eNote 5

The Number Spaces $\mathbb{R}^n$ and $\mathbb{C}^n$

This eNote is about the real number space $\mathbb{R}^n$ and the complex number space $\mathbb{C}^n$, which are essential building blocks in Linear Algebra.

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5.1 Number Spaces

Remark 5.1 The Common Notion $\mathbb{L}$

Defnitions and rules in this eNote are valid both for the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$. The set of real numbers and the set of complex numbers are examples of *fields*. Fields have common calculation rules concerning elementary arithmetic rules (the same rules as those for $\mathbb{C}$ described in Theorem 1.12 in eNote 1). In the following when we use the symbol $\mathbb{L}$ it means that the notion is valid both for the set of real numbers and for the set of complex numbers.

$\mathbb{R}^n$ is the symbol for the set of all n-tuples that contain n real elements. For example,

$$(1,4,5) \text{ and } (1,5,4)$$

are two different 3-tuples that belong to $\mathbb{R}^3$. Similarly $\mathbb{C}^n$ is the symbol for the set of all n-tuples which contains n complex elements, e.g.

$$(1+2i,0,3i,1,1) \text{ and } (1,2,3,4,5)$$

are two different 5-tuples that belong to $\mathbb{C}^5$. Formally we write $\mathbb{L}^n$ in set notation as:

$$\mathbb{L}^n = \{(a_1, a_2, ..., a_n) \mid a_i \in \mathbb{L}\}. \tag{5-1}$$

We introduce addition of elements in $\mathbb{L}^n$ and multiplication of elements in $\mathbb{L}^n$ by an element of $\mathbb{L}$ (a scalar) by the following definition:

|||| Definition 5.2

Let $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ be two elements of $\mathbb{L}^n$ and let k be a number in $\mathbb{L}$ (a scalar). *The sum* of the two n-tuples is defined by

$$(a_1, a_2, ..., a_n) + (b_1, b_2, ..., b_n) = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n), \tag{5-2}$$

and *the product* of $(a_1, a_2, ..., a_n)$ by k by

$$k \cdot (a_1, a_2, ..., a_n) = (a_1, a_2, ..., a_n) \cdot k = (k \cdot a_1, k \cdot a_2, ..., k \cdot a_n). \tag{5-3}$$

$\mathbb{R}^n$ with the operations (5-2) and (5-3) is called the n-dimensional *real number space*. Similarly, $\mathbb{C}^n$ with the operations (5-2) and (5-3), the n-dimensional *complex number space*.

|||| Example 5.3 Addition

An example of the addition of two 4-tuples in $\mathbb{R}^4$ is

$$(1, 2, 3, 4) + (2, 1, -2, -5) = (3, 3, 1, -1)$$

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|||| Example 5.4 Multiplication

Denitio An example of multiplication of a 3-tuple in $\mathbb{R}^3$ by a scalar is

$$5 \cdot (2, 4, 5) = (10, 20, 25).$$

An example of multiplication of a 2-tuple in $\mathbb{C}^2$ by a scalar is

$$i \cdot (2 + i, 4) = (-1 + 2i, 4i).$$

As a short notation for n-tuples we often use small **bold** letters, we write e.g.

$$\mathbf{a} = (3,2,1) \text{ or } \mathbf{b} = (b_1, b_2, ..., b_n)\,.$$

For the n-tuple $(0,0,...,0)$, which is called *the zero element* of $\mathbb{L}^n$, we use the notion

$$\mathbf{0} = (0,0,...,0)\,.$$

When more complicated computational exercises in the number spaces are called for, there is a need for the following arithmetic rules.

|||| Theorem 5.5 Arithmetic Rules in $\mathbb{L}^n$

For all values of n, in the number space $\mathbb{L}^n$ the operations introduced in definition 5.2 obey the following eight rules:

1. $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$ (addition is commutative)
2. $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$ (addition is associative)
3. For all $\mathbf{a}$: $\mathbf{a}+\mathbf{0} = \mathbf{a}$ (i.e. $\mathbf{0}$ is neutral with respect to addition)
4. For all $\mathbf{a}$ there exists an *opposite element* $-\mathbf{a}$ such that $\mathbf{a}+(-\mathbf{a}) = \mathbf{0}$
5. $k_1(k_2\mathbf{a}) = (k_1k_2)\mathbf{a}$ (multiplication by scalars is associative)
6. $(k_1+k_2)\mathbf{a} = k_1\mathbf{a}+k_2\mathbf{a}$ (distributive rule)
7. $k_1(\mathbf{a}+\mathbf{b}) = k_1\mathbf{a}+k_1\mathbf{b}$ (distributive rule)
8. $1\mathbf{a} = \mathbf{a}$ (the number 1 is neutral in a product with a scalar)

|||| Proof

Concerning rule 4: Given two vectors $\mathbf{a} = (a_1, \ldots a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$. Then

$$\mathbf{a}+\mathbf{b} = (a_1+b_1\,, \ldots,\, a_n+b_n) = \mathbf{0} \Leftrightarrow b_1 = -a_1\,, \ldots,\, b_n = -a_n\,.$$

From this we deduce that $\mathbf{a}$ has an opposite vector $-\mathbf{a}$ given by $-\mathbf{a} = (-a_1\,, \ldots, -a_n)$. Moreover, this vector is unique.

The other rules are proved by calculating the left and right hand side of the equations and then comparing the two results.

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From the proof of rule 4 in theorem 5.5 it is evident that for an arbitrary n-tuple $\mathbf{a}$: $-\mathbf{a} = (-1)\mathbf{a}$.

|||| Exercise 5.6

Give a formal proof of rule 2 and rule 5 in Theorem 5.5.

|||| Definition 5.7 Subtraction

Given $\mathbf{a} \in \mathbb{L}^n$ and $\mathbf{b} \in \mathbb{L}^n$. The difference $\mathbf{a} - \mathbf{b}$ is defined as:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) . \tag{5-4}$$

|||| Example 5.8 Subtraction

$$(1+2i,1) - (i,2) = (1+2i,1) + (-(i,2)) = (1+2i,1) + (-i,-2) = (1+i,-1) .$$

|||| Exercise 5.9 The Zero Rule

Show that the following variant of the *zero rule* is valid:

$$k\mathbf{a} = \mathbf{0} \Leftrightarrow k = 0 \text{ or } \mathbf{a} = \mathbf{0} . \tag{5-5}$$

|||| Remark 5.10 n-Tuples as Vectors

Often an n-tuple is written as a ***column vector***. We have two equivalent ways of writing, here with an example from $\mathbb{R}^4$:

$$\mathbf{v} = (1,2,3,4) \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

If in a given context the n-tuple is regarded as a ***row vector*** then a transposition is performed. The transpose of a column vector is a row vector (and vice versa), it has the symbol T:

$$\mathbf{v}^\top = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}.$$