eNote 4

Taylor's Approximation Formulas for Functions of One Variable

In eNotes 19 and 21 it is shown how functions of one and two variables can be approximated by first-degree polynomials at every (development) point and that the graphs for the approximating first-degree polynomial are exactly the tangents and the tangent planes, respectively, for the corresponding graphs of the functions. In this eNote we will show how the functions can be approximated even better by polynomials of higher degree, so if the approximation to a function is sufficiently good then one can use and continue the computations with the approximation polynomial in place of the function itself and hope for a sufficiently small error. But what does it mean that the approximation and the error are sufficiently good *and* sufficiently small*? And how does this depend on the degree of the approximating polynomial? You will find the answers to these questions in this eNote.*
(Updated: 22.09.2021 David Brander).

4.1 Higher Order Derivatives

First we consider functions $f(x)$ of one variable x on an open interval of the real numbers. We will also assume that the functions can be differentiated an arbitrary number of times, that is, all the derivatives exist for every x in the interval: $f'(x_0)$, $f''(x_0)$, $f'''(x_0)$, $f^{(4)}(x_0)$, $f^{(5)}(x_0)$, etc. where $f^{(4)}(x_0)$ means the 4th derivative of $f(x)$ in x_0. These higher order derivatives we will use in the construction of (the coefficients to) the approximating polynomials.

|||| Definition 4.1

If a function $f(x)$ can be differentiated an abitrary number of times at every point x in a given open interval I we say that the function is ***smooth*** on the interval I.

|||| Example 4.2 Higher-Order Derivatives of Some Elementary Functions

Here are some higher-order derivatives of some well-known smooth functions:

$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(5)}(x)$
e^x	e^x	e^x	e^x	e^x	e^x
x^2	$2x$	2	0	0	0
x^3	$3x^2$	$6x$	6	0	0
x^4	$4x^3$	$12x^2$	$24x$	24	0
x^5	$5x^4$	$20x^3$	$60x^2$	$120x$	120
$(x-x_0)^5$	$5\cdot(x-x_0)^4$	$20\cdot(x-x_0)^3$	$60\cdot(x-x_0)^2$	$120\cdot(x-x_0)$	120
$\cos(x)$	$-\sin(x)$	$-\cos(x)$	$\sin(x)$	$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$	$-\sin(x)$	$-\cos(x)$	$\sin(x)$	$\cos(x)$
$\cosh(x)$	$\sinh(x)$	$\cosh(x)$	$\sinh(x)$	$\cosh(x)$	$\sinh(x)$
$\sinh(x)$	$\cosh(x)$	$\sinh(x)$	$\cosh(x)$	$\sinh(x)$	$\cosh(x)$

(4-1)

Note that

1. The nth derivative $f^{(n)}(x)$ of the function $f(x) = (x - x_0)^n$ is

$$f^{(n)}(x) = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n! \quad , \tag{4-2}$$

   Where $n!$ (n factorial) is the short way of writing the product of the natural numbers from and including 1 to and including n, cf. Table 4.2 where $n!$ appears as $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$. Note: by definition $0! = 1$, so $n!$ is well-defined for non-negative integers.

2. By repeated differentiation of $\cos(x)$ we get the same set of functions periodically with the period 4: If $f(x) = \cos(x)$ then

$$f^{(p)}(x) = f^{(p+4)}(x) \quad \text{for all } p \geq 1 \quad . \tag{4-3}$$

   The same applies for $f(x) = \sin(x)$.

3. By repeated differentiation of the hyperbolic cosine function $\cosh(x)$ we again get the same "set" of functions periodically with the period 2: If $f(x) = \cosh(x)$ we get

$$f^{(p)}(x) = f^{(p+2)}(x) \quad \text{for all } p \geq 1 \quad . \tag{4-4}$$

   This applies to the hyperbolic sine function $f(x) = \sinh(x)$, too.

|||| Example 4.3 The Derivatives of a Somewhat Less Elementary Function

A function $f(x)$ can e.g. be given as an integral (that in this case can be expressed by the ordinary elementary functions):

$$f(x) = \int_0^x e^{-t^2}\, dt \quad . \tag{4-5}$$

But we can easily find the higher order derivatives of the function for every x:

$$f'(x) = e^{-x^2} \quad , \quad f''(x) = -2 \cdot x \cdot e^{-x^2} \quad , \quad f'''(x) = -2 \cdot e^{-x^2} + 4 \cdot x^2 \cdot e^{-x^2} \quad \text{etc.} \tag{4-6}$$

|||| Example 4.4 The Derivatives of an Unknown Function

We assume that a function $f(x)$ is given as a solution to a differential equation with the initial conditions at x_0:

$$f''(x) + 3f'(x) + 7f(x) = q(x) \quad , \quad \text{where} \quad f(x_0) = 1 \quad , \quad \text{and} \quad f'(x_0) = -3 \tag{4-7}$$

where $q(x)$ is a given smooth function of x. Again we can fairly easily find the higher order derivatives of the function at x_0 by using the initial conditions directly and by *differentiating the differential equation*. We get the following from the initial conditions and from the differential equation itself:

$$f'(x_0) = -3 \quad , \quad f''(x_0) = q(x_0) - 3f'(x_0) - 7f(x_0) = q(x_0) + 2 \quad . \tag{4-8}$$

The third (and the higher-order) derivatives of $f(x)$ we then obtain by differentiating both sides of the differential equation. E.g. by differentiating once we get:

$$f'''(x) + 3f''(x) + 7f'(x) = q'(x) \quad , \tag{4-9}$$

from which we get:

$$\begin{aligned} f'''(x_0) &= q'(x_0) - 3f''(x_0) - 7f'(x_0) \\ &= q'(x_0) - 3 \cdot (q(x_0) + 2) - 7 \cdot (-3) \\ &= q'(x_0) - 3q(x_0) + 15 \quad . \end{aligned} \tag{4-10}$$

4.2 Approximations by Polynomials

The point of the following to find the polynomial of degree n (e.g. the second-degree polynomial) that best approximates a given smooth function $f(x)$ at and around a given x_0 in the domain of the function $D(f)$.

For the case $n = 2$, we try to write $f(x)$ in the following way:

$$f(x) = a_0 + a_1 \cdot (x - x_0) + a_2 \cdot (x - x_0)^2 + R_{2,x_0}(x) \quad , \tag{4-11}$$

where a_0, a_1, and a_2 are suitable constants that are to be chosen so that the ***remainder function*** also known as the Lagrange remainder term $R_{2,x_0}(x)$ is as small as possible at and around x_0. The remainder function we can express by $f(x)$ and the polynomial we are testing:

$$R_{2,x_0}(x) = f(x) - a_0 - a_1 \cdot (x - x_0) - a_2 \cdot (x - x_0)^2 \quad , \tag{4-12}$$

and it is this function that should be as close as possible to 0 when x is close to x_0 such that the difference between the function $f(x)$ and the second-degree polynomial becomes as small as possible – at least in the vicinity of x_0.

