eNote 3

Elementary Functions

In this eNote we will both repeat some of the basic properties for a selection of the (from high school) well-known functions $f(x)$ of one real variable x, and introduce some new functions, which typically occur in a variety of applications. The basic questions concerning any function are usually the following: How, and for which values of x, is the function defined*? Which values for $f(x)$ do we get when we apply the functions to the x-elements in the domain? Is the function* continuous*? What is the* derivative $f'(x)$ *of the function – if it exists? As a new concept, we will introduce a vast* class *of functions, the* ***epsilon functions****, which are denoted by the common symbol $\varepsilon(x)$ and which we will use generally in order to describe continuity and differentiability – also of functions of more variables, which we introduce in the following eNotes.*

(Updated: 22.9.2021 David Brander)

3.1 Domain and Range

In the description of a real function $f(x)$ both the real numbers x where the function is defined and the values that are obtained by applying the function on the domain are stated. The ***Domain*** we denote $D(f)$ and the ***range***, or *image*, we denote $R(f)$.

Note: in higher mathematics, it is usual to define a function by specifying the domain and *codomain*, (the set where the function in principle takes values) rather than the image. For example: $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$. The codomain is $\mathbb{R}$, but the range is the set of non-negative numbers $[0, \infty[\subset \mathbb{R}$.

|||| Example 3.1 Some Domains and Ranges

Here are domains and the corresponding ranges for some well-known functions.

$$
\begin{array}{llll}
f_1(x) = \exp(x) & , \; D(f_1) = \mathbb{R} =]-\infty, \infty[& , \; R(f_1) =]0, \infty[& \\
f_2(x) = \ln(x) & , \; D(f_2) =]0, \infty[& , \; R(f_2) = \mathbb{R} =]-\infty, \infty[& \\
f_3(x) = \sqrt{x} & , \; D(f_3) = [0, \infty[& , \; R(f_3) = [0, \infty[& \\
f_4(x) = x^2 & , \; D(f_4) = \mathbb{R} =]-\infty, \infty[& , \; R(f_4) = [0, \infty[& \\
f_5(x) = x^7 + 8x^3 + x - 1 & , \; D(f_5) = \mathbb{R} =]-\infty, \infty[& , \; R(f_5) = \mathbb{R} =]-\infty, \infty[& \\
f_6(x) = \exp(\ln(x)) & , \; D(f_6) =]0, \infty[& , \; R(f_6) =]0, \infty[& \\
f_7(x) = \sin(1/x) & , \; D(f_7) =]-\infty, 0[\cup]0, \infty[& , \; R(f_7) = [-1, 1] & \\
f_8(x) = |x|/x & , \; D(f_8) =]-\infty, 0[\cup]0, \infty[& , \; R(f_8) = \{-1\} \cup \{1\} &
\end{array} \tag{3-1}
$$

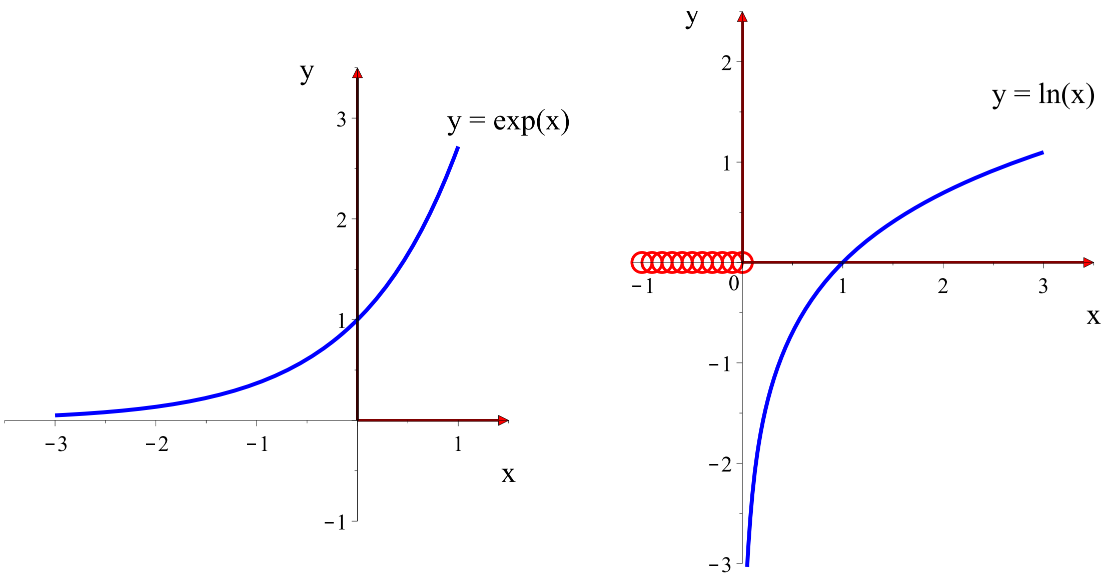

Figure 3.1: The well-known exponential function $e^x = \exp(x)$ and the natural logarithmic function $\ln(x)$. The red circles on the negative x-axis and at 0 indicate that the logarithmic function is not defined on $]-\infty, 0]$.

The function $f_8(x)$ in Example 3.1 is defined using $|x|$, which denotes the absolute value of x, i.e.

$$|x| = \begin{cases} x > 0\,, & \text{for} \quad x > 0 \\ 0\,, & \text{for} \quad x = 0 \\ -x > 0\,, & \text{for} \quad x < 0 \end{cases} \quad . \tag{3-2}$$

From this the domain and range for $f_8(x)$ follow directly.

|||| Example 3.2 Tangent

The function

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)} \tag{3-3}$$

has the domain $D(f) = \mathbb{R} \setminus A$, A denoting those real numbers x for which $\cos(x) = 0$, $\cos(x)$ being the denominator, i.e.

$$D(f) = \mathbb{R} \setminus \{x \,|\, \cos(x) = 0\} = \mathbb{R} \setminus \{(\pi/2) + p \cdot \pi\,,\, p \text{ being an integer}\} \quad . \tag{3-4}$$

The range $R(f)$ is all real numbers, see Figure 3.2.

