eNote 7

Matrices and Matrix Algebra

This eNote introduces matrices and arithmetic operations for matrices and deduces the relevant arithmetic rules. Math knowledge comparable to that of a Danish gymnasium (high school) graduate is the only requirement for benefitting from this note, but it is a good idea to acquaint oneself with the number space $\mathbb{R}^n$ that is described in eNote 5 The Number Spaces*.*

(Updated: 24.09.2021 David Brander)

A ***matrix*** is an array of numbers. Here is an example of a matrix called $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} 1 & 4 & 3 \\ -1 & 2 & 7 \end{bmatrix} \tag{7-1}$$

A matrix is characterized by the number of *rows* and *columns*, and the matrix $\mathbf{M}$ above is therefore called a 2×3 matrix. The matrix $\mathbf{M}$ is said to contain $2 \cdot 3 = 6$ *elements*. In addition to rows and columns a number of further concepts are connected. In order to describe these we write a general matrix, here called $\mathbf{A}$, as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \tag{7-2}$$

The matrix $\mathbf{A}$ has m rows and n columns, and this can indicated by writing $\mathbf{A}_{m \times n}$ or the *$m \times n$ matrix* $\mathbf{A}$. The matrix $\mathbf{A}$ is also said to be *of the type $m \times n$*.

Two $m \times n$-matrices $\mathbf{A}$ and $\mathbf{B}$ are called *equal* if the elements in each matrix are equal, and we then write $\mathbf{A} = \mathbf{B}$.

A matrix with a single column ($n = 1$), is called a *column matrix*. Similarly a matrix with only one row ($m = 1$), a *row matrix*.

A matrix with the same number of row and columns ($m = n$), is called a *square matrix*. Square matrices are investigated in depth in eNote 8 *Square Matrices*.

If all the elements in an $m \times n$-matrix are real numbers, the matrix is called a *real matrix*. The set of these matrices is denoted $\mathbb{R}^{m \times n}$.

7.1 Matrix Sum and the Product of a Matrix by a Scalar

It is possible to add two matrices if they are of the same type. You then add the elements with the same row and column numbers and in this way form a new matrix of the same type. Similarly you can multiply any matrix by a scalar (a number), this is done by multiplying all the elements by the scalar. The matrix in which all elements are equal to 0 is called the *zero matrix* regardless of the type, and is denoted $\mathbf{0}$ or possibly $\mathbf{0}_{m \times n}$. In these notes, all other matrices are called *proper matrices*.

|||| Definition 7.1 Matrix Sum and Multiplication by a Scalar

Given a scalar $k \in \mathbb{R}$ and two real matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{m\times n}$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{og} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} \tag{7-3}$$

The *sum* of the matrices is defined as:

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix} \tag{7-4}$$

The sum is only defined when the matrices are of the same type.

The *product* of the matrix $\mathbf{A}$ by the scalar k is written $k\mathbf{A}$ or $\mathbf{A}k$ and is defined as:

$$k\mathbf{A} = \mathbf{A}k = \begin{bmatrix} k\cdot a_{11} & k\cdot a_{12} & \dots & k\cdot a_{1n} \\ k\cdot a_{21} & k\cdot a_{22} & \dots & k\cdot a_{2n} \\ \vdots & \vdots & & \vdots \\ k\cdot a_{m1} & k\cdot a_{m2} & \dots & k\cdot a_{mn} \end{bmatrix} \tag{7-5}$$

The *opposite matrix* $-\mathbf{A}$ (additive inverse) to a matrix $\mathbf{A}$ is defined by the matrix that results when all the elements in $\mathbf{A}$ are multiplied by -1. It is seen that $-\mathbf{A} = (-1)\mathbf{A}$.

|||| Example 7.2 Simple Matrix Operations

Define two matrices $\mathbf{A}$ and$\mathbf{B}$ by:

$$\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 8 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -4 & 3 \\ 9 & \frac{1}{2} \end{bmatrix} \tag{7-6}$$

The matrices are both of the type 2×2. We wish to determine a third and fourth matrix

$\mathbf{C} = 4\mathbf{A}$ and $\mathbf{D} = 2\mathbf{A} + \mathbf{B}$. This can be done through the use of the definition 7.1.

