eNote 9

Determinants

*In this eNote we look at **square matrices**; that is they are of type $n \times n$ for $n \geq 2$, see eNote 8. It is an advantage but not indispensable to have knowledge about the concept of a determinant for (2×2)-matrices in advance. The matrix algebra from eNote 7 is assumed known (sum, product, transpose and inverse of matrices, plus the general solution method for systems of linear equations from eNote 6.*

Updated: 24.9.21 David Brander.

9.1 Intro to Determinants

The ***determinant*** of a real *square* $(n \times n)$-matrix $\mathbf{A}$ is a real number which we denote by $\det(\mathbf{A})$ or sometimes for short by $|\mathbf{A}|$. The determinant of a matrix can, in a way, be considered as a measure of how much the matrix 'weighs' - with sign; we will illustrate this visually and geometrically for (2×2)-matrices and for (3×3)-matrices in eNote 10.

The determinant is a well defined *function* of the total of n^2 numbers, that constitute the elements of an $(n \times n)$-matrix.

In order to define – and then calculate – the value of the determinant of an $(n \times n)$-matrix directly from the n^2 elements in each of the matrices we need two things: First the well-known formula for the determinant of (2×2)-matrices (see the definition 9.1 below) and secondly a method to cut up an arbitrary $(n \times n)$-matrix into (2×2)-matrices and thereby define and calculate arbitrary determinants from the determinants of these (2×2)-matrices.

9.2 Determinants of $(2 \times 2)-$Matrices

|||| Definition 9.1 Determinants of $(2 \times 2)-$Matrices

Let $\mathbf{A}$ be the arbitrary $(2 \times 2)-$matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{9-1}$$

Then the determinant of $\mathbf{A}$ is defined by:

$$\det(\mathbf{A}) = a_{11} \cdot a_{22} - a_{21} \cdot a_{12} \,. \tag{9-2}$$

|||| Exercise 9.2 Inverse $(2 \times 2)-$Matrix

Remember that the inverse matrix $\mathbf{A}^{-1}$ of a invertible matrix $\mathbf{A}$ has the characteristic property that $\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{E}$. Show directly from (9-1) and (9-2), that the inverse matrix $\mathbf{A}^{-1}$ to a $(2 \times 2)-$matrix $\mathbf{A}$ can be expressed in the following way (when $\det(\mathbf{A}) \neq 0$) :

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}. \tag{9-3}$$

|||| Exercise 9.3 Arithmetic Rules for $(2 \times 2)-$Matrices

For allsquare matrices a number of basic arithmetic rules are valid; they are presented in theorem 9.20 below. Check the three first equations in theorem 9.20 for (2×2)-matrices $\mathbf{A}$ and $\mathbf{B}$. Use direct calculation of both sides of the equations using (9-2).

9.3 Submatrices

|||| Definition 9.4 Submatrices

For an $(n \times n)$-matrix $\mathbf{A}$ we define the (i, j) submatrix, $\widehat{\mathbf{A}}_{ij}$, as the $((n-1) \times (n-1))$-submatrix of $\mathbf{A}$ that emerges by deleting the entire row i and the entire column j from the matrix $\mathbf{A}$.

All the total of n^2 submatrices $\widehat{\mathbf{A}}_{ij}$ (where $1 \leq i \leq n$ and $1 \leq j \leq n$) are less than $\mathbf{A}$ and are of the type $(n-1) \times (n-1)$ and therefore have only $(n-1)^2$ elements.

|||| Example 9.5 Submatrices for a $(3 \times 3)-$Matrix

A (3×3)-matrix $\mathbf{A}$ has total of 9 (2×2)-submatricer $\widehat{\mathbf{A}}_{ij}$. For example, if