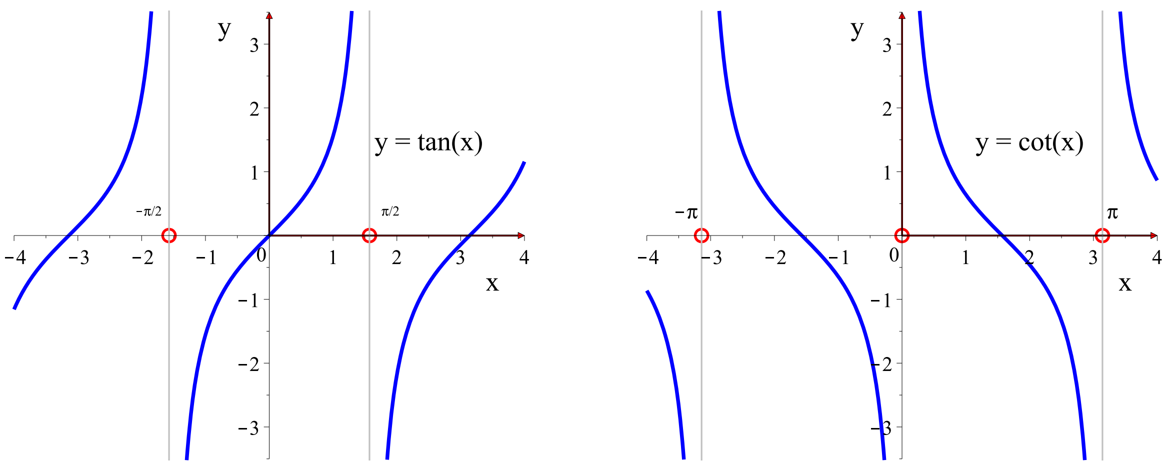

Figure 3.2: The graphs for the functions $\tan(x)$ and $\cot(x)$.

|||| Exercise 3.3

Let $g(x)$ denote the reciprocal function to the function $\tan(x)$:

$$g(x) = \cot(x) = \frac{\cos(x)}{\sin(x)} \tag{3-5}$$

Determine the domain for $g(x)$ and state it in the same way as above for $\tan(x)$, see Figure 3.2.

3.1.1 Extension of the Domain to All of $\mathbb{R}$

A function $f(x)$ that is not defined for all real numbers can easily be *extended* to a function $\widehat{f}(x)$, which has $D(\widehat{f}) = \mathbb{R}$. One way of doing this is by the use of a ***curly bracket*** in the following way:

|||| Definition 3.4

Given a function $f(x)$ with $D(f) \neq \mathbb{R}$. We then define the 0-***extension*** of $f(x)$ by:

$$\widehat{f}(x) = \begin{cases} f(x)\,, & \text{for} \quad x \in D(f) \\ 0\,, & \text{for} \quad x \in \mathbb{R} \setminus D(f) \end{cases} \quad . \tag{3-6}$$

It is evident that depending on the application one can seal and extend the domain for $f(x)$ in many other ways than choosing the constant 0 as the value for the extended function at the points where the original function is not defined.

Naturally, the *Range* $R(\widehat{f})$ for the 0-extended function is the original range for $f(x)$ united with 0, i.e. $R(\widehat{f}) = R(f) \cup \{0\}$.

Hereafter we will assume – unless otherwise stated – that the functions we consider are defined for all $\mathbb{R}$ possibly by extension as above.

3.2 Epsilon Functions

We introduce a special class of functions, which we will use in order to define the important concept of continuity.

|||| Definition 3.5 Epsilon Functions

Every $\varepsilon(x)$ that is defined on an open interval that contains 0 and that assumes the value $\varepsilon(0) = 0$ at $x = 0$ and moreover tends towards 0 when x tends towards 0 is called an ***epsilon function*** of x. Thus epsilon functions are characterized by the properties:

$$\varepsilon(0) = 0 \quad \text{and} \quad \varepsilon(x) \to 0 \quad \text{for} \quad x \to 0 \quad . \tag{3-7}$$

The last condition is equivalent to the fact that the absolute value of $\varepsilon(x)$ can be made as small as possible by choosing the numerical value of x sufficiently small. To be precise the condition means: For every number $a > 0$ there exists a number $b > 0$ such that $|\varepsilon(x)| < a$ for all x satisfying $|x| < b$.

The set of epsilon functions is very large:

|||| Example 3.6 Epsilon Functions

Here are some simple examples of epsilon functions:

$$\begin{aligned}
\varepsilon_1(x) &= x \\
\varepsilon_2(x) &= |x| \\
\varepsilon_3(x) &= \ln(1+x) \\
\varepsilon_4(x) &= \sin(x) \quad .
\end{aligned} \tag{3-8}$$

The quality 'to be an epsilon function' is rather stable: The product of an epsilon function and an arbitrary other function that only has to be bounded is also an epsilon function. The sum and the product of two epsilon functions are again epsilon functions. The absolute value of an epsilon function is an epsilon function.

Functions that are 0 in other places than $x = 0$ can also be epsilon functions:

> If a function $g(x)$ has the properties $g(x_0) = 0$ and $g(x) \to 0$ for $x \to x_0$ then $g(x)$ is an epsilon function of $x - x_0$ i.e. we can write $g(x) = \varepsilon_g(x - x_0)$.

|||| Exercise 3.7

Show that the 0-extension $\widehat{f_8}(x)$ of the function $f_8(x) = |x|/x$ is *not* an epsilon function. Hint: If we choose $k = 10$ then clearly there does *not exist* a value of K such that

$$|f_8(x)| = |\,|x|/x\,| = 1 < \frac{1}{10} \quad , \quad \text{for all } x \text{ with} \quad |x| < \frac{1}{K} \quad . \tag{3-9}$$

Draw the graph for $\widehat{f_8}(x)$. This cannot be drawn without 'lifting the pencil from the paper'!

|||| Exercise 3.8

Show that the 0-extension of the function $f(x) = \sin(1/x)$ is *not* an epsilon function.

3.3 Continuous Functions

We can now formulate the ***concept of continuity*** by use of epsilon functions:

|||| Definition 3.9 Continuity

A function $f(x)$ is continuous at x_0 if there exists an epsilon function $\varepsilon_f(x - x_0)$ such that the following is valid on an open interval that contains x_0:

$$f(x) = f(x_0) + \varepsilon_f(x - x_0) \quad . \tag{3-10}$$

If $f(x)$ is continuous at every x_0 on a given open interval in $D(f)$ we say that $f(x)$ is continuous on the interval.

Note that even though it is clear what the epsilon function precisely is in the definition 3.9, viz. $f(x) - f(x_0)$, then the only property in which we are interested is the following: $\varepsilon_f(x - x_0) \to 0$ for $x \to x_0$ such that $f(x) \to f(x_0)$ for $x \to x_0$, that is precisely as we know the concept of continuity from high school!

|||| Exercise 3.10

According to the above, all epsilon functions are continuous at $x_0 = 0$ (with the value 0 at $x_0 = 0$). Construct an epsilon function that is *not* continuous at any of the points $x_0 = 1/n$ where $n = 1, 2, 3, 4, \cdots$.

Even though the concept of epsilon functions is central to the definition of continuity (and as we shall see below, to the definition of differentiability), epsilon functions need not be continuous for any other values than $x_0 = 0$.

|||| Exercise 3.11

Show that the 0-extension $\widehat{f}(x)$ of the function $f(x) = |x - 7|/(x - 7)$ is *not* continuous on $\mathbb{R}$.

