmetic Rules for the Dot Product

Given three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in the plane or in 3-space and the number k. Observe:

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutative rule)
2. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (associative rule)
3. $(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$
4. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
5. $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}$.

|||| Proof

The Rules 1, 2, 3 follow from a simple coordinate calculation. Rule 4 follows from the Pythagorean Theorem, and Rule 5 is a direct consequence of Rules 1, 2 and 4.

■

In the following three theorems we look at geometric applications of the dot product.

|||| Theorem 10.49 The Length of a Vector

Let $\mathbf{v}$ be an arbitrary vector in the plane or in 3-space. The length of $\mathbf{v}$ satisfies

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}\,. \tag{10-31}$$

|||| Proof

The theorem follows immediately from the arithmetic Rule 4 in 10.48

■

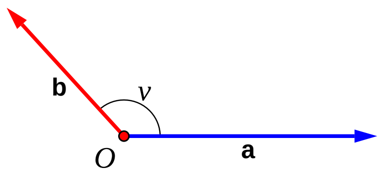


Figure 10.6: Angle between two vectors

|||| Example 10.50 Length of a Vector

Given the vector $\mathbf{v}$ in 3-space and ${}_e\mathbf{v} = (1,2,3)$. We then have

$$|\mathbf{v}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}\,.$$

The following fact concerns the angle between two vectors, see Figure 10.6.

|||| Theorem 10.51 The Angle between Vectors

In the plane or 3-space we are given two proper vectors $\mathbf{a}$ and $\mathbf{b}$. The angle v between $\mathbf{a}$ and $\mathbf{b}$ satisfies

$$\cos(v) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \tag{10-32}$$

|||| Proof

The theorem can be proved using the cosine relation. In carrying out the proof one needs Rule 5 in theorem 10.48. The details are left for the reader.

■

From this theorem it follows directly:

|||| Corollary 10.52 The Size of Angles

Consider the situation in Figure 10.6. We see

1. $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \text{angle}(\mathbf{a}, \mathbf{b}) = \frac{\pi}{2}$
2. $\mathbf{a} \cdot \mathbf{b} > 0 \Leftrightarrow \text{angle}(\mathbf{a}, \mathbf{b}) < \frac{\pi}{2}$
3. $\mathbf{a} \cdot \mathbf{b} < 0 \Leftrightarrow \text{angle}(\mathbf{a}, \mathbf{b}) > \frac{\pi}{2}$

The following theorems are dedicated to ***orthogonal projections***. In Figure 10.7 two vectors $\mathbf{a}$ and $\mathbf{b}$ in the plane or 3-space are drawn from the origin.

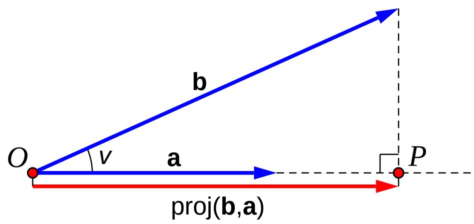


Figure 10.7: Orthogonal projection

Consider P, the foot of the perpendicular from $\mathbf{b}$'s endpoint to the line containing $\mathbf{a}$. By the orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$ we mean the vector $\overrightarrow{OP}$, denoted $\text{proj}(\mathbf{b}, \mathbf{a})$.

|||| Theorem 10.53 The Length of a Projection

Given two proper vectors $\mathbf{a}$ and $\mathbf{b}$ in the plane or 3-space. The length of the orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$ is:

$$|\text{proj}(\mathbf{b}, \mathbf{a})| = \frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{a}|} \tag{10-33}$$

|||| Proof

Using a known theorem about right angled triangles plus Theorem 10.51 we get

$$|\text{proj}(\mathbf{b},\mathbf{a})| = |\cos(v)|\,|\mathbf{b}| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}|}\,.$$

■

|||| Theorem 10.54 Formula for the Projection Vector

Given two proper vectors **a** and **b** in the plane or 3-space. The orthogonal projection of **b** on **a** is:

$$\text{proj}(\mathbf{b},\mathbf{a}) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}\,. \tag{10-34}$$

|||| Proof

If **a** and **b** are orthogonal the theorem is true since the projection in that case is the zero vector. Conversely, let $\text{sign}(\mathbf{a}\cdot\mathbf{b})$ denote the sign of $\mathbf{a}\cdot\mathbf{b}$. We have that $\text{sign}(\mathbf{a}\cdot\mathbf{b})$ is positive exactly when **a** and $\text{proj}(\mathbf{b},\mathbf{a})$ have the same direction and negative exactly when they have the opposite direction. Therefore we get

$$\text{proj}(\mathbf{b},\mathbf{a}) = \text{sign}(\mathbf{a}\cdot\mathbf{b})\cdot|\text{proj}(\mathbf{b},\mathbf{a})|\,\frac{\mathbf{a}}{|a|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}\,,$$

where we have used Theorem 10.53, and the fact that $\frac{\mathbf{a}}{|\mathbf{a}|}$ is a unit vector pointing in the direction of **a**.