The first natural requirement is therefore that:

$$R_{2,x_0}(x_0) = 0 \quad \text{corresponding to} \quad f(x_0) = a_0 \quad , \tag{4-13}$$

by which a_0 is now determined.

The next natural requirement is that the graph of the remainder function has horizontal gradient at x_0 such that the tangent to the remainder function then is identical to the x axis:

$$R'_{2,x_0}(x_0) = 0 \quad \text{such that} \quad f'(x_0) = a_1 \quad , \tag{4-14}$$

by which a_1 is determined.

The next requirement on the remainder function is then:

$$R''_{2,x_0}(x_0) = 0 \quad \text{corresponding to} \quad f''(x_0) = 2 \cdot a_2 \quad , \tag{4-15}$$

by which also $a_2 = \frac{1}{2} f''(x_0)$ then is determined and fixed.

Thus we have found

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} \cdot (x - x_0) + \frac{f''(x_0)}{2!} \cdot (x - x_0)^2 + R_{2,x_0}(x) \tag{4-16}$$

Where the remainder function $R_{2,x_0}(x)$ satisfies the following requirement that makes it very small in the neighborhood of x_0:

$$R_{2,x_0}(x_0) = R'_{2,x_0}(x_0) = R''_{2,x_0}(x_0) = 0 \quad . \tag{4-17}$$

If similarly we had wished to find an approximating n'th degree polynomial for the same function $f(x)$ we would have found:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} \cdot (x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n + R_{n,x_0}(x) \quad , \tag{4-18}$$

where the remainder function $R_{n,x_0}(x)$ is a smooth function that satisfies all the requirements:

$$R_{n,x_0}(x_0) = R'_{n,x_0}(x_0) = \cdots = R^{(n)}_{n,x_0}(x_0) = 0 \quad . \tag{4-19}$$

At this point it is reasonable to expect, on one hand, that these requirements on the remainder functions can be satisfied; on the other, that the remainder function itself must 'appear like' and be as small as a power of $(x - x_0)$ close to x_0.

This is precisely the content of the following Lemma:

|||| Lemma 4.5 Remainder Functions

The remainder function $R_{n,x_0}(x)$ can be expressed from $f(x)$ in two different ways, and we will use both in what follows:

$$R_{n,x_0}(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \cdot (x - x_0)^{n+1} \quad , \tag{4-20}$$

where $\xi(x)$ lies between x and x_0 in the interval I.

The other way is the following one, that contains an epsilon function:

$$R_{n,x_0}(x) = (x - x_0)^n \cdot \varepsilon_f(x - x_0) \quad , \tag{4-21}$$

where $\varepsilon_f(x - x_0)$ is an epsilon function of $(x - x_0)$.

|||| Proof

We will content ourselves by proving the first statement (4-20) in the simplest case, viz. for $n = 0$, i.e. the following : On the interval between (a fixed) x and x_0 we can always find a value ξ such that the following applies:

$$R_{0,x_0}(x) = f(x) - f(x_0) = \frac{f'(\xi)}{(1)!} \cdot (x - x_0) \quad . \tag{4-22}$$

But this is only a form of the ***mean value theorem***: If a smooth function has values $f(a)$ and $f(b)$, respectively, at the end points of an interval $[a, b]$, then the graph for $f(x)$ has at some position a tangent that is parallel to the line segment connecting the two points $(a, f(a))$ and $(b, f(b))$, see Figure 4.1.

The other statement (4-21) follows from the first (4-20) by observing that $\frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot (x - x_0)^{n+1}$ is an epsilon function of $(x - x_0)$, since $\frac{f^{(n+1)}(\xi)}{(n+1)!}$ is bounded and since $(x - x_0)^{n+1}$ is itself an epsilon function.

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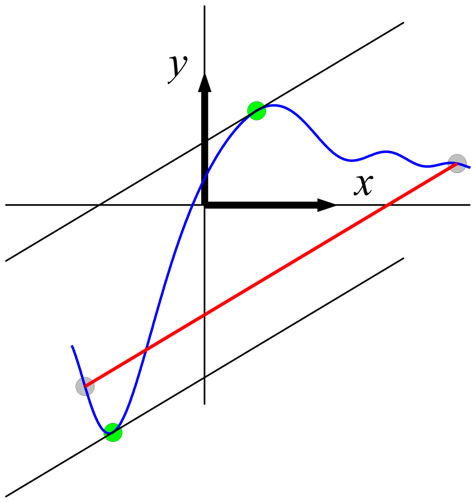

Figure 4.1: Two points on the blue graph curve for a function are connected with a line segment (red). The mean value theorem then says that at least one position exists (in the case shown, exactly two positions, marked in green) on the curve between the two given points where the slope $f'(x)$ for the tangent (black) to the curve is exactly the same as the slope of the straight line segment.

|||| Definition 4.6 Approximating Polynomials

Let $f(x)$ denote a smooth function on an interval I. The polynomial

$$P_{n,x_0}(x) = f(x_0) + \frac{f'(x_0)}{1!} \cdot (x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n \tag{4-23}$$

is called the ***approximating polynomial*** of nth degree for the function $f(x)$ with development point x_0.

To sum up:

|||| Theorem 4.7 Taylor's Formulas

Every smooth function $f(x)$ can for every non-negative integer n be divided into an approximating polynomial of degree n and a remainder function like this:

$$f(x) = P_{n,x_0}(x) + R_{n,x_0}(x) \quad , \tag{4-24}$$

where the remainder function can be expressed in the following ways:

$$R_{n,x_0}(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \cdot (x - x_0)^{n+1} \quad \text{for the } \xi(x) \text{ between } x \text{ and } x_0$$

and

$$R_{n,x_0}(x) = (x - x_0)^n \cdot \varepsilon_f(x - x_0) \quad . \tag{4-25}$$

In particular it is Taylor's Limit Formula (where the remainder function is expressed by an epsilon function) that we will make use of in what follows. We mention this version explicitly:

|||| Theorem 4.8 Taylor's Limit Formula

Let $f(x)$ denote a smooth function on an open interval I that contains a given x_0. Then for all x in the interval and for every integer $n \geq 0$ the following applies

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} \cdot (x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n + (x - x_0)^n \cdot \varepsilon_f(x - x_0) \quad ,$$

where $\varepsilon_f(x - x_0)$ denotes an epsilon function of $(x - x_0)$, i.e. $\varepsilon_f(x - x_0) \to 0$ for $x \to x_0$.