3.4 Differentiable Functions

|||| Definition 3.12 Differentiability

A function $f(x)$ is differentiable at $x_0 \in D(f)$ if both a constant a and an epsilon function $\varepsilon_f(x - x_0)$ exist such that

$$f(x) = f(x_0) + a \cdot (x - x_0) + (x - x_0) \cdot \varepsilon_f(x - x_0) \quad . \tag{3-11}$$

It is the number a that we call $f'(x_0)$ and it is well-defined in the sense that if $f(x)$ *can be* stated at all in the form above (i.e. if $f(x)$ is differentiable at x_0) then there is one and only one value for a that makes this formula true. With this definition of the ***derivative*** $f'(x_0)$ of $f(x)$ at x_0 we then have:

$$f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + (x - x_0) \cdot \varepsilon_f(x - x_0) \quad . \tag{3-12}$$

If $f(x)$ is differentiable for all x_0 in a given open interval in $D(f)$, we then naturally say that $f(x)$ is differentiable on the interval. We often write the derivative of $f(x)$ at x in the following alternative way:

$$f'(x) = \frac{d}{dx} f(x) \quad . \tag{3-13}$$

|||| Explanation 3.13 The Derivative is Unique

We will show that there is only one value of a that fulfills Equation (3-11). Assume that two different values, a_1 and a_2 both fulfill (3-11) possibly with two different epsilon functions:

$$\begin{aligned} f(x) &= f(x_0) + a_1 \cdot (x - x_0) + (x - x_0) \cdot \varepsilon_1(x - x_0) \\ f(x) &= f(x_0) + a_2 \cdot (x - x_0) + (x - x_0) \cdot \varepsilon_2(x - x_0) \quad . \end{aligned} \tag{3-14}$$

By subtracting (3-14) from the uppermost equation we get:

$$0 = 0 + (a_1 - a_2) \cdot (x - x_0) + (x - x_0) \cdot (\varepsilon_1(x - x_0) - \varepsilon_2(x - x_0)) \quad , \tag{3-15}$$

such that

$$a_2 - a_1 = \varepsilon_1(x - x_0) - \varepsilon_2(x - x_0) \tag{3-16}$$

for all $x \neq x_0$ – and clearly this cannot be true; the right hand side tends towards 0 when x tends towards x_0! Therefore the above assumption, i.e. that $a_1 \neq a_2$, is wrong. The two constants a_1 and a_2 must be equal, and this is what we should realize.

The definition above is quite equivalent to the one we know from high school. If we first subtract $f(x_0)$ from both sides of the equality sign in Equation (3-12) and then divide by $(x - x_0)$ we get

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + \varepsilon_f(x - x_0) \to f'(x_0) \quad \text{for} \quad x \to x_0 \quad , \tag{3-17}$$

i.e. the well-known limit value for the *quotient* between the increment in the function $f(x) - f(x_0)$ and the x-increment $x - x_0$. The reason why we do not apply this known definition of $f'(x_0)$ is simply that for functions of more variables the quotient does not make sense – but more about this in a later eNote.

|||| Theorem 3.14 Differentiable Implies Continuous

If a function $f(x)$ is differentiable at x_0, then $f(x)$ is also continuous at x_0.

|||| Proof

We have that

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0) \cdot (x - x_0) + (x - x_0)\varepsilon_f(x - x_0) \\ &= f(x_0) + \left[f'(x_0) \cdot (x - x_0) + (x - x_0)\varepsilon_f(x - x_0)\right], \end{aligned} \tag{3-18}$$

and since the function in the square brackets on the right hand side is an epsilon function of $(x - x_0)$ then $f(x)$ is continuous at x_0.

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But the opposite is not valid – here is an example that shows this:

Example 3.15 Continuous But Not Differentiable

The function $f(x) = |x|$ is continuous but not differentiable at $x_0 = 0$. The function is in itself an epsilon function and therefore $f(x)$ is continuous in 0. But now assume that there exist a constant a and an epsilon function $\varepsilon_f(x - x_0)$ such that

$$f(x) = f(x_0) + a \cdot (x - x_0) + (x - x_0)\varepsilon_f(x - x_0). \tag{3-19}$$

The following will then apply:

$$|x| = 0 + a \cdot x + x \cdot \varepsilon_f(x) \tag{3-20}$$

and hence for all $x \neq 0$:

$$\frac{|x|}{x} = a + \varepsilon_f(x) \quad . \tag{3-21}$$

If so a should both be equal to -1 and to 1 and this is impossible! Therefore the assumption above that there exists a constant a is accordingly wrong; therefore $f(x)$ is not differentiable.

Definition 3.16

The first degree approximating polynomial for $f(x)$ expanded about the point x_0 is defined by:

$$P_{1,x_0}(x) = f(x_0) + f'(x_0) \cdot (x - x_0) \quad . \tag{3-22}$$

Note that $P_{1,x_0}(x)$ really *is* a first degree polynomial in x. The graph for the function $P_{1,x_0}(x)$ is ***the tangent to the graph*** for $f(x)$ at the point $(x_0, f(x_0))$, see Figure 3.3. The equation for the tangent is $y = P_{1,x_0}(x)$, thus $y = f(x_0) + f'(x_0) \cdot (x - x_0)$. The slope of the tangent is clearly $\alpha = f'(x_0)$ and the tangent intersects the y-axis at the point $(0, f(x_0) - x_0 \cdot f'(x_0))$. Later we will find out how we can approximate with polynomials of higher degree n, i.e. polynomials that are then denoted $P_{n,x_0}(x)$.

3.4.1 Differentiation of a Product

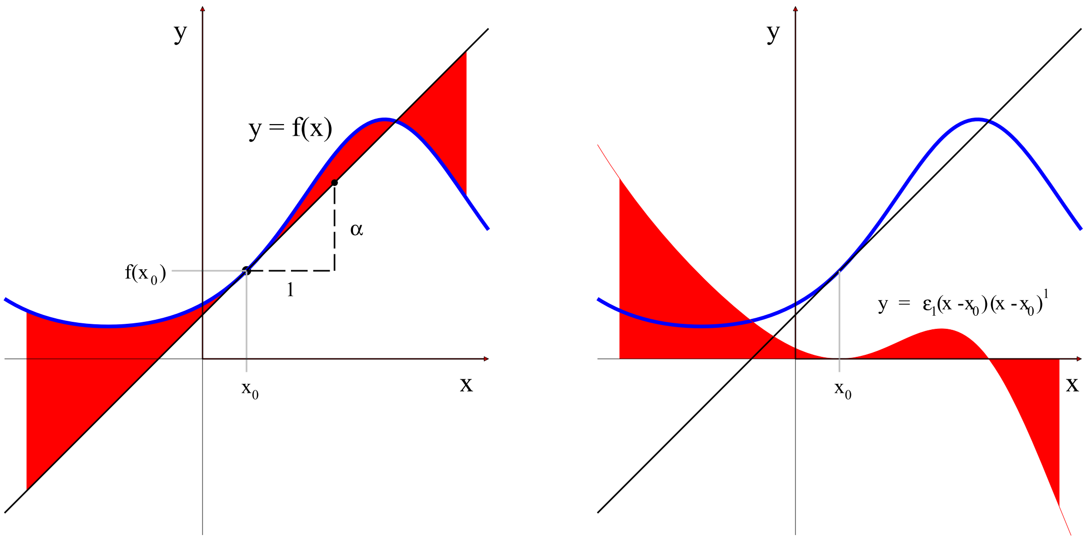

Figure 3.3: Construction of the tangent $y = P_{1,x_0}(x) = f(x_0) + \alpha \cdot (x - x_0)$ with the slope $\alpha = f'(x_0)$ for the function $f(x)$. To the right the difference between $f(x)$ and the 'tangent value' $P_{1,x_0}(x)$.