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \\ 0 & 5 & 1 \end{bmatrix}, \tag{9-4}$$

then the 9 submatrices belonging to $\mathbf{A}$ are given by:

$$\begin{aligned} \widehat{\mathbf{A}}_{11} &= \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}, & \widehat{\mathbf{A}}_{12} &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, & \widehat{\mathbf{A}}_{13} &= \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}, \\ \widehat{\mathbf{A}}_{21} &= \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix}, & \widehat{\mathbf{A}}_{22} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, & \widehat{\mathbf{A}}_{23} &= \begin{bmatrix} 0 & 2 \\ 0 & 5 \end{bmatrix}, \\ \widehat{\mathbf{A}}_{31} &= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, & \widehat{\mathbf{A}}_{32} &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, & \widehat{\mathbf{A}}_{33} &= \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}. \end{aligned} \tag{9-5}$$

The corresponding determinants are determinants of $(2 \times 2)-$matrices and each of these can be calculated directly from the definition 9.1 above:

$$\begin{aligned} &\det(\widehat{\mathbf{A}}_{11}) = -7\,, \det(\widehat{\mathbf{A}}_{12}) = 1\,, && \det(\widehat{\mathbf{A}}_{13}) = 5\,, \\ &\det(\widehat{\mathbf{A}}_{21}) = -3\,, \det(\widehat{\mathbf{A}}_{22}) = 0\,, && \det(\widehat{\mathbf{A}}_{23}) = 0\,, \\ &\det(\widehat{\mathbf{A}}_{31}) = 1\,, \quad \det(\widehat{\mathbf{A}}_{32}) = -1\,, && \det(\widehat{\mathbf{A}}_{33}) = -2\,. \end{aligned} \tag{9-6}$$

9.4 Inductive Definition of Determinants

The determinant of a 3×3 matrix can now be defined from the determinants of 3 of the 9 submatrices, and generally: The determinant of an $n \times n$ matrix is defined by the use of the determinants of the n submatrices that belong to a (freely chosen) row r in the following way, which naturally is called ***expansion along the r-th row***:

|||| Definition 9.6 Determinants are Defined by Expansion

For an arbitrary value of the row index r the determinant of a given $(n \times n)$-matrix $\mathbf{A}$ is defined inductively in the following way:

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{r+j} a_{rj} \det(\widehat{\mathbf{A}}_{rj}) \quad . \tag{9-7}$$

We here and subsequently use the following short notation for the sum and products of many terms, e.g. n given real numbers $c_1, c_2, \ldots, c_{n-1}, c_n$:

$$\begin{aligned} c_1 + c_2 + \cdots + c_{n-1} + c_n &= \sum_{i=1}^{n} c_i \;, \quad \text{and} \\ c_1 \cdot c_2 \cdot \cdots \cdot c_{n-1} \cdot c_n &= \prod_{i=1}^{n} c_i \quad . \end{aligned} \tag{9-8}$$

|||| Example 9.7 Expansion of a Determinant along the 1. Row

We will use Definition 9.6 directly in order to calculate the determinant of the matrix $\mathbf{A}$ that is given in example 9.5. We choose $r = 1$ and we thus need three determinants of the submatrices, $\det(\widehat{\mathbf{A}}_{11}) = -7$, $\det(\widehat{\mathbf{A}}_{12}) = 1$, and $\det(\widehat{\mathbf{A}}_{13}) = 5$, which we calculated already in example 9.5 above:

$$\begin{aligned} \det(\mathbf{A}) &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(\widehat{\mathbf{A}}_{1j}) \\ &= (-1)^{1+1} \cdot 0 \cdot \det(\widehat{\mathbf{A}}_{11}) + (-1)^{1+2} \cdot 2 \cdot \det(\widehat{\mathbf{A}}_{12}) + (-1)^{1+3} \cdot 1 \cdot \det(\widehat{\mathbf{A}}_{13}) \\ &= 0 - 2 + 5 = 3 \quad . \end{aligned} \tag{9-9}$$

Notice that the determinants of the submatrices must be multiplied by the element in **A** that is in entry (r, j) and with the sign-factor $(-1)^{r+j}$ before they are added. And notice that the determinants of the submatrices themselves can be expanded by the use of determinants of even smaller matrices, such that finally we only need to determine weighted sums of determinants of $(2 \times 2)-$matrices!

|||| Exercise 9.8 Choice of 'Expansion Row' Arbitrary

Show by direct calculation that we obtain the same value for the determinant by use of one of the other two rows for the expansion of the determinant in example 9.5.

|||| Definition 9.9 Alternative: Expansion along a Column

The determinant of a given $(n \times n)-$matrix **A** can alternatively be defined inductively by expansion along an arbitrary chosen *column* :

$$\det(\mathbf{A}) = \sum_{i=1}^{n} (-1)^{i+s} a_{i\,s} \det(\widehat{\mathbf{A}}_{i\,s}) \quad . \tag{9-10}$$

Here the expansion is expressed as the expansion along column s.