|||| Example 4.9 The Approximating Polynomials of a Polynomial

One might be led to believe that every polynomial is its own approximating polynomial because every polynomial must be the best approximation to itself. Here is an example that shows that this is *not* that simple. We look at the third-degree polynomial

$$f(x) = 1 + x + x^2 + x^3 \quad . \tag{4-26}$$

The polynomial $f(x)$ has the following quite different approximating polynomials - dependent on the choice of *development point* x_0 and *degree of development* n:

$$\begin{aligned}
P_{7,x_0=0}(x) &= 1 + x + x^2 + x^3 \\
P_{3,x_0=0}(x) &= 1 + x + x^2 + x^3 \\
P_{2,x_0=0}(x) &= 1 + x + x^2 \\
P_{1,x_0=0}(x) &= 1 + x \\
P_{0,x_0=0}(x) &= 1 \\
P_{7,x_0=1}(x) &= 1 + x + x^2 + x^3 \\
P_{3,x_0=1}(x) &= 1 + x + x^2 + x^3 \\
P_{2,x_0=1}(x) &= 2 - 2 \cdot x + 4 \cdot x^2 \\
P_{1,x_0=1}(x) &= -2 + 6 \cdot x \\
P_{0,x_0=1}(x) &= 4 \\
P_{7,x_0=7}(x) &= 1 + x + x^2 + x^3 \\
P_{3,x_0=7}(x) &= 1 + x + x^2 + x^3 \\
P_{2,x_0=7}(x) &= 344 - 146 \cdot x + 22 \cdot x^2 \\
P_{1,x_0=7}(x) &= -734 + 162 \cdot x \\
P_{0,x_0=7}(x) &= 400 \quad .
\end{aligned} \tag{4-27}$$

|||| Exercise 4.10 Remainder Functions for Polynomials

For the function $f(x) = 1 + x + x^2 + x^3$ we consider the following two splittings into approximating polynomials and corresponding remainder functions:

$$\begin{aligned}
f(x) &= P_{2,x_0=1}(x) + R_{2,x_0=1}(x) \quad \text{and} \\
f(x) &= P_{1,x_0=7}(x) + R_{1,x_0=7}(x) \quad ,
\end{aligned} \tag{4-28}$$

where the two approximating polynomials $P_{2,x_0=1}(x)$ and $P_{1,x_0=7}(x)$ already are stated in example 4.9. Determine the two remainder functions $R_{2,x_0=1}(x)$ and $R_{1,x_0=7}(x)$ expressed in both of the two ways shown in 4-25: For each of the two remainder functions the respective expressions for $\xi(x)$ and for $\varepsilon(x - x_0)$ are stated.

|||| Example 4.11 Taylor's Limit Formula with the Development Point $x_0 = 0$

Here are some often-used functions with their respective approximating polynomials (and corresponding remainder functions expressed by epsilon functions) with the common development point $x_0 = 0$ and arbitrarily high degree:

$$\begin{aligned}
\mathrm{e}^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + x^n \cdot \varepsilon(x) \\
\mathrm{e}^{x^2} &= 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{2n}}{n!} + x^{2n} \cdot \varepsilon(x) \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \cdot \frac{x^{2n}}{(2n)!} + x^{2n} \cdot \varepsilon(x) \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} + x^{2n+1} \cdot \varepsilon(x) \\
\ln(1+x) &= x - \frac{x^2}{2} + \cdots + (-1)^{n-1} \cdot \frac{x^n}{n!} + x^n \cdot \varepsilon(x) \\
\ln(1-x) &= -x - \frac{x^2}{2} - \cdots - \frac{x^n}{n!} - x^n \cdot \varepsilon(x) \\
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots + (-1)^{n-1} \cdot x^{n-1} + x^n \cdot \varepsilon(x) \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n \cdot \varepsilon(x)
\end{aligned} \tag{4-29}$$

Note that in Taylor's Limit formula we always end with an epsilon function and with the power of x that is precisely the same as the last power used in the preceding approximating polynomial.

4.3 Continuous Extensions

The function $f(x) = \sin(x)/x$ is not defined at $x = 0$. We will investigate whether we can extend the function to having a value at 0, such that the extended function is continuous at 0. I.e. we will find a value a such that the a-extension

$$\widetilde{f} = \begin{cases} \frac{\sin(x)}{x} & \text{for} \quad x \neq 0 \\ a & \text{for} \quad x = 0 \end{cases} \tag{4-30}$$

is continuous at $x = 0$, i.e. such that

$$\frac{\sin(x)}{x} \to a \quad \text{for} \quad x \to 0 \quad . \tag{4-31}$$

A direct application of Taylor's limit formula appears in the determination of limit values for those quotients $f(x)/g(x)$ where both the functions, i.e. the numerator $f(x)$ and the denominator $g(x)$, tend towards 0 for x tending towards 0. What happens to the quotient as x tends towards 0? We illustrate with a number of examples. Note that even though the numerator function and the denominator function both are continuous at 0, the quotient needs not be continuous.

|||| Example 4.12 Limit Values for Function Fractions

$$\frac{\sin(x)}{x} = \frac{x + x^1 \cdot \varepsilon(x)}{x} = 1 + \varepsilon(x) \to 1 \quad \text{for} \quad x \to 0 \quad . \tag{4-32}$$

$$\frac{\sin(x)}{x^2} = \frac{x - \frac{1}{3!}x^3 + x^3 \cdot \varepsilon(x)}{x^2} = \frac{1}{x} - \frac{x}{3!} + x \cdot \varepsilon(x) \quad , \tag{4-33}$$

that has no limit value for $x \to 0$. Therefore a continuous extension does not exist in this case.

$$\frac{\sin(x^2)}{x^2} \to 1 \quad \text{for} \quad x \to 0 \quad \text{because} \quad \frac{\sin(u)}{u} \to 1 \quad \text{for} \quad u \to 0 \quad . \tag{4-34}$$

$$\frac{\sin(x) - x}{x^2} = \frac{x - \frac{1}{3!}x^3 + x^3 \cdot \varepsilon(x) - x}{x^2} = -\frac{x}{3!} + x \cdot \varepsilon(x) \to 0 \quad \text{for} \quad x \to 0 \quad . \tag{4-35}$$

$$\frac{\sin(x) - x}{x^3} = \frac{x - \frac{1}{3!}x^3 + x^3 \cdot \varepsilon(x) - x}{x^3} = -\frac{1}{3!} + \cdot\varepsilon(x) \to -\frac{1}{6} \quad \text{for} \quad x \to 0 \quad . \tag{4-36}$$

By determination of such limit values the approximating polynomials in the numerator and the denominator are developed to such a high degree that limit value "appears" by dividing both the numerator and the denominator by a power of x.