$$\begin{aligned} \mathbf{C} = 4\mathbf{A} = 4 \cdot \begin{bmatrix} 4 & -1 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} 4 \cdot 4 & 4 \cdot (-1) \\ 4 \cdot 8 & 4 \cdot 0 \end{bmatrix} = \begin{bmatrix} 16 & -4 \\ 32 & 0 \end{bmatrix} \\ \mathbf{D} = 2\mathbf{A} + \mathbf{B} = \begin{bmatrix} 8 & -2 \\ 16 & 0 \end{bmatrix} + \begin{bmatrix} -4 & 3 \\ 9 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 25 & \frac{1}{2} \end{bmatrix} \end{aligned} \tag{7-7}$$

In the following theorem we summarize the arithmetic rules that are valid for sums of matrices and multiplication by a scalar.

|||| Theorem 7.3 Arithmetic Rules for the Matrix Sum and the Product by a Scalar

For arbitrary matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in $\mathbb{R}^{m \times n}$ and likewise arbitrary real numbers k_1 and k_2 the following arithmetic rules are valid:

1.	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	Addition is commutative
2.	$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$	Addition is associative
3.	$\mathbf{A} + \mathbf{0} = \mathbf{A}$	$\mathbf{0}$ is a neutral matrix for addition in $\mathbb{R}^{m \times n}$
4.	$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$	Every matrices in $\mathbb{R}^{m \times n}$ has an opposite matrix
5.	$k_1(k_2\mathbf{A}) = (k_1 k_2)\mathbf{A}$	Product of a matrix by scalars is associative
6.	$(k_1 + k_2)\mathbf{A} = k_1\mathbf{A} + k_2\mathbf{A}$	} The distributive rules are valid
7.	$k_1(\mathbf{A} + \mathbf{B}) = k_1\mathbf{A} + k_1\mathbf{B}$	
8.	$1\mathbf{A} = \mathbf{A}$	The scalar 1 is neutral in the product by a matrix

The arithmetic rules in Theorem 7.3 can be proved by applying the ordinary arithmetic rules for real numbers. The method is demonstrated for two of the rules in the following example.

|||| Example 7.4 Demonstration of Arithmetic Rule

Given the two matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{7-8}$$

plus the constants k_1 and k_2. We now try by way of example to show the distributive rules in

Theorem 7.3. First we have:

$$\begin{aligned} (k_1+k_2)\mathbf{A} &= (k_1+k_2)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} (k_1+k_2)a_{11} & (k_1+k_2)a_{12} \\ (k_1+k_2)a_{21} & (k_1+k_2)a_{22} \end{bmatrix} \\ k_1\mathbf{A}+k_2\mathbf{A} &= \begin{bmatrix} k_1a_{11} & k_1a_{12} \\ k_1a_{21} & k_1a_{22} \end{bmatrix} + \begin{bmatrix} k_2a_{11} & k_2a_{12} \\ k_2a_{21} & k_2a_{22} \end{bmatrix} = \begin{bmatrix} k_1a_{11}+k_2a_{11} & k_1a_{12}+k_2a_{12} \\ k_1a_{21}+k_2a_{21} & k_1a_{22}+k_2a_{22} \end{bmatrix} \end{aligned} \tag{7-9}$$

If you take a_{11}, a_{12}, a_{21} anda_{22} outside the parentheses in each of the elements in the last expression, it is seen that $(k_1+k_2)\mathbf{A} = k_1\mathbf{A}+k_2\mathbf{A}$ in this case. The operation of taking the a-elements outside the parentheses is exactly equivalent to be using the distributive rule for the real numbers.

The second distributive rule is demonstrated for given matrices and constants:

$$\begin{aligned} k_1(\mathbf{A}+\mathbf{B}) &= k_1\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} = \begin{bmatrix} k_1(a_{11}+b_{11}) & k_1(a_{12}+b_{12}) \\ k_1(a_{21}+b_{21}) & k_1(a_{22}+b_{22}) \end{bmatrix} \\ k_1\mathbf{A}+k_1\mathbf{B} &= \begin{bmatrix} k_1a_{11} & k_1a_{12} \\ k_1a_{21} & k_1a_{22} \end{bmatrix} + \begin{bmatrix} k_1b_{11} & k_1b_{12} \\ k_1b_{21} & k_1b_{22} \end{bmatrix} = \begin{bmatrix} k_1a_{11}+k_1b_{11} & k_1a_{12}+k_1b_{12} \\ k_1a_{21}+k_1b_{21} & k_1a_{22}+k_1b_{22} \end{bmatrix} \end{aligned} \tag{7-10}$$

If k_1 is taken outside of the parenthesis in each of the elements in the matrix in the last expression it is seen that the second distributive rule also is valid in this case: $k_1(\mathbf{A}+\mathbf{B}) = k_1\mathbf{A}+k_1\mathbf{B}$. The distributive rule for real numbers is again used for each element.