As is already hinted with the definitions and as shown in the concrete case of the matrix in example 9.5, it doesn't matter which row (or column) defines the expansion:

|||| Theorem 9.10 Choice of Row or Column for the Expansions Immaterial

The two definitions, 9.6 and 9.9, of the determinant of a square matrix give the same value and this they do without regard to the choice of row or column in the corresponding expansions.

|||| Exercise 9.11 Choice of Column for the Expansion is Immaterial

Show by direct calculation that we get the same value for the determinant in 9.5 by using expansion along any of the three columns in $\mathbf{A}$.

It is of course wisest to expand along a row (or a column) that contains many 0's.

|||| Exercise 9.12 Determinants of Some Larger Matrices

Use the above instructions and results to find the determinants of each of the following matrices:

$$\begin{bmatrix} 0 & 2 & 7 & 1 \\ 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & 8 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 & 0 & 5 \\ 1 & 3 & 2 & 0 & 2 \\ 0 & 5 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 5 & 2 & 7 & 1 & 9 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2 & 1 & 5 & 3 \\ 0 & 1 & 3 & 2 & 2 & 1 \\ 0 & 0 & 5 & 1 & 1 & 4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 5 & 2 & 7 & 1 & 9 \end{bmatrix}. \tag{9-11}$$

If there are many 0's in a matrix then it is much easier to calculate its determinant! Especially if all the elements in a row (or a column) are 0 except one element then it is clearly wisest to expand along that row (or column). And we are allowed to 'obtain' a lot of 0's by application of the well-known row operations, if you keep record of the constants used for divisions and how often you swap rows. See theorem 9.16 and example 9.17 below.

9.5 Computational Properties of Determinants

We collect some of the most important tools that are often used for the calculation and inspection of the matrix determinants.

It is not difficult to prove the following theorem, e.g. by expansion first along the first column or the first row, after which the pattern shows:

|||| Theorem 9.13 Matrices with 0 above or below the Diagonal

If an $(n \times n)-$matrix has only 0's above or below the diagonal, then the determinant is given by the products of the elements on the diagonal.

As a special case of this theorem we have:

|||| Theorem 9.14 The Determinant of a Diagonal Matrix

Let $\mathbf{\Lambda}$ denote an $(n \times n)$-***diagonal matrix*** with the elements in the diagonal $\lambda_1, \lambda_2, ..., \lambda_n$ and 0's outside the diagonal:

$$\mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} . \tag{9-12}$$

Then the determinant is

$$\det(\mathbf{\Lambda}) = \lambda_1 \, \lambda_2 \cdots \lambda_n = \prod_{i=1}^{n} \lambda_i \tag{9-13}$$

|||| Exercise 9.15 Determinant of a Bi-diagonal Matrix

Determine the determinant of the $(n \times n)-$ ***bi-diagonal matrix*** with arbitrarily given values $\mu_1, \ldots, \mu_n$ in the bi-diagonal and 0's elsewhere:

$$\mathbf{M} = \mathbf{bidiag}(\mu_1, \mu_2, \cdots, \mu_n) = \begin{bmatrix} 0 & \cdots & 0 & \mu_1 \\ 0 & \cdots & \mu_2 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ \mu_n & \ldots & 0 & 0 \end{bmatrix} . \tag{9-14}$$

General matrices (including square matrices), as known from eNote 6, can be reduced to reduced row echelon form by the use of row operations. If you keep an eye on what happens in every step in this reduction then the determinant of the matrix can be read directly from the process. The determinant of a matrix behaves 'nicely' even if you perform row operations on the matrix:

|||| Theorem 9.16 The Influence of Row Operations on the Determinant

The determinant has the following properties:

