eNote 2

Polynomials of One Variable

In this eNote complex polynomials of one variable are introduced. An elementary knowledge of complex numbers is a prerequisite, and knowlege of real polynomials of one real variable is recommended.

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2.1 Introduction

Polynomials are omnipresent in the technical literature about mathematical models of physical problems. A great advantage of polynomials is the simplicity of computation since only addition, multiplication and powers are needed. Because of this polynomials are especially applicable as approximations to more complicated types of functions.

Knowledge about the *roots* of polynomials is the main road to understanding their properties and efficient usage, and is therefore a major subject in the following. But first we introduce some general properties.

|||| Definition 2.1

By a *polynomial* of degree n we understand a function that can be written in the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \tag{2-1}$$

where $a_0, a_1, \ldots, a_n$ are complex constants with $a_n \neq 0$, and z is a complex variable.

a_k is called the *coefficient* of z^k, $k = 0, 1, \ldots, n$, and a_n is the *leading coefficient*.
A *real polynomial* is a polynomial in which all the coefficients are real.
A *real polynomial of a real variable* is a real polynomial in which we assume $z \in \mathbb{R}$.

Polynomials are often denoted by a capital P or similar letter $Q, R, S \ldots$ If the situation requires you to include the variable name, the polynomial is written as $P(z)$ where it is understood that z is an independent complex variable.

|||| Example 2.2 Examples of Polynomials

$P(z) = 2z^3 + (1+i)z + 5$ is a polynomial of the third degree.
$Q(z) = z^2 + 1$ is a real quadratic polynomial.
$R(z) = 17$ is a polynomial of the $0'$th degree.
$S(z) = 0$ is called the 0-polynomial and is not assigned any degree.
$T(z) = 2z^3 + 5\sqrt{z} - 4$ is not a polynomial.

If you multiply a polynomial by a constant, or when you add, subtract, multiply and compose polynomials with each other, you get a new polynomial. This polynomial can be simplified by gathering terms of the same degree and written in the form (2.1).

|||| Example 2.3 Addition and Multiplication of Polynomials

Two polynomials P and Q are given by $P(z) = z^2 - 1$ and $Q(z) = 2z^2 - z + 2$. The polynomials $R = P + Q$ and $S = P \cdot Q$ are determined like this:

$$\begin{aligned}
R(z) &= (z^2 - 1) + (2z^2 - z + 2) = (z^2 + 2z^2) + (-z) + (-1 + 2) = 3z^2 - z + 1\,.\\
S(z) &= (z^2 - 1) \cdot (2z^2 - z + 2) = (2z^4 - z^3 + 2z^2) + (-2z^2 + z - 2)\\
&= 2z^4 - z^3 + (2z^2 - 2z^2) + z - 2 = 2z^4 - z^3 + z - 2\,.
\end{aligned}$$

2.2 The Roots of Polynomials

|||| Definition 2.4 Root of a Polynomial

By a *root* of a polynomial $P(z)$ we understand a number z_0 such that $P(z_0) = 0$.

|||| Example 2.5 Whether a Given Number Is a Root of a Polynomial

Show that 3 is a root of $P(z) = z^3 - 5z - 12$, and that 1 is not a root.

Since $P(3) = 3^3 - 5 \cdot 3 - 12 = 0$, 3 is a root of P.
Since $P(1) = 1^3 - 5 \cdot 1 - 12 = -16 \neq 0$, 1 is not a root of P.

To develop the theory we shall need the following Lemma.

|||| Lemma 2.6 The Theorem of Descent

A polynomial P of the degree n is given by

$$P(z) = a_n\, z^n + a_{n-1}\, z^{n-1} + \cdots + a_1\, z + a_0\,. \tag{2-2}$$

If z_0 is an arbitrary number, and Q is the polynomial of the n-$1'$ degree given by the coefficients

$$b_{n-1} = a_n \tag{2-3}$$
$$b_k = a_{k+1} + z_0 \cdot b_{k+1} \quad \text{for} \quad k = n{-}2, \ldots, 0 \quad , \tag{2-4}$$

then P can be written in the factorized form

$$P(z) = (z - z_0)Q(z) \tag{2-5}$$

if and only if z_0 is a root of P.