|||| Theorem 3.17 Differentiation of $f(x) \cdot g(x)$

A product $h(x) = f(x) \cdot g(x)$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable and its derivative is as follows:

$$\frac{d}{dx}(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \quad . \tag{3-23}$$

Even though this formula is rather well known from high school we shall give a short sketch of a proof – to illustrate the use of epsilon functions.

|||| Proof

Since $f(x)$ and $g(x)$ are differentiable in x_0, we have:

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0) \cdot (x - x_0) + (x - x_0)\varepsilon_f(x - x_0) \\ g(x) &= g(x_0) + g'(x_0) \cdot (x - x_0) + (x - x_0)\varepsilon_g(x - x_0) \quad , \end{aligned} \tag{3-24}$$

resulting in the product of the two right hand sides:

$$\begin{aligned} h(x) &= f(x) \cdot g(x) \\ &= f(x_0) \cdot g(x_0) + (f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)) \cdot (x - x_0) + (x - x_0)\varepsilon_h(x - x_0) \, , \end{aligned} \tag{3-25}$$

where we have used $(x - x_0)\varepsilon_h(x - x_0)$ as short for the remaining part of the product sum. Furthermore any of the addends in the remaining part contains the factor $(x - x_0)^2$ or the product of $(x - x_0)$ with an epsilon function and therefore *can* be written in the stated form. But then the product formula follows directly from the factor in front of $(x - x_0)$ in Equation (3-25):

$$h'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0) \quad . \tag{3-26}$$

■

3.4.2 Differentiation of a Quotient

The following differentiation rule is also well known from high school:

|||| Theorem 3.18 Differentiation of $f(x)/g(x)$

A quotient $h(x) = f(x)/g(x)$ involving two differentiable functions $f(x)$ and $g(x)$, is differentiable everywhere that $g(x) \neq 0$, and the derivative is given in this well-known fashion:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)}{g(x)} - \frac{f(x) \cdot g'(x)}{g^2(x)} = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} \quad . \tag{3-27}$$

|||| Exercise 3.19

Use the epsilon function argument in the same way as in the differentiation rule for a product to show Equation 3.18.

3.4.3 Differentiation of Composite Functions

|||| Theorem 3.20 The Chain Rule for Composite Functions

A function $h(x) = f(g(x))$ that is composed of two differentiable functions $f(x)$ and $g(x)$ is in itself differentiable at every x_0 with the derivative

$$h'(x_0) = f'(g(x_0)) \cdot g'(x_0) \tag{3-28}$$

|||| Proof

We exploit that the two functions $f(x)$ and $g(x)$ are differentiable. In particular $g(x)$ is differentiable at x_0:

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + (x - x_0) \cdot \varepsilon_g(x - x_0) \quad , \tag{3-29}$$

and the function $f(u)$ is differentiable at $u_0 = g(x_0)$:

$$f(u) = f(u_0) + f'(u_0)(u - u_0) + (u - u_0) \cdot \varepsilon_f(u - u_0) \quad . \tag{3-30}$$

From this we get, setting $u = g(x)$ and $u_0 = g(x_0)$:

$$\begin{aligned} h(x) &= f(g(x)) \\ &= f(g(x_0)) + f'(g(x_0))(g(x) - g(x_0) + (g(x) - g(x_0) \cdot \varepsilon_f(g(x) - g(x_0) \\ &= h(x_0) + f'(g(x_0))(g'(x_0)(x - x_0) + (x - x_0) \cdot \varepsilon_g(x - x_0)) \\ &\qquad + (g'(x_0)(x - x_0) + (x - x_0) \cdot \varepsilon_g(x - x_0)) \cdot \varepsilon_f(g(x) - g(x_0) \\ &= h(x_0) + f'(g(x_0))g'(x_0) \cdot (x - x_0) + (x - x_0) \cdot \varepsilon_h(x - x_0) \quad , \end{aligned} \tag{3-31}$$

from which we directly read that $h'(x_0) = f'(g(x_0))g'(x_0)$ – because this is exactly the unique coefficient of $(x - x_0)$ in the above expression.

■

|||| Exercise 3.21

Above we have used – at the end of Equation (3-31) – that

$$f'(g(x_0)) \cdot \varepsilon_g(x - x_0) + (g'(x_0) + \cdot \varepsilon_g(x - x_0)) \cdot \varepsilon_f(g(x) - g(x_0)) \tag{3-32}$$

is an epsilon function, which we accordingly can call (and have called) $\varepsilon_h(x - x_0)$. Consider why this is entirely OK.

|||| Exercise 3.22

Find the derivatives of the following functions for every x-value in their respective domains:

$$\begin{aligned} f_1(x) &= (x^2+1) \cdot \sin(x) \\ f_2(x) &= \sin(x)/(x^2+1) \\ f_3(x) &= \sin(x^2+1) \quad . \end{aligned} \tag{3-33}$$

3.5 Inverse Functions

The exponential function $\exp(x)$ and the logarithmic function $\ln(x)$ are ***inverse functions*** to each other – as is well known the following is valid:

$$\begin{aligned} \exp(\ln(x)) = x \quad &\text{for} \quad x \in D(\ln) =]0, \infty[= R(\exp) \\ \ln(\exp(x)) = x \quad &\text{for} \quad x \in D(\exp) =]-\infty, \infty[= R(\ln) \quad . \end{aligned} \tag{3-34}$$

Note that even though $\exp(x)$ is defined for all x, the inverse function $\ln(x)$ is only defined for $x > 0$ – and vice versa (!).