Here is a somewhat more complicated example:

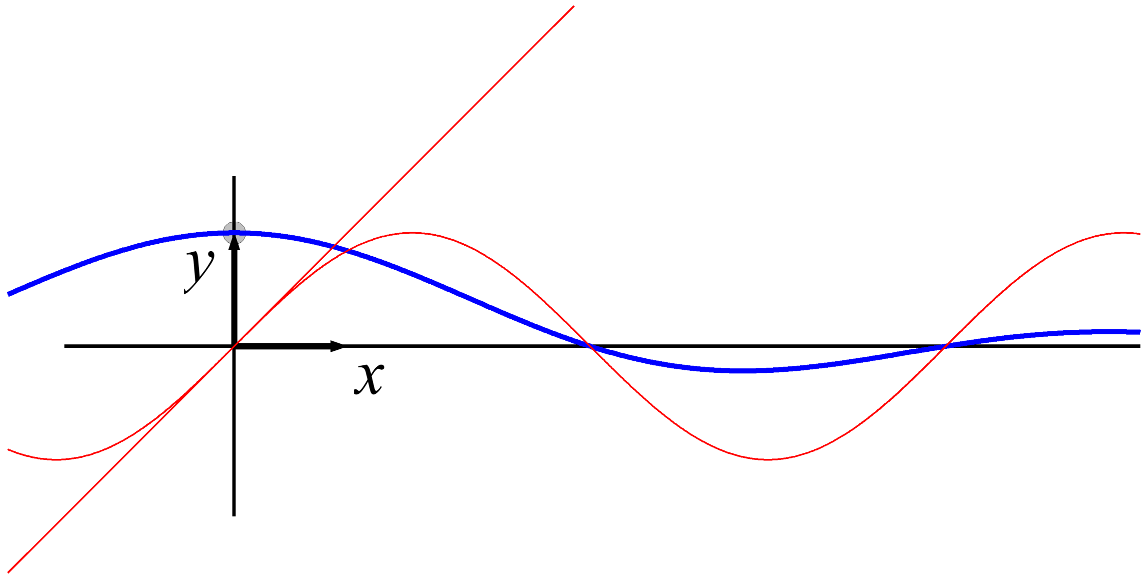

Figure 4.2: The function $f(x) = \sin(x)/x$ (blue) together with the numerator function $\sin(x)$ (red) and the denominator function x (also red). The function $f(x)$ is continuous at $x = 0$ exactly when we use the value $f(1) = 1$.

|||| Example 4.13 The Limit Value for a Fraction Between Functions

$$
\begin{aligned}
\frac{2\cos(x) - 2 + x^2}{x \cdot \sin(x) - x^2} &= \frac{2 \cdot (1 - \frac{1}{2!} \cdot x^2 + \frac{1}{4!} \cdot x^4 + x^4 \cdot \varepsilon_1(x)) - 2 + x^2}{x \cdot (x - \frac{1}{3!} \cdot x^3 + \frac{1}{5!} \cdot x^5 + x^5 \cdot \varepsilon_2(x)) - x^2} \\
&= \frac{2 - x^2 + \frac{1}{12} \cdot x^4 + 2 \cdot x^4 \cdot \varepsilon_1(x) - 2 + x^2}{x^2 - \frac{1}{3!} \cdot x^4 + \frac{1}{5!} \cdot x^6 + x^6 \cdot \varepsilon_2(x) - x^2} \\
&= \frac{\frac{1}{12} \cdot x^4 + 2 \cdot x^4 \cdot \varepsilon_1(x)}{-\frac{1}{3!} \cdot x^4 + \frac{1}{5!} \cdot x^6 + x^6 \cdot \varepsilon_2(x)} \\
&= \frac{\frac{1}{12} + 2 \cdot \varepsilon_1(x)}{-\frac{1}{6} + \frac{1}{5!} \cdot x^2 + x^2 \cdot \varepsilon_2(x)} \\
&\to -\frac{1}{2} \quad \text{for} \quad x \to 0 \quad ,
\end{aligned}
\tag{4-37}
$$

since the numerator tends towards $\frac{1}{12}$ for $x \to 0$ and the denominator tends towards $-\frac{1}{6}$ for $x \to 0$.

4.4 Estimation of the Remainder Functions

How large is the error committed by using the approximating polynomial (which it is easy to compute) instead of the function itself (that can be difficult to compute) on a given (typically small) interval around the development point? The remainder function can of course give the answer to this question. We give here a couple of examples that show how the remainder function can be used for such error estimations for given functions.

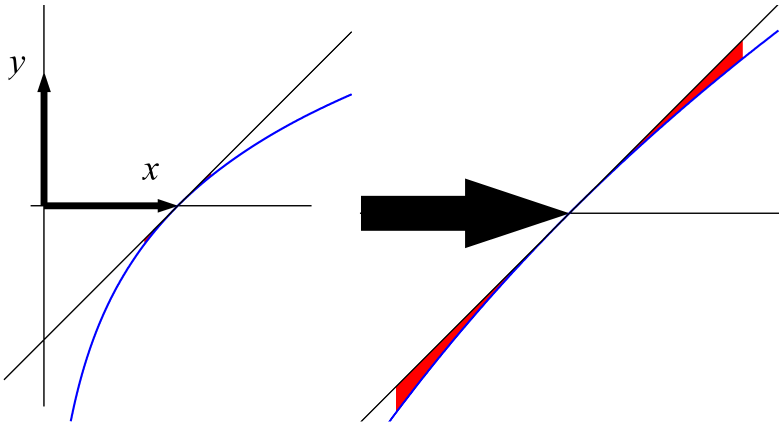

Figure 4.3: The function $f(x) = \ln(x)$ from Example 4.14 (blue), the approximating first-degree polynomial (black) with development point $x_0 = 1$ and the corresponding remainder function (red) illustrated as the difference between $f(x)$ and the approximating polynomial on the interval $\left[\frac{3}{4}, \frac{5}{4}\right]$. To the right is shown the figure around the point $(1, 0)$ close-up.

|||| Example 4.14 Approximation of an Elementary Function

The logarithmic function $\ln(x)$ is defined for positive values of x. We approximate with the approximating first-degree polynomial with the development point at $x_0 = 1$ and will estimate the remainder term on a suitably small interval around $x_0 = 1$, i.e. the starting point is the following:

$$f(x) = \ln(x) \quad , \quad x_0 = 1 \quad , \quad n = 1 \quad , \quad x \in \left[\frac{3}{4}, \frac{5}{4}\right] \quad . \tag{4-38}$$