1. If all the elements in a row in $\mathbf{A}$ are 0 then the determinant is 0, $\det(\mathbf{A}) = 0$.
2. If two rows are swapped in $\mathbf{A}$, $R_i \leftrightarrow R_j$, then the sign of the determinant is shifted.
3. If all the elements in a row in $\mathbf{A}$ are multiplied by a constant k, $k \cdot R_i$, then the determinant is multiplied by k.
4. If two rows in a matrix $\mathbf{A}$ are equal then $\det(\mathbf{A}) = 0$.
5. A row operation of the type $R_j + k \cdot R_i$, $i \neq j$ does not change the determinant.

As indicated above it follows from these properties of the determinant function that the well-known reduction of a given matrix $\mathbf{A}$ to the reduced row echelon form, $\mathrm{rref}(\mathbf{A})$, through row operations as described in eNote 6, in fact *comprises* a totally explicit calculation of the determinant of $\mathbf{A}$. We illustrate with a simple example:

|||| Example 9.17 Inspection of Determinant by Reduction to the Reduced Row Echelon Form

We consider the $(3 \times 3)-$matrix $\mathbf{A}_1 = \mathbf{A}$ from example 9.5:

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \\ 0 & 5 & 1 \end{bmatrix} \tag{9-15}$$

We reduce $\mathbf{A}_1$ to the reduced row echelon form in the usual way by Gauss–Jordan row operations and all the time we keep an eye on what happens to the determinant by using the rules in 9.16 (and possibly by checking the results by direct calculations):

Operation: Swap row 1 and row 2, $R_1 \leftrightarrow R_2$: The determinant changes sign :

$$\det(\mathbf{A}_2) = -\det(\mathbf{A}_1) \ : \tag{9-16}$$

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 5 & 1 \end{bmatrix} \tag{9-17}$$

Operation: $\frac{1}{2}R_2$, row 2 is multiplied by $\frac{1}{2}$: The determinant is multiplied by $\frac{1}{2}$:

$$\det(\mathbf{A}_3) = \frac{1}{2}\det(\mathbf{A}_2) = -\frac{1}{2}\det(\mathbf{A}_1) \ : \tag{9-18}$$

$$\mathbf{A}_3 = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1/2 \\ 0 & 5 & 1 \end{bmatrix} \tag{9-19}$$

Operation: $R_1 - 3R_2$: The determinant is unchanged:

$$\det(\mathbf{A}_4) = \det(\mathbf{A}_3) = \frac{1}{2}\det(\mathbf{A}_2) = -\frac{1}{2}\det(\mathbf{A}_1) \ : \tag{9-20}$$

$$\mathbf{A}_4 = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 5 & 1 \end{bmatrix} \tag{9-21}$$

Operation: $R_3 - 5R_2$: The determinant is unchanged:

$$\det(\mathbf{A}_5) = \det(\mathbf{A}_4) = \det(\mathbf{A}_3) = \frac{1}{2}\det(\mathbf{A}_2) = -\frac{1}{2}\det(\mathbf{A}_1) \ : \tag{9-22}$$

$$\mathbf{A}_5 = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -3/2 \end{bmatrix} \tag{9-23}$$

Now the determinant is the product of the elements in the diagonal because all the elements below the diagonal are 0, see theorem 9.13. All in all we therefore have:

$$-\frac{3}{2} = \det(\mathbf{A}_5) = \det(\mathbf{A}_4) = \det(\mathbf{A}_3) = \frac{1}{2}\det(\mathbf{A}_2) = -\frac{1}{2}\det(\mathbf{A}_1) \ : \tag{9-24}$$

From this we obtain directly – by reading 'backwards':

$$-\frac{1}{2}\det(\mathbf{A}_1) = -\frac{3}{2}\,, \tag{9-25}$$

such that

$$\det(\mathbf{A}_1) = 3\,. \tag{9-26}$$

In addition we have the following relation between the rank and the determinant of a matrix; the determinant reveals whether the matrix is singular or invertible:

|||| Theorem 9.18 Rank versus Determinant

The rank of a square $(n \times n)$-matrix $\mathbf{A}$ is less than n if and only if the determinant of $\mathbf{A}$ is 0. In other words, $\mathbf{A}$ is singular if and only if $\det(\mathbf{A}) = 0$.