The function $f(x) = x^2$ has an inverse function in its respective intervals of monotony, i.e. where $f(x)$ is either increasing or decreasing: The inverse function on the interval $[0, \infty[$ where $f(x)$ is increasing is the well-known function $g(x) = \sqrt{x}$. Thus the function $f(x)$ maps the interval $A = [0, \infty[$ one-to-one onto the interval $B = [0, \infty[$, and the inverse function $g(x)$ maps the interval B one-to-one onto the interval A such that:

$$\begin{aligned} f(g(x)) &= (\sqrt{x})^2 = x \quad \text{for} \quad x \in B = [0, \infty[\\ g(f(x)) &= \sqrt{x^2} = x \quad \text{for} \quad x \in A = [0, \infty[\quad . \end{aligned} \tag{3-35}$$

The inverse function to $f(x)$ on the interval $]-\infty, 0]$ where $f(x)$ is decreasing is the function $h(x) = -\sqrt{x}$, which is not defined on the *same* interval as $f(x)$. The function $f(x)$ maps the interval $C =]-\infty, 0]$ one-to-one onto the interval $D = [0, \infty[$, and the inverse function $h(x)$ maps the interval D one-to-one onto the interval C such that:

$$\begin{aligned} f(h(x)) &= (-\sqrt{x})^2 = x \quad \text{for} \quad x \in D = [0, \infty[\\ h(f(x)) &= -\sqrt{x^2} = x \quad \text{for} \quad x \in C =]-\infty, 0] \quad . \end{aligned} \tag{3-36}$$

If $f(x)$ is not monotonic on an interval, it means that we can obtain the *same function-value* $f(x)$ for more x-values – in the same way as $x^2 = 1$ both for $x = 1$ and for $x = -1$, and then the function is not one-to-one on the interval. The functions $\cos(x)$ and $\sin(x)$ are only monotonic on certain subintervals of the x-axis, see Figure 3.7. If we wish to define inverse functions to the functions we must choose the interval with care, see Section 3.8 and Figure 3.8.

|||| Definition 3.23 Notation for Inverse Functions

We denote the inverse function for a given function $f(x)$ by $f^{\circ -1}(x)$. The inverse function is generally defined by the following properties on suitably chosen intervals A and B that are part of $D(f)$ and $D(f^{\circ -1})$, respectively

$$\begin{aligned} f^{\circ -1}(f(x)) &= x \quad \text{for} \quad x \in A \subset D(f) \\ f(f^{\circ -1}(x)) &= x \quad \text{for} \quad x \in B \subset D(f^{\circ -1}) \quad . \end{aligned} \tag{3-37}$$

We use here the symbol $f^{\circ -1}(x)$ in order to avoid confusion with $(f(x))^{-1} = 1/f(x)$. However the reader should note that the standard notation is simply f^{-1} for the inverse function. The graph for the inverse function $g(x) = f^{\circ -1}(x)$ to a function $f(x)$ can be obtained by ***mirroring the graph*** for $f(x)$ in the diagonal in the first quadrant in the (x, y)-coordinate system – i.e. the line with the equation $y = x$ – see Figure 3.4.

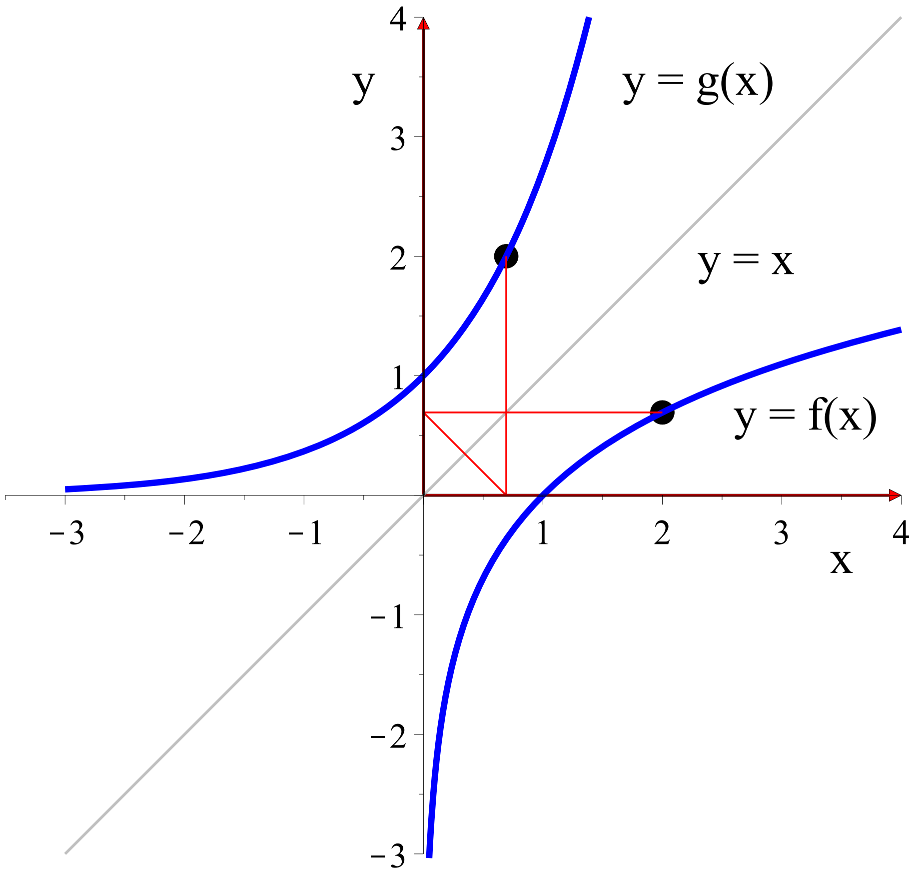

Figure 3.4: The graph for a function $f(x)$ and the graph for the inverse function $g(x)$. It is valid that $g(x) = f^{\circ -1}(x)$ and $f(x) = g^{\circ -1}(x)$, but they each have their own definition intervals.

3.5.1 Differentiation of Inverse Functions

|||| Theorem 3.24 Differentiation of Inverse Functions

If a differentiable function $f(x)$ has the inverse function $f^{\circ -1}(x)$ and if $f'(f^{\circ -1}(x_0)) \neq 0$, then the inverse function$f^{\circ -1}(x)$ is itself differentiable at x_0:

$$(f^{\circ -1})'(x_0) = \frac{1}{f'(f^{\circ -1}(x_0))} \tag{3-38}$$

|||| **Proof**

From the definition of inverse functions we have

$$h(x) = f(f^{\circ -1}(x)) = x \quad , \tag{3-39}$$

so $h'(x_0) = 1$, but we also have from the chain rule in (3-28):

$$h'(x_0) = f'(f^{\circ -1}(x_0)) \cdot (f^{\circ -1})'(x_0) = 1 \quad , \tag{3-40}$$

from which we get the result by dividing by $f'(f^{\circ -1}(x_0))$.

