eNote 8

Square Matrices

In this eNote we explore the basic characteristics of the set of square matrices and introduce the notion of the inverse of certain square matrices. We presume that the reader has a knowledge of basic matrix operations, see e.g. eNote 7, Matrices and Matrix Algebra.

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Square matrices are simply matrices with *equal number of rows and columns*, that is they are of the type $n \times n$. This note will introduce some of the basic operations that apply to square matrices.

A square $n \times n$ matrix $\mathbf{A}$ looks like this:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{8-1}$$

The elements $a_{11}, a_{22}, \dots, a_{nn}$ are said to be placed on the ***main diagonal*** or just the *diagonal* of $\mathbf{A}$.

A square matrix $\mathbf{D}$, the non-zero elements of which lie exclusively on the main diagonal, is termed a ***diagonal matrix***, and one can write $\mathbf{D} = \mathbf{diag}(a_{11}, a_{22}, \dots, a_{nn})$.

A ***symmetric matrix*** $\mathbf{A}$ is a square matrix that is equal to its own transpose, thus $\mathbf{A} = \mathbf{A}^\top$.

The square matrix with 1's in the main diagonal and zeroes elsewhere, is called the

identity matrix regardless of the number of rows and columns. The identity matrix is here denoted **E**, (more commonly in the literature as **I**). Accordingly

$$\mathbf{E} = \mathbf{E}_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{8-2}$$

Internationally accepted usage is to denote the identity matrix by **I**.

Theorem 8.1 Identity Matrix

The identity matrix **E** in $\mathbb{R}^{n\times n}$ is the only matrix in $\mathbb{R}^{n\times n}$ that satisfies the following relations:

$$\mathbf{AE} = \mathbf{EA} = \mathbf{A} \tag{8-3}$$

for an arbitrary matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Proof

Suppose another matrix **D**, satisfies the same relations as **E**, that is $\mathbf{AD} = \mathbf{DA} = \mathbf{A}$ for all $n \times n$ matrices **A**. This arbitrary matrix **A** could be the identity matrix, combining the two equations we get : $\mathbf{D} = \mathbf{ED} = \mathbf{DE} = \mathbf{E}$.

Since $\mathbf{E} = \mathbf{D}$ there is no other matrix than the identity matrix **E** that can be a neutral element for the matrix product.

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The identity matrix can be regarded as the "1 for matrices": A scalar is not altered by multiplication by 1, likewise a matrix is not altered by the matrix product of the matrix with the identity matrix of the same type.

As is evident from the following it is often crucial for the manipulation of square matrices whether they have full rank or not. Therefore we now introduce a special concept to express this.

|||| **Definition 8.2 Invertible and Singular Matrix**

A square matrix is called *regular* or *non-singular* (or ***invertible***) if it is of full rank, that is $\rho(\mathbf{A}_{n\times n}) = n$.

A square matrix is called ***singular*** if it not of full rank, that is $\rho(\mathbf{A}_{n\times n}) < n$.

8.1 Inverse Matrix

The reciprocal of a scalar $a \neq 0$ satisfies the following equation: $a \cdot x = 1$, where x is the reciprocal. This can be rewritten as $x = a^{-1}$. This idea will now be generalized to square matrices. Notice that you cannot determine the reciprocal of a scalar a if $a = 0$. A similar exception emerges when we generalize to square matrices.

In order to determine the "reciprocal matrix" to a matrix $\mathbf{A}$, termed the inverse matrix, a *matrix equation* similar to $a \cdot x = 1$ for a scalar:

$$\mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{A} = \mathbf{E} \tag{8-4}$$

The unknown $\mathbf{X}$ is a matrix. If there is a solution $\mathbf{X}$, it is denoted $\mathbf{A}^{-1}$ and is called the ***inverse matrix*** to $\mathbf{A}$. Hence we wish to find a certain matrix called $\mathbf{A}^{-1}$, for which the matrix product of $\mathbf{A}$ with this matrix yields the identity matrix.

It is not all square matrices that possess an inverse. This is postulated in the following theorem.

|||| **Theorem 8.3 Inverse Matrix**

A square matrix $\mathbf{A}_{n\times n}$ has an inverse matrix $\mathbf{A}^{-1}$, that satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{E}$, if and only if $\mathbf{A}$ is non-singular.

The inverse matrix is uniquely determined by the solution of the matrix equation $\mathbf{A}\mathbf{X} = \mathbf{E}$, where $\mathbf{X}$ is the unknown.