Note that the zero matrix in $\mathbb{R}^{m\times n}$ is the only matrix $\mathbb{R}^{m\times n}$ that is neutral with respect to addition, and that $-\mathbf{A}$ is the only solution to the equation $\mathbf{A}+\mathbf{X}=\mathbf{0}$.

|||| Definition 7.5 Difference Between Matrices

The difference $\mathbf{A}-\mathbf{B}$ between two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same type is introduced by:

$$\mathbf{A}-\mathbf{B} = \mathbf{A}+(-\mathbf{B}). \tag{7-11}$$

In other words $\mathbf{B}$ is subtracted from $\mathbf{A}$ by subtracting each element in $\mathbf{B}$ from the corresponding element in $\mathbf{A}$.

|||| Example 7.6 Simple Matrix Operation with Difference

With the matrices given in Example 7.2 we get

$$\mathbf{D} = 2\mathbf{A} - \mathbf{B} = 2\mathbf{A} + (-1)\mathbf{B} = \begin{bmatrix} 8 & -2 \\ 16 & 0 \end{bmatrix} + \begin{bmatrix} 4 & -3 \\ -9 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 12 & -5 \\ 7 & -\frac{1}{2} \end{bmatrix} \tag{7-12}$$

7.2 Matrix-Vector Products and Matrix-Matrix Products

In this subsection we describe the multiplication of a matrix with a vector and then the multiplication of matrix by another matrix.

A vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ can be written as a column matrix and is then called a *column vector*:

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{7-13}$$

Using this concept you can divide a matrix $\mathbf{A}_{m\times n}$ into its column vectors. This is written like this:

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} & \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} & \ldots & \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \end{aligned} \tag{7-14}$$

Accordingly there are n column vectors with m elements each.

Notice that the square brackets around the column vectors can be removed just like that! This can be done in all dealings with matrices, where double square brackets occur. It is always the innermost brackets that are removed. In this way there is no difference between the two expressions. The last expression is always preferred, because it is the easier to read.

We now define the product of a matrix and a vector, in which the matrix has as many columns as the vector has elements:

|||| Definition 7.7 Matrix-Vector Product

Let $\mathbf{A}$ be an arbitrary matrix in $\mathbb{R}^{m\times n}$, and let $\mathbf{v}$ be an arbitrary vector in $\mathbb{R}^n$.

The *matrix-vector product* of $\mathbf{A}$ with $\mathbf{v}$ is defined as:

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots + v_n\mathbf{a}_n \end{bmatrix} \tag{7-15}$$

The result is a column vector with m elements. It is the sum of the products of the k'th column in the matrix and the k'th element in the column vector for all $k = 1, 2, \ldots, n$.

It is necessary that there are as many columns in the matrix as there are rows in the column vector, here n.

Notice the order in the matrix-vector product: first matrix, then vector! It is not a vector-matrix product so to speak. The number of rows and columns will not match in the other configuration unless the matrix is of the type 1×1.

|||| Example 7.8 Matrix-Vector Product

The following matrix and vector (a column vector) are given:

$$\mathbf{A} = \mathbf{A}_{2\times 3} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}. \tag{7-16}$$

We now form the matrix-vector product of $\mathbf{A}$ with $\mathbf{v}$ by use of definition 7.7:

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} = \left[3\begin{bmatrix} a \\ d \end{bmatrix} + 4\begin{bmatrix} b \\ e \end{bmatrix} + (-1)\begin{bmatrix} c \\ f \end{bmatrix} \right] = \begin{bmatrix} 3a + 4b - c \\ 3d + 4e - f \end{bmatrix} \tag{7-17}$$

If $\mathbf{A}$ is given like this

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix}, \tag{7-18}$$

you get the product

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} 3\cdot(-1) + 4\cdot 2 - 6 \\ 3\cdot 2 + 4\cdot 1 - 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \tag{7-19}$$

It is seen that the result (in both cases) is a column vector with as many rows as there are rows in **A**.

|||| Exercise 7.9 Matrix-Vector Product

Form the matrix-vector product **A** with **x** in the equation $\mathbf{Ax} = \mathbf{b}$, when it is given that

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad , \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{7-20}$$

Is this something you have seen before? From where does it come?