■

3.6 Hyperbolic Functions

|||| **Definition 3.25 Hyperbolic Cosine and Hyperbolic Sine**

We will define two new functions $\cosh(x)$ and $\sinh(x)$ as the unique solution to the following system of differential equations with initial conditions. The two solutions are denoted ***hyperbolic cosine*** and ***hyperbolic sine***, respectively:

$$\begin{aligned} \cosh'(x) &= \sinh(x) \quad , \quad \cosh(0) = 1 \\ \sinh'(x) &= \cosh(x) \quad , \quad \sinh(0) = 0 \quad . \end{aligned} \tag{3-41}$$

The *names* $\cosh(x)$ and $\sinh(x)$ (often spoken as "cosh" and "sinsh") look like $\cos(x)$ and $\sin(x)$, but the functions are very different, as we shall demonstrate below.

Yet there are also fundamental structural similarities between the two pairs of functions and this is what motivates the names. In the system of differential equations for $\cos(x)$

and $\sin(x)$ only a single minus sign separates this from (3-41):

$$\begin{aligned} \cos'(x) &= -\sin(x) \quad , \quad \cos(0) = 1 \\ \sin'(x) &= \cos(x) \quad , \quad \sin(0) = 0 \quad . \end{aligned} \tag{3-42}$$

In addition (again with the decisive minus sign as the only difference) the following simple analogy to the well-known and often used relation $\cos^2(x) + \sin^2(x) = 1$ applies:

|||| Theorem 3.26 Fundamental Relation of $\cosh(x)$ and $\sinh(x)$

$$\cosh^2(x) - \sinh^2(x) = 1 \quad . \tag{3-43}$$

|||| Proof

Make the derivative with respect to x on both sides of the equation (3-43) and conclude that $\cosh^2(x) - \sinh^2(x)$ is a constant. Finally use the initial conditions.

■

|||| Exercise 3.27

Show directly from the system of differential equations (3-41) that the two "new" functions are in fact not so new:

$$\begin{aligned} \cosh(x) &= \frac{e^x + e^{-x}}{2} \quad , \quad D(\cosh) = \mathbb{R} \quad , \quad R(\cosh) = [1, \infty[\\ \sinh(x) &= \frac{e^x - e^{-x}}{2} \quad , \quad D(\sinh) = \mathbb{R} \quad , \quad R(\sinh) =]-\infty, \infty[\end{aligned} \tag{3-44}$$

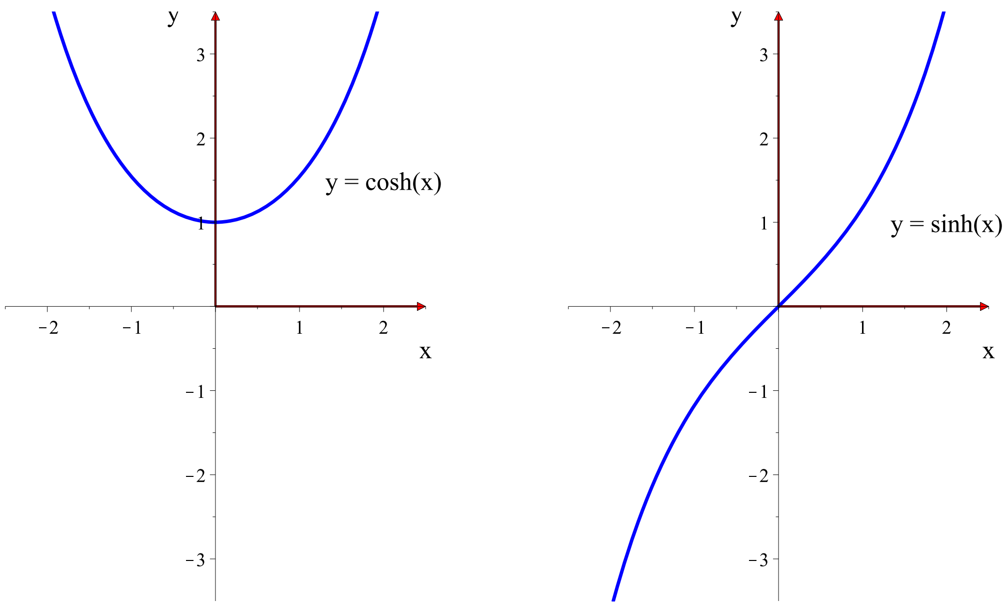

Figure 3.5: Hyperbolic cosine, $\cosh(x)$, and hyperbolic sine, $\sinh(x)$.

|||| Exercise 3.28

Show *directly* from the expressions found in Exercise 3.27, that

$$\cosh^2(x) - \sinh^2(x) = 1 \quad . \tag{3-45}$$

|||| Exercise 3.29

The graph for the function $f(x) = \cosh(x)$ looks a lot like a parabola, viz. the graph for the function $g(x) = 1 + (x^2/2)$ when we plot both functions on a suitably small interval around $x_0 = 0$. Try this! If we instead plot the two graphs in very large x-interval, we learn that the two functions have very different graphical behaviours. Try this, i.e. try to plot both functions on the interval $[-50, 50]$. Comment upon and explain the qualitative differences. Similarly compare the two functions $\sinh(x)$ and $x + (x^3/6)$ in the same way.

It is natural and useful to define hyperbolic analogies to $\tan(x)$ and $\cot(x)$. This is done as follows:

|||| Definition 3.30 Hyperbolic Tangent and Hyperbolic Cotangent

$$
\begin{aligned}
\tanh(x) &= \frac{\sinh(x)}{\cosh(x)} = \frac{e^{2x}-1}{e^{2x}+1} \,,\; D(\tanh) = \mathbb{R} \,,\; R(\tanh) =]-1,1[\\
\coth(x) &= \frac{\cosh(x)}{\sinh(x)} = \frac{e^{2x}+1}{e^{2x}-1} \,,\; D(\coth) = \mathbb{R} - \{0\} \,, \\
&\qquad R(\coth) =]-\infty,-1[\cup]1,\infty[\quad .
\end{aligned}
\tag{3-46}
$$

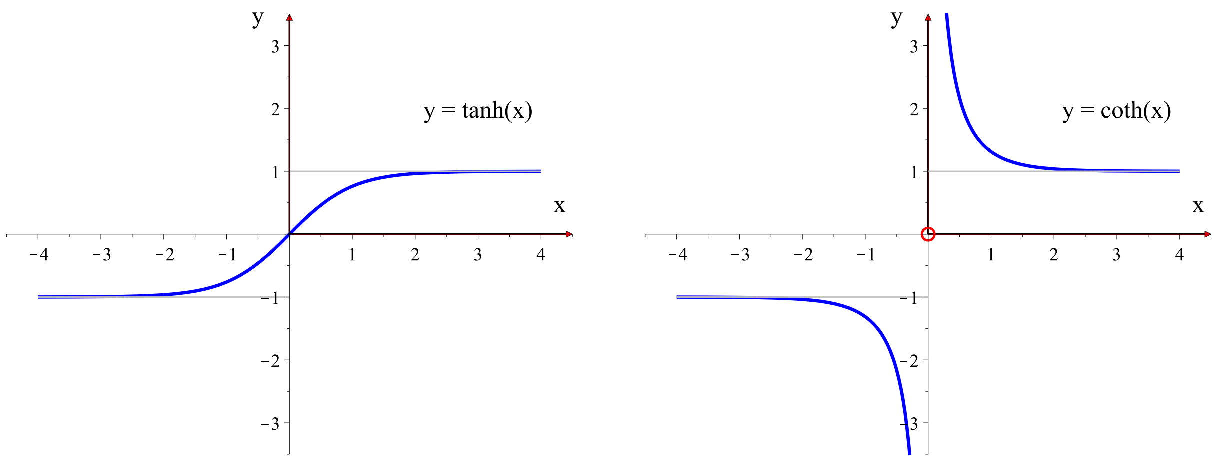

Figure 3.6: Hyperbolic tangent, $\tanh(x)$, and hyperbolic cotangent, $\coth(x)$.