According to Taylor's formula with the remainder function we have - using the development point $x_0 = 1$ where $f(1) = 0$ and $f'(1) = 1$ and using $f''(x) = -1/x^2$ for all x in the domain:

$$f(x) = \ln(x) = \ln(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(\xi)}{2!} \cdot (x-1)^2 = x - 1 - \frac{1}{2 \cdot \xi^2} \cdot (x-1)^2 \quad \text{(4-39)}$$

for a value of ξ between x and 1. Thus we have found:

$$P_{1,x_0=1}(x) = x - 1 \quad , \quad \text{and} \quad R_{1,x_0=1}(x) = -\frac{1}{2 \cdot \xi^2} \cdot (x-1)^2 \quad . \quad \text{(4-40)}$$

The *absolute value of the remainder function* on the given interval can now be evaluated for all x in the given interval - even if we do not know very much about the position of ξ in the interval apart from the fact that ξ lies between x and 1:

We have

$$|R_{1,x_0=1}(x)| = |-\frac{1}{2 \cdot \xi^2} \cdot (x-1)^2| \leq |\frac{1}{2 \cdot \xi^2} \cdot \left(\frac{1}{4}\right)^2| \quad . \quad \text{(4-41)}$$

Here the minus sign has been removed because we only look at the absolute value and we have also used that $(x-1)^2$ clearly is largest (with the value $(1/4)^2$) for $x = 3/4$ and for $x = 5/4$ in the interval. In addition ξ is *smallest* and thus $(1/\xi)^2$ *largest* on the interval for $\xi = 3/4$. (Note that here we do not use the fact of ξ lying between x and 1 - we simply use the fact of ξ lying in the interval!) I.e.

$$|R_{1,x_0=1}(x)| \leq |\frac{1}{32 \cdot \xi^2}| \leq |\frac{1}{32 \cdot \left(\frac{3}{4}\right)^2}| = \frac{1}{18} \quad , \quad \text{(4-42)}$$

thus we have proved that

$$|\ln(x) - (x-1)| \leq \frac{1}{18} \quad \text{for all} \quad x \in \left[\frac{3}{4}, \frac{5}{4}\right] \quad . \quad \text{(4-43)}$$

One may well wonder why the remainder function estimation of such a simple function as $f(x) = \ln(x)$ in Example 4.14 should be so complicated, when it is evident to everybody (!) that the red remainder function in that case assumes its largest numerical (absolute) value at one of the end points of the actual interval, see Figure 4.3 – a statement, moreover, which we can prove by a quite ordinary function investigation.

By differentiation of the remainder function we get:

$$R'_{1,x_0=1}(x) = \frac{d}{dx}\left(\ln(x) - (x-1)\right) = \frac{1}{x} - 1 \quad , \tag{4-44}$$

that is less than 0 precisely for $x > 1$ (such that $R_{1,x_0=1}(x)$ to the right of $x = 1$ is negative and decreasing from the value 0 at $x = 1$) and greater than 0 for $x < 1$ (such that $R_{1,x_0=1}(x)$ to the left of $x = 1$ is negative and increasing towards the value 0 at $x = 1$). But the problem is that we *in principle do not know* what the value of $\ln(x)$ in fact is – neither at $x = 3/4$ nor at $x = 5/4$ unless we use Maple or some other tool as help. The remainder function estimate uses *only* the defined properties of $f(x) = \ln(x)$, i.e. $f'(x) = 1/x$ and $f(1) = 0$ and the estimation gives the values (also at the end points of the interval) with a (numerical) error of at most $1/18$ in this case.

If we actually get the information that $\ln(3/4) = -0.2877$ and $\ln(5/4) = 0.2223$ we then of course get a direct estimate of the largest value of $|R_{1,x_0=1}(x)|$ in the interval $\left[\frac{3}{4}, \frac{5}{4}\right]$:

$$\begin{aligned} |R_{1,x_0=1}(x)| &\leq \max\{|-0.2877 + 0.25|\,,\ |0.2223 - 0.25|\} \\ &= 0.0377 < 1/18 = 0.0556 \quad . \end{aligned} \tag{4-45}$$

With the ordinary function analysis we get a somewhat better estimate of the remainder function – but only because we beforehand can estimate the function value at the end points.

|||| Exercise 4.15 Approximation of a Non-Elementary Function

Given the function from Example 4.3:

$$f(x) = \int_0^x e^{-t^2}\, dt \tag{4-46}$$

An estimate of the magnitude of the difference between $f(x)$ and the approximating first-degree polynomial $P_{1,x_0=0}(x)$ with the development point at $x_0 = 0$ is wished for. The exercise is about determining the largest absolute value that the remainder function $|R_{1,x_0=0}(x)|$

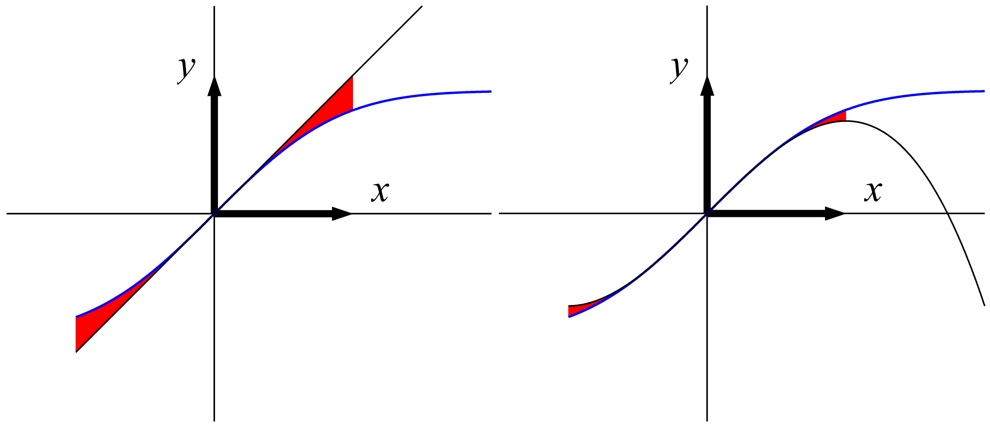

Figure 4.4: The function $f(x)$ from Example 4.15 (blue), the approximating first- and third-degree polynomials (black) with development points $x_0 = 0$ and the corresponding remainder functions (red) on the interval $[-1,1]$.

assumes on the interval $[-1,1]$.
Hint: Use higher order derivatives of $f(x)$ evaluated at $x_0 = 0$ found earlier, cf. Example 4.3: $f(0) = 0$, $f'(0) = 1$, $f''(x) = -2 \cdot x \cdot \mathrm{e}^{-x^2}$. See Figure 4.4.