Note: this is why a non-singular square matrix is also called *invertible*.

In the following method it is explained, how the matrix equation described above (8-4) is solved, and thus how the inverse of an invertible matrix is found.

|||| Method 8.4 Determining the Inverse Matrix

You determine the inverse matrix denoted $\mathbf{A}^{-1}$, for the invertible square matrix $\mathbf{A}$ by use of the *matrix equation*

$$\mathbf{AX} = \mathbf{E}. \tag{8-5}$$

The equation is solved with respect to the unknown $\mathbf{X}$ in the following way:

1. The augmented matrix $\mathbf{T} = \left[\,\mathbf{A} \mid \mathbf{E}\,\right]$ is formed.
2. By ordinary Gauss-Jordan elimination the reduced row echelon form $\mathrm{rref}(\mathbf{T})$ of $\mathbf{T}$ is determined.
3. In the elimination process the identity matrix is finally formed on the left hand side of the vertical line, while the solution (the inverse of $\mathbf{A}$) can be read on the right hand side: $\mathrm{rref}(\mathbf{T}) = \left[\,\mathbf{E} \mid \mathbf{X}\,\right] = \left[\,\mathbf{E} \mid \mathbf{A}^{-1}\,\right]$.

|||| Example 8.5 Inverse Matrix

We wish to find the inverse matrix $\mathbf{A}^{-1}$ to the matrix $\mathbf{A}$, given in this way:

$$\mathbf{A} = \begin{bmatrix} -16 & 9 & -10 \\ 9 & -5 & 6 \\ 2 & -1 & 1 \end{bmatrix} \tag{8-6}$$

This can be done using method 8.4. First the augmented matrix is formed:

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} -16 & 9 & -10 & 1 & 0 & 0 \\ 9 & -5 & 6 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \tag{8-7}$$

Now we form the leading 1 in the first row: First the row operation $R_1 + R_2$ and then $R_1 + 4 \cdot R_3$. This yields

$$\left[\begin{array}{ccc|ccc} -7 & 4 & -4 & 1 & 1 & 0 \\ 9 & -5 & 6 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 4 \\ 9 & -5 & 6 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \tag{8-8}$$

Then the numbers in the 1st column of the 2nd and 3rd row are eliminated: $R_2 - 9 \cdot R_1$ and

$R_3 - 2 \cdot R_1$. Furthermore the 2nd and 3rd rows are swapped: $R_2 \leftrightarrow R_3$. We then get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 4 \\ 0 & -5 & 6 & -9 & -8 & -36 \\ 0 & -1 & 1 & -2 & -2 & -7 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 4 \\ 0 & -1 & 1 & -2 & -2 & -7 \\ 0 & -5 & 6 & -9 & -8 & -36 \end{array}\right] \tag{8-9}$$

Now we change the sign in row 2: $(-1) \cdot R_2$ and then we eliminate the number in the 2nd column of the 3rd row: $R_3 + 5 \cdot R_2$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 4 \\ 0 & 1 & -1 & 2 & 2 & 7 \\ 0 & -5 & 6 & -9 & -8 & -36 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 4 \\ 0 & 1 & -1 & 2 & 2 & 7 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right] \tag{8-10}$$

The last step is then evident: $R_2 + R_3$:

$$\operatorname{rref}(\mathbf{T}) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & 3 & 4 & 6 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right] \tag{8-11}$$

We see that $\rho(\mathbf{A}) = \rho(\mathbf{T}) = 3$, thus $\mathbf{A}$ is of full rank, and therefore one can read the inverse to $\mathbf{A}$ on the right hand side of the vertical line:

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 1 & 4 \\ 3 & 4 & 6 \\ 1 & 2 & -1 \end{bmatrix} \tag{8-12}$$

Notice that the left hand side of the augmented matrix is the identity matrix. It is so to speak "moved" from the right to the left hand side of the equality signs (the vertical line).

Finally we check whether $\mathbf{A}^{-1}$, as is expected, satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{E}$ and$\mathbf{A}^{-1}\mathbf{A} = \mathbf{E}$:

$$\begin{aligned} \mathbf{A}\mathbf{A}^{-1} &= \begin{bmatrix} -16 & 9 & -10 \\ 9 & -5 & 6 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 4 \\ 3 & 4 & 6 \\ 1 & 2 & -1 \end{bmatrix} \\ &= \left[\begin{bmatrix} -16 & 9 & -10 \\ 9 & -5 & 6 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} -16 & 9 & -10 \\ 9 & -5 & 6 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} -16 & 9 & -10 \\ 9 & -5 & 6 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ -1 \end{bmatrix} \right] \\ &= \begin{bmatrix} -16+27-10 & -16+36-20 & -64+54+10 \\ 9-15+6 & 9-20+12 & 36-30-6 \\ 2-3+1 & 2-4+2 & 8-6-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E} \end{aligned} \tag{8-13}$$

It is true! By use of the same procedure it is seen that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{E}$ is also true.