As we have remarked above a matrix can be viewed as a number of column vectors aligned after one another. This is used in the following definition of a matrix-matrix product as a series of matrix-vector products.

|||| Definition 7.10 Matrix-Matrix Product

Let **A** be an arbitrary matrix in $\mathbb{R}^{m \times n}$, and let **B** be an arbitrary matrix in $\mathbb{R}^{n \times p}$.

The *matrix-matrix product* or just the ***matrix product*** of **A** and **B** is defined like this:

$$\mathbf{AB} = \mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \dots & \mathbf{Ab}_p \end{bmatrix} \tag{7-21}$$

The result is matrix of type $m \times p$. The k'th column in the resulting matrix is a matrix-vector product of the first matrix (here **A**) and the k'th column vector in the last matrix (here **B**), cf. definition 7.7.

There must be as many columns in the first matrix as there are rows in the last matrix.

|||| Example 7.11 Matrix-Matrix Product

Given two matrices $\mathbf{A}_{2\times 2}$ and $\mathbf{B}_{2\times 3}$:

$$\mathbf{A} = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} \quad \text{og} \quad \mathbf{B} = \begin{bmatrix} -8 & 3 & 3 \\ 2 & 9 & -9 \end{bmatrix} \tag{7-22}$$

We wish to form the matrix-matrix product of **A** and **B**. This is done by use of definition 7.10.

$$\begin{aligned}\mathbf{AB} &= \left[\begin{bmatrix}4 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}-8\\2\end{bmatrix}\ \begin{bmatrix}4 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}3\\9\end{bmatrix}\ \begin{bmatrix}4 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}3\\-9\end{bmatrix}\right]\\ &= \begin{bmatrix}4\cdot(-8)+5\cdot 2 & 4\cdot 3+5\cdot 9 & 4\cdot 3+5\cdot(-9)\\ -8+2\cdot 2 & 3+2\cdot 9 & 3+2\cdot(-9)\end{bmatrix} = \begin{bmatrix}-22 & 57 & -33\\ -4 & 21 & -15\end{bmatrix}\end{aligned} \tag{7-23}$$

NB: It is *not* possible to form the matrix-matrix product **BA**, because there are not as many columns in **B** as there are rows in **A** ($3 \neq 2$).

|||| Example 7.12 Matrix-Matrix Product two Ways

Given the two matrices $\mathbf{A}_{2\times 2}$ and $\mathbf{B}_{2\times 2}$:

$$\mathbf{A} = \begin{bmatrix}3 & 2\\-5 & 1\end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix}4 & 4\\-1 & 0\end{bmatrix} \tag{7-24}$$

Because the two matrices are square matrices of the same type both matrix-matrix products **AB** and **BA** can be calculated. We use the definition 7.10.

$$\begin{aligned}\mathbf{AB} &= \left[\begin{bmatrix}3 & 2\\-5 & 1\end{bmatrix}\begin{bmatrix}4\\-1\end{bmatrix}\ \begin{bmatrix}3 & 2\\-5 & 1\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix}\right]\\ &= \begin{bmatrix}3\cdot 4+2\cdot(-1) & 3\cdot 4+2\cdot 0\\ -5\cdot 4+1\cdot(-1) & -5\cdot 4+1\cdot 0\end{bmatrix} = \begin{bmatrix}10 & 12\\-21 & -20\end{bmatrix}\\ \mathbf{BA} &= \left[\begin{bmatrix}4 & 4\\-1 & 0\end{bmatrix}\begin{bmatrix}3\\-5\end{bmatrix}\ \begin{bmatrix}4 & 4\\-1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\right]\\ &= \begin{bmatrix}4\cdot 3+4\cdot(-5) & 4\cdot 2+4\\ -1\cdot 3 & -1\cdot 2\end{bmatrix} = \begin{bmatrix}-8 & 12\\-3 & -2\end{bmatrix}\end{aligned} \tag{7-25}$$

We see that $\mathbf{AB} \neq \mathbf{BA}$. The factors are **not** interchangeable!