The derivatives of $\cosh(x)$ and of $\sinh(x)$ are already given by the defining system in (3-41).

$$
\begin{aligned}
\frac{d}{dx}\cosh(x) &= \sinh(x) \\
\frac{d}{dx}\sinh(x) &= \cosh(x) \\
\frac{d}{dx}\tanh(x) &= \frac{1}{\cosh^2(x)} = 1 - \tanh^2(x) \\
\frac{d}{dx}\coth(x) &= \frac{-1}{\sinh^2(x)} = 1 - \coth^2(x) \quad .
\end{aligned}
\tag{3-47}
$$

|||| **Exercise 3.31**

Show the last two expressions for the derivatives for $\tanh(x)$ and $\coth(x)$ in (3-47) by the use of the differentiation rule in Theorem 3.18.

3.7 The Area Functions

The inverse functions to the hyperbolic functions are called ***area functions*** and are named $\cosh^{\circ -1}(x) = \text{arcosh}(x)$, $\sinh^{\circ -1}(x) = \text{arsinh}(x)$, $\tanh^{\circ -1}(x) = \text{artanh}(x)$, and $\coth^{\circ -1}(x) = \text{arcoth}(x)$, respectively.

Since the functions $\cosh(x)$, $\sinh(x)$, $\tanh(x)$, and $\coth(x)$ all can be expressed in terms of exponential functions it is no surprise that the inverse functions and their derivatives can be expressed by logarithmic functions. We gather the information here:

$$
\begin{aligned}
\text{arcosh}(x) &= \ln(x + \sqrt{x^2 - 1}) \text{ for } \quad x \in [1, \infty[\\
\text{arsinh}(x) &= \ln(x + \sqrt{x^2 + 1}) \text{ for } \quad x \in \mathbb{R} \\
\text{artanh}(x) &= \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right) \quad \text{for} \quad x \in] - 1, 1[\\
\text{arcoth}(x) &= \frac{1}{2} \ln\left(\frac{x - 1}{x + 1}\right) \quad \text{for} \quad x \in] - \infty, 1[\cup]1, \infty[\quad .
\end{aligned}
\tag{3-48}
$$

$$
\begin{aligned}
\frac{d}{dx}\text{arcosh}(x) &= \frac{1}{\sqrt{x^2 - 1}} \quad \text{for} \quad x \in]1, \infty[\\
\frac{d}{dx}\text{arsinh}(x) &= \frac{1}{\sqrt{x^2 + 1}} \quad \text{for} \quad x \in \mathbb{R} \\
\frac{d}{dx}\text{artanh}(x) &= \frac{1}{1 - x^2} \quad \text{for} \quad x \in] - 1, 1[\\
\frac{d}{dx}\text{arcoth}(x) &= \frac{1}{1 - x^2} \quad \text{for} \quad x \in] - \infty\, 1[\cup]1, \infty[\quad .
\end{aligned}
\tag{3-49}
$$

3.8 The Arc Functions

The inverse functions to the trigonometric functions are a bit more complicated. As mentioned earlier here we must choose for each trigonometric function an interval

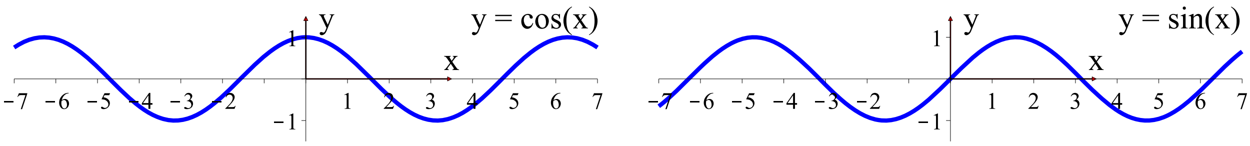

Figure 3.7: Cosine and Sine Functions.

where the function in question is monotonic. In return, once we *have* chosen such an interval, it is clear how the inverse function should be defined and how it should then be differentiated. The inverse functions to $\cos(x)$, $\sin(x)$, $\tan(x)$, and $\cot(x)$ are usually written $\arccos(x)$, $\arcsin(x)$, $\arctan(x)$, and $\text{arccot}(x)$, respectively; their names are arccosine, arcsine, arctangent, and arccotangent. As above we gather the results here:

$$\begin{aligned}
\cos^{\circ -1}(x) &= \arccos(x) \in [0,\pi] \text{ for } \quad x \in [-1,1] \\
\sin^{\circ -1}(x) &= \arcsin(x) \in [-\pi/2,\pi/2] \text{ for } \quad x \in [-1,1] \\
\tan^{\circ -1}(x) &= \arctan(x) \in [-\pi/2,\pi/2] \text{ for } \quad x \in \mathbb{R} \\
\cot^{\circ -1}(x) &= \text{arccot}(x) \in]0,\pi[\text{ for } \quad x \in \mathbb{R} \quad .
\end{aligned} \tag{3-50}$$

$$\begin{aligned}
\frac{d}{dx}\arccos(x) &= \frac{-1}{\sqrt{1-x^2}} \text{ for } \quad x \in]-1,1[\\
\frac{d}{dx}\arcsin(x) &= \frac{1}{\sqrt{1-x^2}} \text{ for } \quad x \in]-1,1[\\
\frac{d}{dx}\arctan(x) &= \frac{1}{1+x^2} \text{ for } \quad x \in \mathbb{R} \\
\frac{d}{dx}\text{arccot}(x) &= \frac{-1}{1+x^2} \text{ for } \quad x \in \mathbb{R} \quad .
\end{aligned} \tag{3-51}$$

Note that the derivatives for $\arccos(x)$ and $\arcsin(x)$ are not defined at $x_0 = 1$ or at $x_0 = -1$. This is partly because, if the function we consider is only defined on a bounded interval then we cannot say that the function is differentiable at the end-points of the interval. Moreover the formulas for $\arccos'(x)$ and $\arcsin'(x)$ show that they are not defined at $x_0 = 1$ or $x_0 = -1$; these values give 0 in the denominators.