As can be seen in the next example, the inverse can be used in the solution of matrix equations with square matrices. In matrix equations one can *interchange terms* and *multiply by scalars* in order to isolate the unknown just as one would do in ordinary scalar equations. Moreover one can *multiply all terms* by matrices – this can be done either from right or the left on all terms in the equation, yielding different results.

|||| Example 8.6 Matrix Equation

We wish to solve the matrix equation

$$\mathbf{AX} = \mathbf{B} - \mathbf{CX} \tag{8-14}$$

where

$$\mathbf{A} = \begin{bmatrix} -4 & 2 & -1 \\ 9 & 5 & -5 \\ 2 & 0 & 7 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 8 & -12 & 5 \\ 5 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} -12 & 7 & -9 \\ 0 & -10 & 11 \\ 0 & -1 & -6 \end{bmatrix} \tag{8-15}$$

First the equation is reduced as far as possible, see e.g. Theorem 7.13, without using the values:

$$\mathbf{AX} = \mathbf{B} - \mathbf{CX} \Leftrightarrow \mathbf{AX} + \mathbf{CX} = \mathbf{B} - \mathbf{CX} + \mathbf{CX} \Leftrightarrow (\mathbf{A} + \mathbf{C})\mathbf{X} = \mathbf{B} \tag{8-16}$$

Since $\mathbf{X}$ is the unknown we try to isolate this matrix totally. If $(\mathbf{A} + \mathbf{C})$ is an invertible matrix, one can multiply by the inverse to $(\mathbf{A} + \mathbf{C})$ from the left on both sides of the equality sign. Thus:

$$(\mathbf{A} + \mathbf{C})^{-1}(\mathbf{A} + \mathbf{C})\mathbf{X} = (\mathbf{A} + \mathbf{C})^{-1}\mathbf{B} \Leftrightarrow \mathbf{EX} = \mathbf{X} = (\mathbf{A} + \mathbf{C})^{-1}\mathbf{B}, \tag{8-17}$$

because $(\mathbf{A} + \mathbf{C})^{-1}(\mathbf{A} + \mathbf{C}) = \mathbf{E}$ according to the definition of inverse matrices. We now form $\mathbf{A} + \mathbf{C}$ and determine whether the matrix is invertible:

$$\mathbf{A} + \mathbf{C} = \begin{bmatrix} -4 & 2 & -1 \\ 9 & 5 & -5 \\ 2 & 0 & 7 \end{bmatrix} + \begin{bmatrix} -12 & 7 & -9 \\ 0 & -10 & 11 \\ 0 & -1 & -6 \end{bmatrix} = \begin{bmatrix} -16 & 9 & -10 \\ 9 & -5 & 6 \\ 2 & -1 & 1 \end{bmatrix} \tag{8-18}$$

The inverse of this matrix is already determined in Example 8.5, and this part of the procedure is therefor skipped. $\mathbf{X}$ is determined as:

$$\begin{aligned} \mathbf{X} = (\mathbf{A} + \mathbf{C})^{-1}\mathbf{B} &= \begin{bmatrix} -16 & 9 & -10 \\ 9 & -5 & 6 \\ 2 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 8 & -12 & 5 \\ 5 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 4 \\ 3 & 4 & 6 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 8 & -12 & 5 \\ 5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 28 & -11 & 5 \\ 62 & -45 & 20 \\ 11 & -23 & 10 \end{bmatrix} \end{aligned} \tag{8-19}$$

In the further investigation of the invertibility of the transpose or the inverse of an invertible matrix plus the invertibility of the product of two or more invertible matrices we will need the following corollary, which is stated without proof (see eNote 9, in particular, Theorem 9.20 for one way to prove it).

|||| Lemma 8.7 Inherited Invertibility

1. If $\mathbf{A}$ is an invertible square matrix, both $\mathbf{A}^\top$ and $\mathbf{A}^{-1}$ are invertible.

2. The product $\mathbf{A}\,\mathbf{B}$ of two square matrices is invertible if and only if both $\mathbf{A}$ and $\mathbf{B}$ are invertible.