If a matrix contains a variable, a parameter, then the determinant of the matrix is a function of this parameter; in the applications of matrix-algebra it is often crucial to be able to find the zeroes of this function – exactly because the corresponding matrix is singular for those values of the parameter, and hence there might not be a (unique) solution to the corresponding system of linear equations with the matrix as the coefficient matrix.

|||| Exercise 9.19 Determinant of a Matrix with a Variable

Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & 0 & a^2 & a^3 \\ 1 & a & a & a^3 \\ 1 & a & a^2 & a \end{bmatrix}, \text{ where } a \in \mathbb{R}. \tag{9-27}$$

1. Determine the determinant of $\mathbf{A}$ as a polynomium in a.
2. Determine the roots of this polynomium.
3. Find the rank of $\mathbf{A}$ for $a \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. What does the rank have to do with the roots of the determinant?
4. Find the rank of $\mathbf{A}$ for all a.

|||| Theorem 9.20 Arithmetic Rules for Determinants

Let $\mathbf{A}$ and $\mathbf{B}$ denote two $(n \times n)-$matrices. Then:

1. $\det(\mathbf{A}) = \det(\mathbf{A}^\top)$
2. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$
3. $\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}$, when $\mathbf{A}$ is invertible, that is $\det(\mathbf{A}) \neq 0$
4. $\det(\mathbf{A}^k) = (\det(\mathbf{A}))^k$, for all $k \geq 1$.
5. $\det(\mathbf{B}^{-1}\mathbf{AB}) = \det(\mathbf{A})$, when $\mathbf{B}$ is invertible, that is $\det(\mathbf{B}) \neq 0$.

|||| Exercise 9.21

Prove the last 3 equations in theorem 9.20 by the use of $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

|||| Exercise 9.22 The Determinant of a Sum is not the Sum of the Determinants

Show by the most simple example, that the determinant-function det() is *not* additive. That is, find two $(n \times n)-$matrices $\mathbf{A}$ and $\mathbf{B}$ such that

$$\det(\mathbf{A} + \mathbf{B}) \neq \det(\mathbf{A}) + \det(\mathbf{B})\ . \tag{9-28}$$

|||| Exercise 9.23 Use of Arithmetic Rules for Determinants

Let a denote a real number. The following matrices are given:

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 4 \\ 1 & a & 2 \\ 2 & 3 & 3 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 2 & -1 & 0 \\ -5 & 3 & -1 \\ 0 & 1 & a \end{bmatrix}. \tag{9-29}$$

1. Find $\det(\mathbf{A})$ and $\det(\mathbf{B})$.
2. Find $\det(\mathbf{A}\,\mathbf{B})$ and $\det\left((\mathbf{A}^\top\mathbf{B})^4\right)$.

3. Determine those values of a for which $\mathbf{A}$ is invertible and find for these values of a the expression for $\det(\mathbf{A}^{-1})$.

9.6 Advanced: Cramer's Solution Method

If $\mathbf{A}$ is a invertible $n \times n$ matrix and $\mathbf{b} = (b_1, ..., b_n)$ is an arbitrary vector in $\mathbb{R}^n$, then there exists (as is known from eNote 6 (invertible coefficient matrix)) exactly one solution $\mathbf{x} = (x_1, ..., x_n)$ to the system of linear equations $\mathbf{Ax} = \mathbf{b}$ and we found in that eNote method for finding the solution.