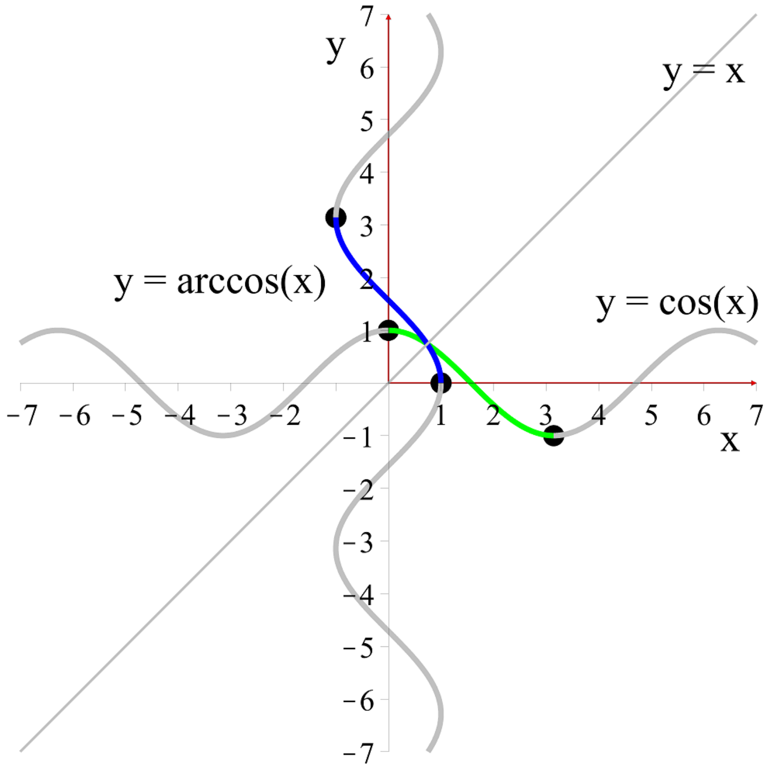

Figure 3.8: The arccosine function is defined here.

|||| Exercise 3.32

Use a suitable modification of $\arctan(x)$ in order to determine a new differentiable (and hence continuous) function $f(x)$ that looks like the 0-extension of $|x|/x$ (which is neither continuous nor differentiable), i.e. we want a function $f(x)$ with the following properties: $1 > f(x) > 0.999$ for $x > 0.001$ while $-0.999 > f(x) > -1$ for $x < -0.001$. See Figure 3.10. Hint: Try to plot $\arctan(1000x)$.

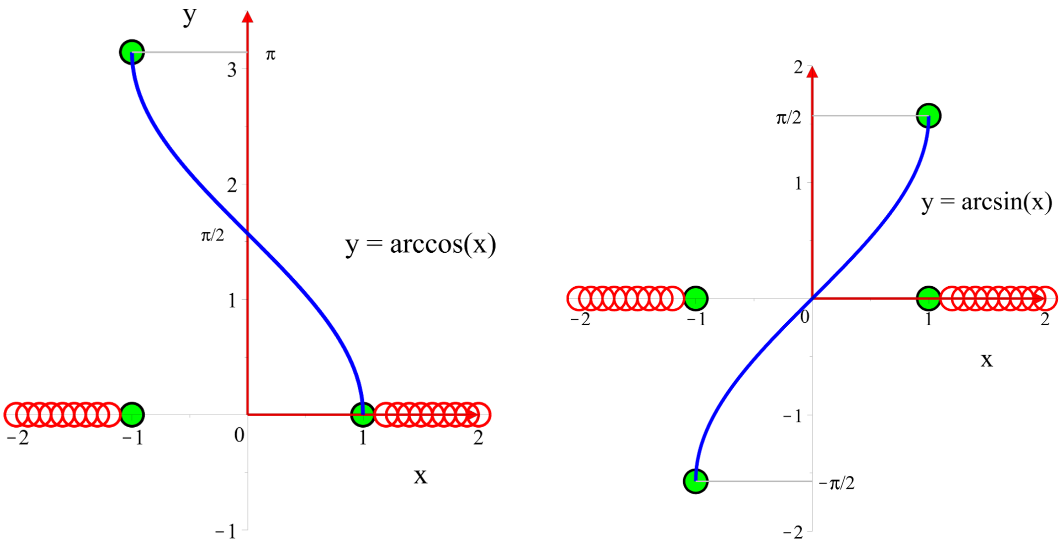

Figure 3.9: Arccosine and arcsine. Again the red circles indicate that the arc-functions are not defined outside the interval $[-1, 1]$. Similarly the green circular disks indicate that the arc-functions *are* defined at the end-points $x = 1$ and $x = -1$.

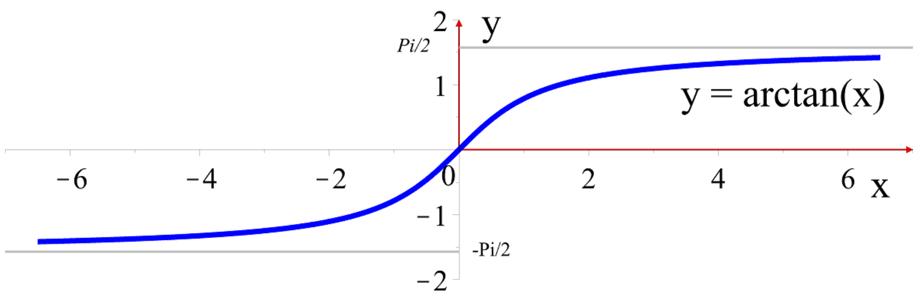

Figure 3.10: The arctangent function.

3.9 Summary

We have treated some of the fundamental properties of some well-known and some not so well-known functions. How are they defined, what are their domains, are they continuous, are they differentiable, and if so what are their derivatives?

- A function $f(x)$ is continuous at x_0 if $f(x) - f(x_0)$ is an epsilon function of $(x - x_0)$, i.e.
$$f(x) = f(x_0) + \varepsilon_f(x - x_0) \quad . \tag{3-52}$$
- A function $f(x)$ is differentiable at x_0 with the derivative $f'(x_0)$ if
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + (x - x_0)\varepsilon_f(x - x_0) \quad .$$
- If a function is differentiable at x_0, then it is also continuous at x_0. The converse does *not* apply.
- The derivative of a product of two functions is
$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \quad . \tag{3-53}$$
- The derivative of a quotient of two functions is
$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)}{g(x)} - \frac{f(x) \cdot g'(x)}{g^2(x)} = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} \quad . \tag{3-54}$$
- The derivative of a composite function is
$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \quad . \tag{3-55}$$
- The derivative of the inverse function $f^{\circ -1}(x)$ is
$$\left(f^{\circ -1}\right)'(x) = \frac{1}{f'(f^{\circ -1}(x))} \quad . \tag{3-56}$$