|||| Example 4.16 Approximation of an Unknown (But Elementary) Function

Given the function from example 4.4, i.e. the function satisfies the following differential equation with initial conditions:

$$f''(x) + 3f'(x) + 7f(x) = x^2 \quad , \quad \text{where} \quad f(0) = 1 \quad , \quad \text{and} \quad f'(0) = -3 \quad , \tag{4-47}$$

where we have assumed that the right-hand side of the equation is $q(x) = x^2$ and that the development point is $x_0 = 0$. By this we now get:

$$f'(0) = -3 \quad , \quad f''(0) = 2 \quad , \quad f'''(0) = 15 \quad . \tag{4-48}$$

We have

$$\begin{aligned} f(x) &= f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2 + \frac{f'''(0)}{6} \cdot x^3 + x^3 \cdot \varepsilon(x) \\ &= 1 - 3 \cdot x + x^2 + \frac{5}{2} \cdot x^3 + x^3 \cdot \varepsilon(x) \quad , \end{aligned} \tag{4-49}$$

such that the approximating third-degree polynomial for $f(x)$ with development point $x_0 = 0$ is

$$P_{3,x_0=0}(x) = 1 - 3 \cdot x + x^2 + \frac{5}{2} \cdot x^3 \quad . \tag{4-50}$$

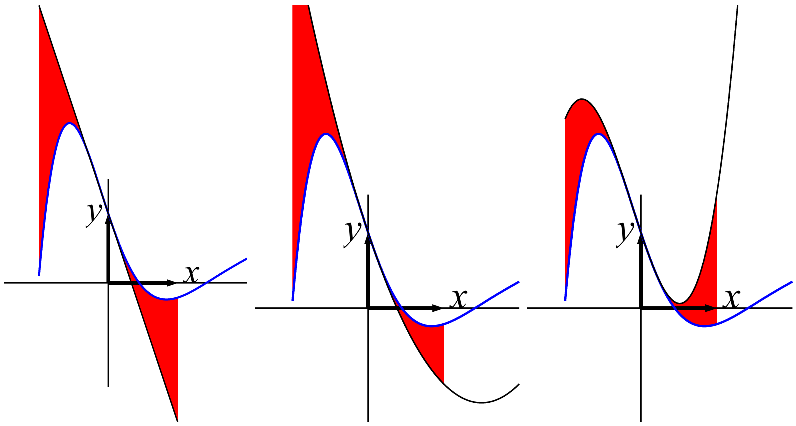

Figure 4.5: The function $f(x)$ from Example 4.16 (blue), the approximating first-, second-, and third-degree polynomials (black) with the development point $x_0 = 0$. The corresponding respective remainder functions (red) are illustrated as the differences between $f(x)$ and the approximating polynomials.

Note that $P_{3,x_0=0}(x)$ satisfies the initial conditions in (4-47) but the polynomial $P_{3,x_0=0}(x)$ is not a solution to the differential equation itself!

4.5 Functional Investigations

A very important property of continuous functions is the following, which means one can control how large and how small values a continuous function can assume on an interval, as long as the interval is sufficiently nice:

|||| Theorem 4.17 Main Theorem for Continuous Functions of One Variable

Let $f(x)$ denote a function that is continuous on all of its domain $D(f) \subset \mathbb{R}$. Let $I = [\,a\,,\,b\,]$ be a bounded, closed, and connected interval in $D(f)$.

Then the range for the function $f(x)$ on the interval I is also a bounded, closed and connected interval $[\,A\,,\,B\,] \subset \mathbb{R}$, thus denoted:

$$R(f_{|_I}) = f(I) = \{f(x)\,|\,x \in I\} = [\,A\,,\,B\,] \quad , \tag{4-51}$$

where the possibility that $A = B$ is allowed and this happens precisely when $f(x)$ is constant on the whole interval I.

|||| Definition 4.18 Global Minimum and Global Maximum

When a function $f(x)$ has the range $R(f_{|_I}) = f(I) = [\,A\,,\,B\,]$ on an interval $I = [a, b]$ we say that

1. A is the ***global minimum value*** for $f(x)$ on I, and if $f(x_0) = A$ for $x_0 \in I$ then x_0 is a ***global minimum point*** for $f(x)$ on I.

2. B is the ***global maximum value*** for $f(x)$ on I, and if $f(x_0) = B$ for $x_0 \in I$ then x_0 is a ***global maximum point*** for $f(x)$ on I.

A well-known and important task is to find the global maximum and minimum values for given functions $f(x)$ on given intervals and to determine the x-values for which these maximum and minimum values are *assumed*, that is, the minimum and maximum points. To solve this task the following is an invaluable help – see Figure 4.6:

|||| Lemma 4.19 Maxima and Minima at Stationary Points

Let x_0 be a global maximum or minimum value for $f(x)$ on I. Assume that x_0 is not an end point for the interval I and that $f(x)$ is differentiable at x_0.
Then x_0 is a ***stationary point*** for $f(x)$,i.e. $f'(x_0) = 0$.

|||| Proof

We outline the argument. Since $f(x)$ is assumed differentiable, we have:

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0) \cdot (x - x_0) + (x - x_0) \cdot \varepsilon_f(x - x_0) \\ &= f(x_0) + (x - x_0) \cdot (f'(x_0) + \varepsilon_f(x - x_0)) \quad . \end{aligned} \tag{4-52}$$

Now if we assume that $f'(x_0)$ is positive then the parenthesis $(f'(x_0) + \varepsilon_f(x - x_0))$ is also positive for x sufficiently close to x_0 (since $\varepsilon_f(x - x_0) \to 0$ for $x \to x_0$), but then $(x - x_0) \cdot (f'(x_0) + \varepsilon_f(x - x_0))$ is also positive for x sufficiently close to x_0 and then $f(x) > f(x_0)$ for $x > x_0$, and $f(x) < f(x_0)$ for $x < x_0$. Therefore $f(x_0)$ can not be neither a maximum value nor a minimum value for $f(x)$. A similar conclusion appears when the assumption is $f'(x_0) < 0$. If x_0 is a global maximum or minimum value for $f(x)$ on I this assumption must imply that $f'(x_0) = 0$.

■

Hereby we have the following investigation method at our disposal:

|||| Method 4.20 Method of Investigation

Let $f(x)$ be a continuous function and $I = [a, b]$ an interval in the domain $D(f)$.

Maximum and minimum values for the function $f(x)$, $x \in I$, i.e. A and B in the range $[A, B]$ for $f(x)$ restricted to I, are found by finding and comparing the function values at the following points:

1. Interval end points (the boundary points a and b for the interval I).
2. Exception points, i.e. the points in the open interval $]a, b[$ where the function is *not* differentiable.
3. The stationary points, i.e. all the points x_0 in the open interval $]a, b[$ where $f'(x_0) = 0$.