Here we summarize the arithmetic rules that apply to matrix-matrix products and matrix sums. Because the matrix-vector product is a special case of the matrix-matrix product, the rules also apply for matrix-vector products.

|||| Theorem 7.13 Arithmetic Rules for Matrix Sum and Product

For arbitrary matrices **A**, **B** and **C** and likewise an arbitrary real number k the following arithmetic rules are valid, in so far as the matrix-matrix products can be formed:

$$(k\mathbf{A})\mathbf{B} = \mathbf{A}(k\mathbf{B}) = k(\mathbf{A}\mathbf{B}) \quad \text{Product with a scalar is associative}$$

$$\left.\begin{aligned} \mathbf{A}(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\mathbf{B}+\mathbf{A}\mathbf{C} \\ (\mathbf{A}+\mathbf{B})\mathbf{C} &= \mathbf{A}\mathbf{C}+\mathbf{B}\mathbf{C} \end{aligned}\right\} \text{the distributive rules apply}$$

$$\mathbf{A}(\mathbf{B}\mathbf{C}) = (\mathbf{A}\mathbf{B})\mathbf{C} \quad \text{Matrix-matrix products are associative}$$

Analogous to the demonstration of the arithmetic rules in Theorem 7.3 we demonstrate the last arithmetic rule in Theorem 7.13:

|||| Example 7.14 Are Matrix Products Associative?

The last arithmetic rule in 7.13 is tested on the three matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} , \quad \mathbf{B} = \begin{bmatrix} -3 & -2 & -1 \\ 0 & 0 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 4 & -5 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \tag{7-26}$$

First we calculate **AB** and **BC**:

$$\begin{aligned} \mathbf{AB} &= \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \end{bmatrix} \ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} \ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 7 \end{bmatrix} \right] = \begin{bmatrix} -3 & -2 & 13 \\ -9 & -6 & 25 \end{bmatrix} \\ \mathbf{BC} &= \left[\begin{bmatrix} -3 & -2 & -1 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \ \begin{bmatrix} -3 & -2 & -1 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix} \right] = \begin{bmatrix} -17 & 16 \\ 7 & -21 \end{bmatrix} \end{aligned} \tag{7-27}$$

Then we calculate $\mathbf{A}(\mathbf{BC})$ and $(\mathbf{AB})\mathbf{C}$:

$$\begin{aligned} \mathbf{A}(\mathbf{BC}) &= \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -17 \\ 7 \end{bmatrix} \ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 16 \\ -21 \end{bmatrix} \right] = \begin{bmatrix} -3 & -26 \\ -23 & -36 \end{bmatrix} \\ (\mathbf{AB})\mathbf{C} &= \left[\begin{bmatrix} -3 & -2 & 13 \\ -9 & -6 & 25 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \ \begin{bmatrix} -3 & -2 & 13 \\ -9 & -6 & 25 \end{bmatrix}\begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix} \right] = \begin{bmatrix} -3 & -26 \\ -23 & -36 \end{bmatrix} \end{aligned} \tag{7-28}$$

We see that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$, and therefore it doesn't matter which of the matrix products **AB** and **BC** we calculate first. This is valid for all matrices (although not proved here).

As is done in example 7.14 we can demonstrate the other arithmetic rules. By writing down carefully the formula for each element of a matrix in the final product, in terms of the elements of the other matrices, one can prove the rules properly.

|||| Exercise 7.15 Demonstration of Arithmetic Rule

Demonstrate the first arithmetic rule in Theorem 7.13 with two real matrices $\mathbf{A}_{2\times 2}$ and $\mathbf{B}_{2\times 2}$ and the constant k.

7.3 Transpose of a Matrix

By interchanging rows and columns in a matrix the ***transpose matrix*** is formed as in this example:

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \text{has the transpose} \quad \mathbf{A}^\top = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \tag{7-29}$$

$\mathbf{A}^\top$ is '$\mathbf{A}$ *transpose*'. In addition you have that $(\mathbf{A}^\top)^\top = \mathbf{A}$. Here is a useful arithmetic rule for the transpose of a matrix-matrix product.

|||| Theorem 7.16 Transpose of a Matrix

Let there be given two arbitrary matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{n\times p}$. You form the transposed matrices , $\mathbf{A}^\top$ and $\mathbf{B}^\top$ respectively, by interchanging the columns and rows of each matrix.