We can now give arithmetic rules for inverse matrices.

|||| Theorem 8.8 Arithmetic Rules for Inverse Matrices

For the invertible square matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ the following arithmetic rules apply:

1. The inverse of the inverse of a matrix is equal to the matrix itself:

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A} \tag{8-20}$$

2. The transpose of an inverse matrix is equal to the inverse of the transpose of the matrix:

$$(\mathbf{A}^\top)^{-1} = (\mathbf{A}^{-1})^\top \tag{8-21}$$

   $\mathbf{A}^\top$ is invertible if and only if $\mathbf{A}$ is invertible.

3. In matrix equations we can multiply all terms by the inverse of a matrix. This can be done either from the right or the left hand side on both sides of the equality sign:

$$\mathbf{AX} = \mathbf{B} \Leftrightarrow \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} \quad \text{and} \quad \mathbf{XC} = \mathbf{D} \Leftrightarrow \mathbf{X} = \mathbf{DC}^{-1} \tag{8-22}$$

4. The inverse of a matrix product of two matrices is equal to the product of the corresponding inverse matrices in reverse order:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{8-23}$$

All the arithmetic rules in theorem 8.8 are easily proven by checking.

Below one of the rules is tested in an example. The arithmetic rule in equation (8-22) has already been used in example 8.6.

|||| Example 8.9 Checking of Arithmetic Rule for an Inverse Matrix

Two square matrices are given

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \tag{8-24}$$

We wish to test the last arithmetic rule in theorem 8.8, viz. that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. First $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ are determined by use of method 8.4.

$$[\,\mathbf{A} \mid \mathbf{E}\,] = \left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 6 & 10 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -\frac{5}{2} & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right] \tag{8-25}$$

Similarly with $\mathbf{B}$:

$$[\,\mathbf{B} \mid \mathbf{E}\,] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right] \tag{8-26}$$

Since we have obtained the identity matrix on the left hand side of the vertical line in both cases , we get

$$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{5}{2} & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad \mathbf{B}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \tag{8-27}$$

$\mathbf{B}^{-1}\mathbf{A}^{-1}$ is determined:

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \left[\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{5}{2} \\ \frac{3}{2} \end{bmatrix} \; \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \right] = \begin{bmatrix} -9 & \frac{7}{2} \\ \frac{13}{2} & -\frac{5}{2} \end{bmatrix} \tag{8-28}$$

On the other side of the equality sign in the arithmetic rule we first calculate $\mathbf{AB}$:

$$\mathbf{AB} = \left[\begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \; \begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right] = \begin{bmatrix} 10 & 14 \\ 26 & 36 \end{bmatrix} \tag{8-29}$$

Now the inverse of $\mathbf{AB}$ is determined:

$$[\,\mathbf{AB} \mid \mathbf{E}\,] = \left[\begin{array}{cc|cc} 10 & 14 & 1 & 0 \\ 26 & 36 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{7}{5} & \frac{1}{10} & 0 \\ 0 & -\frac{2}{5} & -\frac{13}{5} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -9 & \frac{7}{2} \\ 0 & 1 & \frac{13}{2} & -\frac{5}{2} \end{array}\right] \tag{8-30}$$

Finally we arrive at

$$(\mathbf{AB})^{-1} = \begin{bmatrix} -9 & \frac{7}{2} \\ \frac{13}{2} & -\frac{5}{2} \end{bmatrix}, \tag{8-31}$$

Comparison of equations (8-28) and (8-31) yields the identity: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

|||| Exercise 8.10 Inverse Matrix

Given the (not square!) matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \tag{8-32}$$

a) Determine $(\mathbf{BA})^{-1}$.

b) Show that $\mathbf{AB}$ is not invertible and therefore one cannot determine $(\mathbf{AB})^{-1}$.

|||| Exercise 8.11 Inverse Matrix

Given the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \tag{8-33}$$

a) Calculate $\mathbf{AC}$, $\mathbf{BD}$ and $\mathbf{DC}$.

b) Determine if possible, $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$ and $(\mathbf{AB})^{-1}$.

c) Is it possible to decide whether $(\mathbf{AB})^{-1}$ exists after you have tried to determine $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$? If yes, how?