With this method of investigation we not only find the global maximum and minimum values but also the x-values in I for which the global maximum and the global minimum are assumed i.e. maximum and minimum points in the actual interval.

|||| Example 4.21 A Continuous Function Is Investigated

A Continuous function $f(x)$ is defined for all x in the following way:

$$f(x) = \begin{cases} 0.75 & \text{for} \quad x \leq -1.5 \\ 0.5 + (x+1)^2 & \text{for} \quad -1.5 \leq x \leq 0 \\ 1.5 \cdot (1 - x^3) & \text{for} \quad 0 \leq x \leq 1 \\ x - 1 & \text{for} \quad 1 \leq x \leq 2 \\ 1 & \text{for} \quad x > 2 \end{cases} \tag{4-53}$$

See Figure 4.6, where we only consider the function on the interval $I = [-1.5, 2.0]$. There are two exception points where the function is not differentiable: $x_0 = 0$ and $x_0 = 1$. There is one stationary point in $]-1.5, 2.0[$ where $f'(x_0) = 0$ viz. $x_0 = -1$. And finally there are two boundary points (the interval end points $x_0 = -1.5$ and $x_0 = 2$) that need to be investigated.

Therefore we have the following candidates for global maximum and minimum values for f on I:

$x_0 =$	-1.5	-1	0	1	2
$f(x_0) =$	0.75	0.5	1.5	0	1

(4-54)

In conclusion we read from this that the maximum value for $f(x)$ is $B = 1.5$ which is assumed at the maximum point $x_0 = 0$. The minimum value is $A = 0$, assumed at the minimum point $x_0 = 1$. There are no other maximum or minimum points for f on I.

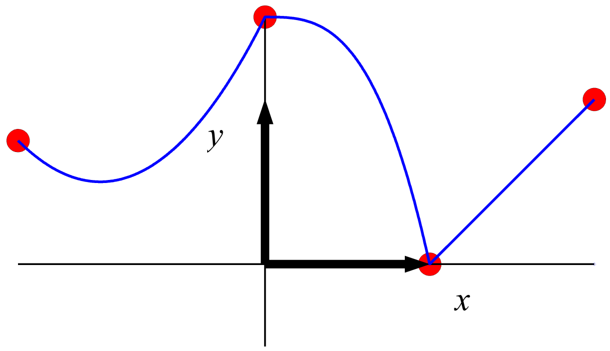

Figure 4.6: The continuous function $f(x)$ from example 4.21 (blue). On the graph we have marked (in red) the 5 points that need to be investigated particularly in order to determine the range for f in the interval $[-1.5, 2]$, cf. Method 4.20.

|||| Definition 4.22 Local Minima and Local Maxima

Let $f(x)$ denote a function on an interval $I = [a, b]$ containing a given $x_0 \in]a, b[$.

1. If $f(x) \geq f(x_0)$ for all x in a (as small as you like) neighborhood of x_0 then $f(x_0)$ is called a ***local minimum value*** for $f(x)$ in I and x_0 is a ***local minimum point*** for $f(x)$ in I. If actually $f(x) > f(x_0)$ for all x in the neighborhood apart from the point x_0 itself then $f(x_0)$ is called a ***proper local minimum value***.

2. If $f(x) \leq f(x_0)$ for all x in a (as small as you like) neighborhood of x_0 then $f(x_0)$ is called a ***local maximum value*** for $f(x)$ in I and x_0 is a ***local maximum point*** for f on I. If actually $f(x) < f(x_0)$ for all x in the neighborhood apart from the point x_0 itself then $f(x_0)$ is called a ***proper local maximum value***.

If the function we want to investigate is smooth at its stationary points then we can qualify the Method 4.20 even better, since the approximating polynomial of degree 2 with development point at the stationary point can help in the decision whether the value of $f(x)$ at the stationary point is a candidate to be a maximum value or a minimum value.

|||| Lemma 4.23 Local Analysis at a Stationary Point

Let $f(x)$ be a smooth function and assume that x_0 is a stationary point for $f(x)$ on an interval $I =]a, b[$. Then the following applies:

1. If $f''(x_0) > 0$ then $f(x_0)$ is a proper local minimum value for $f(x)$.
2. If $f''(x_0) < 0$ then $f(x_0)$ is a proper local maximum value for $f(x)$.
3. If $f''(x_0) = 0$ then this is not sufficient information to decide whether $f(x_0)$ is a local minimum value or a local maximum value or neither.

|||| Exercise 4.24

Prove Lemma 4.23 by using Taylor's limit formula with the approximating second-degree polynomial for $f(x)$ and with the development point x_0. Remember that x_0 is a stationary point, such that $f'(x_0) = 0$.

|||| Example 4.25 Local Maxima and Minima

The continuous function $f(x)$

$$f(x) = \begin{cases} 0.75 & \text{for} \quad x \leq -1.5 \\ 0.5 + (x+1)^2 & \text{for} \quad -1.5 \leq x \leq 0 \\ 1.5 \cdot (1 - x^3) & \text{for} \quad 0 \leq x \leq 1 \\ x - 1 & \text{for} \quad 1 \leq x \leq 2 \\ 1 & \text{for} \quad x \geq 2 \end{cases} \tag{4-55}$$

is shown in Figure 4.6. On the interval $I = [-1.5, 2.0]$ the function has the proper local minimum values 0.5 and 0 in the respective proper local minimum points $x_0 = -1$ and $x_0 = 1$ and the function has a proper local maximum value 1.5 at the proper local maximum point $x_0 = 0$. If we extend the interval to $J = [-7, 7]$ and note that the function values by definition are constant outside the interval I we get the new local maximum values 0.75 and 1 for f on J – not one of them is a *proper* local maximum value. All $x_0 \in]-7, -1.5]$ and all $x_0 \in [2, 7[$ are local maximum points for f on J but not one of them is a *proper* local maximum point. All x_0 in the *open interval* $x_0 \in]-7, -1.5[$ and all x_0 in the *open interval* $x_0 \in]2, 7[$ in addition also local minimum points for $f(x)$ i J but not one of them is a *proper* local minimum point.