Cramer's method for solving such systems of equations is a *direct* method. Essentially it consists of calculating suitable determinants of matrices constructed from $\mathbf{A}$ and $\mathbf{b}$ and then writing down the solution directly from the calculated determinants.

|||| Theorem 9.24 Cramer's Solution Formula

Let $\mathbf{A}$ be a invertible $n \times n$ matrix and let $\mathbf{b} = (b_1, ..., b_n)$ denote an arbitrary vector in $\mathbb{R}^n$. Then there exists (as is known from eNote 6 (invertible coefficient matrix)) exactly one solution $\mathbf{x} = (x_1, ..., x_n)$ to the system of linear equations

$$\mathbf{Ax} = \mathbf{b} \,, \tag{9-30}$$

and the elements in the solution are given by:

$$x_j = \frac{1}{\det(\mathbf{A})} \det(\mathbf{A}\dagger_j^{\mathbf{b}}) \,, \tag{9-31}$$

where $\mathbf{A}\dagger_j^{\mathbf{b}}$ denotes the $(n \times n)-$matrix that emerges by replacing column j in $\mathbf{A}$ with $\mathbf{b}$.

|||| Explanation 9.25 What † Means

If $\mathbf{A}$ is the following matrix (from example 9.5)

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \\ 0 & 5 & 1 \end{bmatrix}, \tag{9-32}$$

and if we let $\mathbf{b} = (b_1, b_2, b_3)$, then

$$\mathbf{A}\dagger_1^{\mathbf{b}} = \begin{bmatrix} b_1 & 2 & 1 \\ b_2 & 3 & 2 \\ b_3 & 5 & 1 \end{bmatrix}, \quad \mathbf{A}\dagger_2^{\mathbf{b}} = \begin{bmatrix} 0 & b_1 & 1 \\ 1 & b_2 & 2 \\ 0 & b_3 & 1 \end{bmatrix}, \quad \mathbf{A}\dagger_3^{\mathbf{b}} = \begin{bmatrix} 0 & 2 & b_1 \\ 1 & 3 & b_2 \\ 0 & 5 & b_3 \end{bmatrix}. \tag{9-33}$$

|||| Exercise 9.26 Use Cramer's Solution Formula

If in particular we let $\mathbf{A}$ be the same matrix as above and now let $\mathbf{b} = (1, 3, 2)$, then we get by substitution of $\mathbf{b}$ in (9-33) and then computing the relevant determinants:

$$\begin{aligned} \det(\mathbf{A}\dagger_1^{\mathbf{b}}) &= \det\left(\begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 2 \\ 2 & 5 & 1 \end{bmatrix}\right) = 4 \\ \det(\mathbf{A}\dagger_2^{\mathbf{b}}) &= \det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 2 \\ 0 & 2 & 1 \end{bmatrix}\right) = 1 \\ \det(\mathbf{A}\dagger_3^{\mathbf{b}}) &= \det\left(\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 5 & 2 \end{bmatrix}\right) = 1 \,. \end{aligned} \tag{9-34}$$

Since we also know $\det(\mathbf{A}) = 3$ we have now constructed the solution to the system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, through (9-31):

$$\mathbf{x} = (x_1, x_2, x_3) = \left(\frac{1}{3}\cdot 4, \frac{1}{3}\cdot 1, \frac{1}{3}\cdot 1\right) = \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{9-35}$$

1. Check by direct isubstitution, that $\mathbf{x}$ is a solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$.
2. Determine $\mathbf{A}^{-1}$ and use it directly for the solution of the system of equations.

3. Solve the system of equations by reduction of the augmented matrix to the reduced row echelon form as in eNote 2 followed by a reading of the solution.

In order to show what is actually going on in Cramer's solution formula we first define the ***adjoint matrix*** for a matrix **A**:

|||| Definition 9.27 The Adjoint Matrix

The *(classical) adjoint matrix* adj(**A**) (also called the *adjugate matrix*) is defined by the elements that are used in the definition 9.6 of the determinant of **A** :

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} (-1)^{1+1}\det(\widehat{\mathbf{A}}_{11}) & \cdot & \cdot & (-1)^{1+n}\det(\widehat{\mathbf{A}}_{1n}) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (-1)^{n+1}\det(\widehat{\mathbf{A}}_{n1}) & \cdot & \cdot & (-1)^{n+n}\det(\widehat{\mathbf{A}}_{nn}) \end{bmatrix}^{\top} \tag{9-36}$$