The transpose of a matrix-matrix product $\mathbf{AB}$ is equal to the matrix-matrix product of $\mathbf{B}^\top$ with $\mathbf{A}^\top$ (that is, in reverse order):

$$(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top \tag{7-30}$$

In the following example Theorem 7.16 is tested.

|||| Example 7.17 Demonstration of Theorem 7.16

Given the two matrices $\mathbf{A} = \begin{bmatrix} 0 & 1 & 6 \\ 7 & -3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 9 & 1 \\ 1 & 0 \\ -6 & 3 \end{bmatrix}$. Then

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} \begin{bmatrix} 0 & 1 & 6 \\ 7 & -3 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \\ -6 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 6 \\ 7 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 1\cdot 1 - 6\cdot 6 & 6\cdot 3 \\ 7\cdot 9 - 3\cdot 1 - 2\cdot 6 & 7\cdot 1 + 2\cdot 3 \end{bmatrix} = \begin{bmatrix} -35 & 18 \\ 48 & 13 \end{bmatrix} \end{aligned} \tag{7-31}$$

We now try to form the matrix-matrix product $\mathbf{B}^\top \mathbf{A}^\top$ and we find

$$\mathbf{A}^\top = \begin{bmatrix} 0 & 7 \\ 1 & -3 \\ 6 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^\top = \begin{bmatrix} 9 & 1 & -6 \\ 1 & 0 & 3 \end{bmatrix} \tag{7-32}$$

and then

$$\begin{aligned} \mathbf{B}^\top \mathbf{A}^\top &= \begin{bmatrix} \begin{bmatrix} 9 & 1 & -6 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix} & \begin{bmatrix} 9 & 1 & -6 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 1\cdot 1 - 6\cdot 6 & 9\cdot 7 - 1\cdot 3 - 6\cdot 2 \\ 3\cdot 6 & 1\cdot 7 + 3\cdot 2 \end{bmatrix} = \begin{bmatrix} -35 & 48 \\ 18 & 13 \end{bmatrix} \end{aligned} \tag{7-33}$$

The two results look identical:

$$\begin{bmatrix} -35 & 18 \\ 48 & 13 \end{bmatrix}^\top = \begin{bmatrix} -35 & 48 \\ 18 & 13 \end{bmatrix} \quad \Leftrightarrow \quad (\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top , \tag{7-34}$$

in agreement with Theorem 7.16

|||| Exercise 7.18 Matrix Product and the Transpose

Given the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \tag{7-35}$$

Calculate if possible the following:

a) $2\mathbf{A} - 3\mathbf{B}$, b) $2\mathbf{A}^\top - 3\mathbf{B}^\top$, c) $2\mathbf{A} - 3\mathbf{B}^\top$, d) $\mathbf{AB}$,
e) $\mathbf{AB}^\top$, f) $\mathbf{BA}^\top$, g) $\mathbf{B}^\top\mathbf{A}$, h) $\mathbf{A}^\top\mathbf{B}$.

7.4 Summary

- Matrices are arrays characterized by the number of *columns* and *rows*, determining the *type* of the matrix. An entry in the matrix is called an *element*.
- The type of a matrix is denoted as: $\mathbf{A}_{m\times n}$. The matrix $\mathbf{A}$ has m rows and n columns.
- Matrices can be multiplied by a scalar by multiplying each element in the matrix by the scalar.
- Matrices can be added if they are of the same type. This is done by adding corresponding elements.
- The matrix-vector product, of $\mathbf{A}_{m\times n}$ with the vector $\mathbf{v}$ with n elements, is defined as:

$$\mathbf{A}_{m\times n}\mathbf{v} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 v_1 + \mathbf{a}_2 v_2 + \dots + \mathbf{a}_n v_n \end{bmatrix}, \tag{7-36}$$

  where $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are the *column vectors* in $\mathbf{A}$.

- The matrix-matrix product (or just the matrix product) is defined as a series of matrix-vector products:

$$\mathbf{A}_{m\times n}\mathbf{B}_{n\times p} = \mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{b}_1 & \mathbf{A}\mathbf{b}_2 & \dots & \mathbf{A}\mathbf{b}_p \end{bmatrix} \tag{7-37}$$

- More arithmetic rules for matrix sums, matrix products and matrix-scalar products are found in Theorem 7.3 and Theorem 7.13.
- The *transpose* $\mathbf{A}^\top$ of a matrix $\mathbf{A}$ is determined by interchanging rows and columns in the matrix.