■

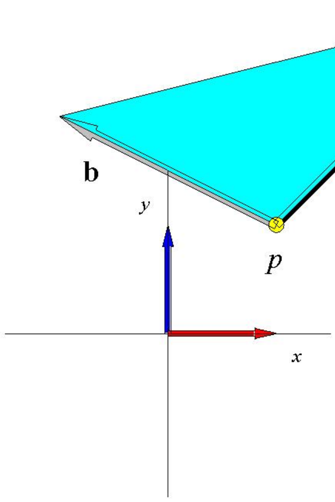


Figure 10.8: A triangle spanned by two vectors in the plane

10.8.2 Geometric Interpretation of the Determinant of a 2×2 Matrix

A triangle $\triangle = \triangle(p, \mathbf{a}, \mathbf{b})$ is determined by two vectors drawn from a common initial point, see the triangle $\triangle = \triangle(p, \mathbf{a}, \mathbf{b})$ in Figure 10.8.

The area of a triangle is known to be half the base times its height. We can choose the length $|\mathbf{a}|$ of $\mathbf{a}$ as the base. And the height in the triangle is

$$|\mathbf{b}| \sin(\theta) = \frac{|\mathbf{b} \cdot \widehat{\mathbf{a}}|}{|\widehat{\mathbf{a}}|} \,, \tag{10-35}$$

where θ is the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$, and where $\widehat{\mathbf{a}}$ denotes the ***hat vector*** in the plane to $\mathbf{a}$, that is in coordinates we have $\widehat{\mathbf{a}} = (-a_2, a_1)$. Hence the area is:

$$\begin{aligned}
\text{Area}(\triangle(p, \mathbf{a}, \mathbf{b})) &= \frac{1}{2}|\mathbf{b} \cdot \widehat{\mathbf{a}}| \\
&= \frac{1}{2}|a_1 b_2 - a_2 b_1| \\
&= |\,\frac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\,| \\
&= \frac{1}{2}|\,\det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right)\,| \\
&= \frac{1}{2}|\,\det\left([\mathbf{a}\ \mathbf{b}]\right)\,| \;.
\end{aligned} \tag{10-36}$$

Thus we have proven the theorem:

|||| Theorem 10.55 Area of a Triangle as a Determinant

The area of the triangle $\triangle(p, \mathbf{a}, \mathbf{b})$ is the absolute value of half the determinant of the 2×2 matrix that is obtained by insertion of **a** and **b** as columns in the matrix.

10.8.3 The Cross Product and the Scalar Triple Product

The *cross product* of two vectors and the *scalar triple product* of three vectors are introduced using determinants:

|||| Definition 10.56 Cross Product

In 3-space two vectors are given ${}_e\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and ${}_e\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

By the *cross product* (or the *vector product*) $\mathbf{a} \times \mathbf{b}$ of **a** and **b** is understood the vector **v** given by

$$
{}_e\mathbf{v} = \begin{bmatrix} \det \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \\ \det \begin{bmatrix} a_3 & b_3 \\ a_1 & b_1 \end{bmatrix} \\ \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \end{bmatrix} \tag{10-37}
$$

The cross product has a geometric significance. Consider Figure 10.9 and the following theorem:

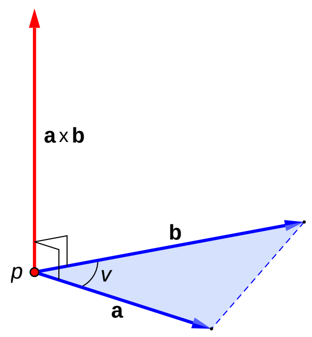


Figure 10.9: Geometry of the cross-product

|||| Theorem 10.57 The Area of a Triangle by the Cross Product

For two linearly independent vectors $\mathbf{a}$ and $\mathbf{b}$ that form the angle v with each other, the cross product $\mathbf{a} \times \mathbf{b}$ satisfies

1. $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
2. $|\mathbf{a} \times \mathbf{b}| = 2 \cdot \text{Area}(\triangle(p, \mathbf{a}, \mathbf{b}))$.
3. The vector set $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ follows the *right hand rule*: seen from the tip of $\mathbf{a} \times \mathbf{b}$ the direction from $\mathbf{a}$ to $\mathbf{b}$ is counter-clockwise.