In other words: The element in entry (j,i) in the adjoint matrix adj(**A**) is the sign-modified determinant of the (i,j) submatrix, that is: $(-1)^{i+j}\det(\widehat{\mathbf{A}}_{ij})$. Notice the use of the transpose in (9-36).

|||| Example 9.28 An Adjoint Matrix

In example 9.5 we looked at the following matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \\ 0 & 5 & 1 \end{bmatrix} . \tag{9-37}$$

The matrix **A** has the following adjoint matrix:

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} -7 & 3 & 1 \\ -1 & 0 & 1 \\ 5 & 0 & -2 \end{bmatrix} , \tag{9-38}$$

that is obtained directly from earlier computations of the determinants of the submatrices, remembering that each element is given a sign that depends on the 'entry', and that the expression (9-36) is to be transposed.

$$\begin{aligned} &\det(\widehat{\mathbf{A}}_{11}) = -7\,, \det(\widehat{\mathbf{A}}_{12}) = 1\,, \quad \det(\widehat{\mathbf{A}}_{13}) = 5\,, \\ &\det(\widehat{\mathbf{A}}_{21}) = -3\,, \det(\widehat{\mathbf{A}}_{22}) = 0\,, \quad \det(\widehat{\mathbf{A}}_{23}) = 0\,, \\ &\det(\widehat{\mathbf{A}}_{31}) = 1\,, \quad \det(\widehat{\mathbf{A}}_{32}) = -1\,, \det(\widehat{\mathbf{A}}_{33}) = -2\,. \end{aligned} \tag{9-39}$$

|||| Exercise 9.29 Adjoint Versus Inverse Matrix

Show that all square matrices $\mathbf{A}$ fulfil the following

$$\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{E} \tag{9-40}$$

such that the inverse matrix to $\mathbf{A}$ (which exists precisely if $\det(\mathbf{A}) \neq 0$) can be found in the following way:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\,\mathrm{adj}(\mathbf{A}) \quad . \tag{9-41}$$

Hint: The exercise it not easy. It is recommended to practice on a (2×2)-matrix. The zeroes of the identity matrix in equation (9-40) are obtained by using the property that the determinant of a matrix with two identical columns is 0.

The proof of theorem 9.24 is now rather short:

|||| Proof

By multiplying both sides of equation (9-30) with $\mathbf{A}^{-1}$ we get:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{\det(\mathbf{A})}\,\mathrm{adj}(\mathbf{A})\mathbf{b} \quad , \tag{9-42}$$

and thus – if we denote the elements in $\mathrm{adj}(\mathbf{A})$, α_{ij} :

$$\begin{bmatrix} x_1 \\ \cdot \\ x_n \end{bmatrix} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} \alpha_{11} & \cdot & \alpha_{1n} \\ \cdot & \cdot & \cdot \\ \alpha_{n1} & \cdot & \alpha_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ \cdot \\ b_n \end{bmatrix} \quad . \tag{9-43}$$

From this we read directly

$$\begin{aligned} x_j &= \frac{1}{\det(\mathbf{A})} \sum_{i=1}^{n} \alpha_{ji} b_i \\ &= \frac{1}{\det(\mathbf{A})} \sum_{i=1}^{n} (-1)^{i+j} b_i \det(\widehat{\mathbf{A}}_{ij}) \\ &= \frac{1}{\det(\mathbf{A})} \det(\mathbf{A}\dagger_j^{\mathbf{b}}) \,, \end{aligned} \tag{9-44}$$

where we in the establishment of the last equality sign have used that

$$\sum_{i=1}^{n} (-1)^{i+j} b_i \det(\widehat{\mathbf{A}}_{ij}) \tag{9-45}$$

is exactly the expansion of $\det(\mathbf{A}\dagger_j^{\mathbf{b}})$ along *column* number j, that is the expansion along the **b**-column in $\det(\mathbf{A}\dagger_j^{\mathbf{b}})$, see the definition in equation (9-10).

■