8.2 Powers of Matrices

We have now seen how the inverse of an invertible matrix is determined and we say that it has the power -1. Similarly we define arbitrary integer ***powers of square matrices***.

|||| Definition 8.12 Powers of a Matrix

For an arbitrary square matrix $\mathbf{A}$ the following natural powers are defined:

$$\mathbf{A}^0 = \mathbf{E} \quad \text{and} \quad \mathbf{A}^n = \overbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}^{n \text{ times}}, \text{ for } n \in \mathbb{N} \tag{8-34}$$

Furthermore for an arbitrary **invertible** square matrix $\mathbf{B}$ the negative powers are defined:

$$\mathbf{B}^{-n} = (\mathbf{B}^{-1})^n = \overbrace{\mathbf{B}^{-1}\mathbf{B}^{-1}\cdots\mathbf{B}^{-1}}^{n \text{ times}}, \text{ for } n \in \mathbb{N} \tag{8-35}$$

As a consequence of the definition of powers, some arithmetic rules can be given.

|||| Theorem 8.13 Arithmetic Rules for Powers of Matrices

For an arbitrary square matrix $\mathbf{A}$ and two arbitrary non-negative integers a and b the following arithmetic rules for powers are valid

$$\mathbf{A}^a\mathbf{A}^b = \mathbf{A}^{a+b} \quad \text{and} \quad (\mathbf{A}^a)^b = \mathbf{A}^{ab} \tag{8-36}$$

If $\mathbf{A}$ is invertible, these arithmetic rules are also valid for negative integers a and b.

Below is an example of two (simple) matrices that possess some funny characteristics. The characteristics are *not* typical for matrices!

|||| Example 8.14 Two Funny Matrices with Respect to Powers

Given the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \tag{8-37}$$

By use of both the Definition 8.12 and Theorem 8.13 the following calculations are performed. $\mathbf{A}^2$ is determined:

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{E} \tag{8-38}$$

Following the addendum to the fourth arithmetic rule in Theorem 8.8, $\mathbf{A}$ is invertible, and moreover $\mathbf{A} = \mathbf{A}^{-1}$. This gives

$$\begin{array}{ll} \vdots & \vdots \\ \mathbf{A}^{-3} = (\mathbf{A}\mathbf{A}^2)^{-1} = \mathbf{A}^{-1} = \mathbf{A} & \mathbf{A}^{-2} = (\mathbf{A}^2)^{-1} = \mathbf{E} \\ \mathbf{A}^{-1} = \mathbf{A} & \mathbf{A}^{0} = \mathbf{E} \\ \mathbf{A}^{1} = \mathbf{A} & \mathbf{A}^{2} = \mathbf{E} \\ \vdots & \vdots \end{array} \tag{8-39}$$

Thus all odd powers of $\mathbf{A}$ give $\mathbf{A}$, while even powers give the identity matrix:

$$\mathbf{A}^{2n} = \mathbf{E} \;\text{ and }\; \mathbf{A}^{2n+1} = \mathbf{A} \;\text{ for }\; n \in \mathbb{Z} \tag{8-40}$$

$\mathbf{B}^2$ is determined:

$$\mathbf{B}^2 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0} \tag{8-41}$$

According to the same arithmetic rule $\mathbf{B}$ is singular. Then it follows

$$\mathbf{B}^0 = \mathbf{E}\,,\; \mathbf{B}^1 = \mathbf{B}\,,\; \mathbf{B}^2 = \mathbf{0}\,,\; \mathbf{B}^n = \mathbf{0} \;\text{ for }\; n \geq 2 \tag{8-42}$$

8.3 Summary

- Square matrices are matrices where the number of rows equals the number of columns.
- The unit matrix $\mathbf{E}$ is a square matrix with the number one in the diagonal and zeros elsewhere:

$$\mathbf{E} = \mathbf{E}_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{8-43}$$

- If a square matrix has full rank, it is called regular, otherwise it is called singular.
- A square matrix, the entries of which are all zero except for those on the diagonal, is called a diagonal matrix.
- A square matrix, that is equal to the transpose of itself, is called a symmetric matrix.
- For a *regular* matrix $\mathbf{A}$ there exists a unique inverse, denoted $\mathbf{A}^{-1}$, satisfying:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{E} \tag{8-44}$$

  The inverse can be determined by Method 8.4.
- Rules of computation with square and inverse matrices exist, see Theorem 8.8.
- Powers of suare matrices are defined, see Definition 8.12. In addition some arithmetic rules exist.
- Inverse matrices are e.g. used in connection with *change of basis* and the *eigenvalue problem*. Moreover the *determinant* of a square matrix is defined in eNote 9, *Determinants*.