|||| Definition 10.58 Scalar Triple Product

The *scalar triple product* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of the vectors ${}_e\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, ${}_e\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and ${}_e\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ a_3 \end{bmatrix}$ is defined by:

$$
\begin{aligned}
[\mathbf{a}, \mathbf{b}, \mathbf{c}] &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \\
&= (c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1) \\
&= \det\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \right) \\
&= \det\left([{}_e\mathbf{a}\; {}_e\mathbf{b}\; {}_e\mathbf{c}] \right) \quad .
\end{aligned}
\tag{10-38}
$$

10.8.4 Geometric Interpretation of the Determinant of a 3×3 Matrix

From elementary Euclidean space geometry we know that the volume of a tetrahedron is one third of the area of the base times the height. Consider the tetrahedron $\boxtimes = \boxtimes(p, \mathbf{a}, \mathbf{b}, \mathbf{c})$ spanned by the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ drawn from the point p, in Figure 10.10. The area of the base, $\triangle(p, \mathbf{a}, \mathbf{b})$ has been determined in the second part of Theorem 10.57.

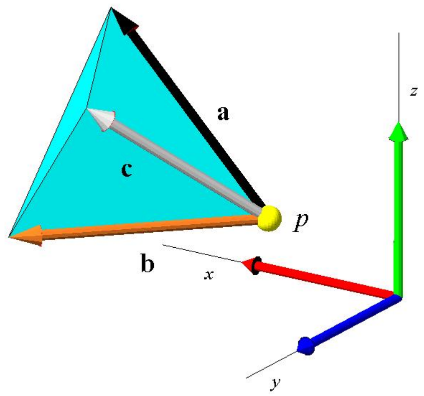


Figure 10.10: A tetrahedron spanned by three vectors in 3-space

The height can then be determined as the scalar product of the third edge vector $\mathbf{c}$ with a unit vector, perpendicular to the base triangle.

But $\mathbf{a} \times \mathbf{b}$ is exactly perpendicular to the base triangle (because the cross product is perpendicular to the edge vectors of the base triangle, see part 2 of Theorem (10.57), so we use this:

$$\begin{aligned}\mathrm{Vol}(\boxtimes(p,\mathbf{a},\mathbf{b},\mathbf{c})) &= |\frac{1}{3}\,\mathrm{Area}(\triangle(p,\mathbf{a},\mathbf{b}))\,\frac{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}}{|\mathbf{a}\times\mathbf{b}|}| \\ &= \frac{1}{6}|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|\end{aligned} \tag{10-39}$$

where we have used part 2 of Theorem 10.57.

By comparing this to the definition of *scalar triple product*, see 10.58, we now get the volume of a tetrahedron written in 'determinant-form':

|||| Theorem 10.59 Volume of a Tetrahedron as a Scalar Triple Product

The volume of the tetrahedron $\boxtimes = \boxtimes(p,\mathbf{a},\mathbf{b},\mathbf{c})$ is:

$$\mathrm{Vol}(\boxtimes(p,\mathbf{a},\mathbf{b},\mathbf{c})) = \frac{1}{6}|\det\left([\mathbf{a}\,\mathbf{b}\,\mathbf{c}]\right)| \quad . \tag{10-40}$$

A tetrahedron has the volume 0, is collapsed, exactly when the determinant in (10-40) is 0, and this occurs exactly when one of the vectors can be written as a linear combination of the two others (why is that?).

|||| Definition 10.60 Regular Tetrahedron

A ***regular tetrahedron*** is a tetrahedron that has a proper volume, that is a volume, that is strictly greater than 0.

|||| Exercise 10.61

Let $\mathbf{A}$ denote a (2×2)-matrix with the column vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{A} = [\,\mathbf{a}\ \mathbf{b}\,] \quad . \tag{10-41}$$

Show that the determinant of $\mathbf{A}$ is 0 if and only if the column vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent in $\mathbb{R}^2$.

|||| Exercise 10.62

Let $\mathbf{A}$ denote a $(3 \times 3)-$matrix with the column vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$:

$$\mathbf{A} = [\,\mathbf{a}\ \mathbf{b}\ \mathbf{c}\,] \quad . \tag{10-42}$$

Show, that the determinant of $\mathbf{A}$ is 0 if and only if the column vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ constitute a linearly dependent set of vectors in $\mathbb{R}^3$.

|||| Exercise 10.63

Use the geometric interpretations of the determinant above to show the following Hadamard's inequality for $(2 \times 2)-$matrices and for $(3 \times 3)-$matrices (in fact the inequality is true for all square matrices):

$$(\det(\mathbf{A}))^2 \leq \prod_{j}^{n} \left(\sum_{i}^{n} a_{ij}^2 \right) \quad . \tag{10-43}$$

When is the equality sign valid in (10-43)?