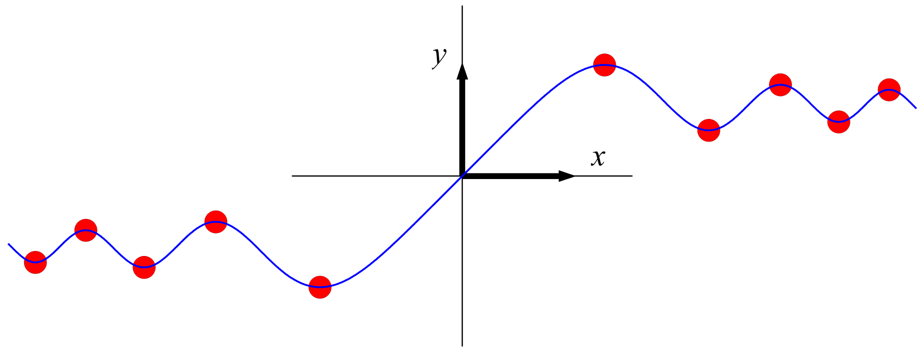

Figure 4.7: Proper local maxima and proper local minima for the function from Example 4.26 are here indicated on the graph for the function. We note: The local maximum and minimum *points* for the function are the x-coordinates of the graph points shown in red, and the local maximum and minimum *values* for the function are the y-coordinates of the graph-points shown in red.

Example 4.26 A Non-Elementary Function

The function $f(x)$

$$f(x) = \int_0^x \cos(t^2)\, dt \tag{4-56}$$

has stationary points at those values of x_0 satisfying:

$$f'(x_0) = \cos(x_0^2) = 0 \quad , \quad \text{dvs.} \quad x_0^2 = \frac{\pi}{2} + p \cdot \pi \quad \text{where } p \text{ is an integer} \quad . \tag{4-57}$$

Since we also have that

$$f''(x) = -2 \cdot x \cdot \sin(x^2) \quad , \tag{4-58}$$

such that at the stated stationary points it applies

$$f''(x_0) = -2 \cdot x_0 \cdot (-1)^p \quad . \tag{4-59}$$

From this it follows – via Lemma 4.23 – that every other stationary point x_0 along the x-axis is a proper local maximum point for $f(x)$ and the other points proper local minimum points. See Figure 4.7. In Figure 4.8 are shown graphs (parabolas) for a pair of the approximating second-degree polynomials for $f(x)$ with the development points at chosen stationary points.

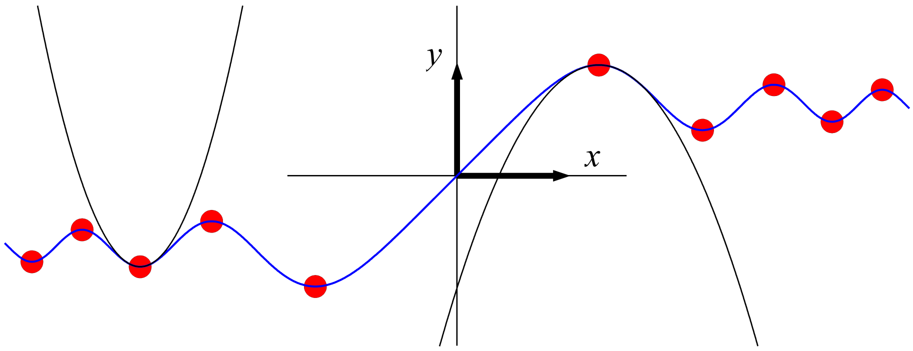

Figure 4.8: The graph for the function in Example 4.26 and two approximating parabolas with development points in two stationary points, which are a proper local minimum point and a proper local maximum point for $f(x)$.

|||| Example 4.27 When the Approximation to Degree 2 is Not Good Enough

As stated in Lemma 4.23 one cannot from $f'(x_0) = f''(x_0) = 0$ decide whether the function has a local maximum or minimum at x_0. This is shown in the three simple functions in Figure 4.9 with all the clarity one could wish for: $f_1(x) = x^4$, $f_2(x) = -x^4$ and $f_3(x) = x^3$. All three functions have a stationary point at $x_0 = 0$ and all have $f''(x_0) = 0$, but $f_1(x)$ has a proper local minimum point at 0, $f_2(x)$ has a proper local maximum point at 0, and $f_3(x)$ has neither a local minimum point nor a local maximum point at 0.

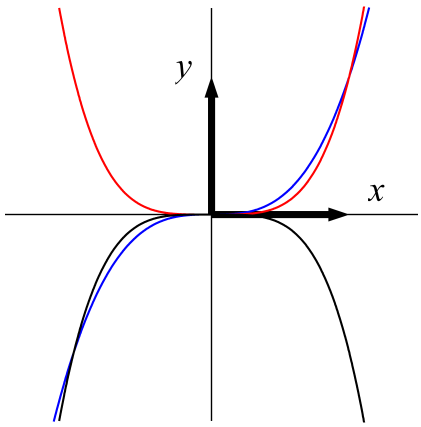

Figure 4.9: Three elementary functions with approximating second-degree polynomials $P_{2,x_0=0}(x) = 0$ for all x. The functions are: $f_1(x) = x^4$ (red), $f_2(x) = -x^4$ (black) and $f_3(x) = x^3$ (blue).

4.6 Summary

In this eNote we have studied how one can approximate smooth functions using polynomials.

- Every smooth function $f(x)$ on an interval I can be split into an approximating n'th-degree polynomial $P_{n,x_0}(x)$ with the development point x_0 and a corresponding remainder function $R_{n,x_0}(x)$ like this:

$$f(x) = P_{n,x_0}(x) + R_{n,x_0}(x) \quad , \tag{4-60}$$

  where the polynomial and the remainder function in Taylor's limit formula are written like this:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} \cdot (x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n + (x - x_0)^n \cdot \varepsilon_f(x - x_0) \quad ,$$

  with $\varepsilon_f(x - x_0)$ denoting an epsilon function of $(x - x_0)$, i.e. $\varepsilon_f(x - x_0) \to 0$ for $x \to x_0$.

- Taylor's limit formula can be used to find the continuous extension of quotients of functions by finding (if possible) their limit values for $x \to x_0$ where x_0 are the values where the numerator function is 0 such that the quotient at the starting point is not defined at x_0:

$$\frac{\sin(x)}{x} = \frac{x + x^1 \cdot \varepsilon(x)}{x} = 1 + \varepsilon(x) \to 1 \quad \text{for} \quad x \to 0 \quad . \tag{4-61}$$

- Estimation of the remainder function gives an upper bound for the largest numerical difference between a given function and the approximating polynomial of a suitable degree and with a suitable development point on a given interval of investigation. Such an estimation can also be made for functions that are possibly only "known" via a differential equation or as a non-elementary integral:

$$|\ln(x) - (x - 1)| \leq \frac{1}{18} \quad \text{for all} \quad x \in \left[\frac{3}{4}, \frac{5}{4}\right] \quad . \tag{4-62}$$

- Taylor's limit formula with approximating second-degree polynomials is used for efficient functional investigation, including determination of range, global and local maxima and minima for given functions.