|||| Proof

Let the polynomial P be given as in the theorem, and let α be an arbitrary number. Consider an arbitrary $(n-1)$-degree polynomial

$$Q(z) = b_{n-1}\, z^{n-1} + b_{n-2}\, z^{n-2} + \cdots + b_1\, z + b_0\,.$$

By simple calculation we get

$$(z-\alpha)Q(z) = b_{n-1}\, z^n + (b_{n-2} - \alpha b_{n-1})\, z^{n-1} + \cdots + (b_0 - \alpha b_1)z - \alpha b_0\,.$$

It is seen that the polynomials $(z-\alpha)Q(z)$ and $P(z)$ have the same representation if we in succession write the b_k-coefficients for Q as given in (2-3) and (2-4), and if at the same time the following is valid:

$$-\alpha b_0 = a_0 \;\Leftrightarrow\; b_0\, \alpha = -a_0\,.$$

We investigate whether this condition is satisfied by using (2-3) and (2-4) in the opposite order:

$$\begin{aligned} b_0\,\alpha &= (a_1+\alpha b_1)\alpha = b_1\alpha^2 + a_1\alpha \\ &= (a_2+\alpha b_2)\alpha^2 + a_1\alpha = b_2\alpha^3 + a_2\alpha^2 + a_1\alpha \\ &\vdots \\ &= b_{n-1}\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_2\alpha^2 + a_1\alpha \\ &= a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_2\alpha^2 + a_1\alpha = -a_0 \\ \Leftrightarrow\ P(\alpha) &= a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_2\alpha^2 + a_1\alpha + a_0 = 0\,. \end{aligned}$$

It is seen that the condition is only satisfied if and only if α is a root of P. By this the proof is complete.

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|||| Example 2.7 Descent

Given the polynomial $P(z) = 2z^4 - 12z^3 + 19z^2 - 6z + 9$. It is seen that 3 is a root since $P(3) = 0$. Determine a third-degree polynomial Q such that

$$P(z) = (z-3)Q(z)\,.$$

We set $a_4 = 2,\ a_3 = -12,\ a_2 = 19,\ a_1 = -6$ og $a_0 = 9$ and find the coefficients for Q by the use of (2-3) and (2-4):

$$\begin{aligned} b_3 &= a_4 = 2 \\ b_2 &= a_3 + 3b_3 = -12 + 3\cdot 2 = -6 \\ b_1 &= a_2 + 3b_2 = 19 + 3\cdot(-6) = 1 \\ b_0 &= a_1 + 3b_1 = -6 + 3\cdot 1 = -3\,. \end{aligned}$$

We conclude that

$$Q(z) = 2z^3 - 6z^2 + z - 3$$

so

$$P(z) = (z-3)\,(2z^3 - 6z^2 + z - 3)\,.$$

When a polynomial P with the root z_0 is written in the form $P(z) = (z-z_0)Q_1(z)$, where Q_1 is a polynomial, it is possible that z_0 is also a root of Q_1. Then Q_1 can similarly be written as $Q_1(z) = (z - z_0)Q_2(z)$ where Q_2 is a polynomial. And in this way the descent can successively be carried out as $P(z) = (z - z_0)^m R(z)$ where R is a polynomial in which z_0 is not a root. We will now show that this factorization is unique.

|||| Theorem 2.8 The Multiplicity of a Root

If z_0 is a root of the polynomial P, it can in exactly one way be written in factorised form as:

$$P(z) = (z - z_0)^m \, R(z) \tag{2-6}$$

where $R(z)$ is a polynomial for which z_0 is not a root.

The exponent m is called the *algebraic multiplicity* of the root z_0.

|||| Proof

Assume that α is a root of P, and that (contrary to the statement in the theorem) there exist two different factorisations

$$P(z) = (z - \alpha)^r \, R(z) = (z - \alpha)^s \, S(z)$$

where $r > s$, and $R(z)$ and $S(z)$ are polynomials of which α is not a root. We then get

$$(z - \alpha)^r \, R(z) - (z - \alpha)^s \, S(z) = (z - \alpha)^s \big((z - \alpha)^k R(z) - S(z) \big) = 0 \,, \text{ for all } z \in \mathbb{C}$$

where $k = r - s$. This equation is only satisfied if

$$(z - \alpha)^k \, R(z) = S(z) \quad \text{for all} \quad z \neq \alpha \,.$$

Since both the left-hand and the right-hand sides are continuous functions, they must have the same value at $z = \alpha$. From this we get that

$$S(\alpha) = (z - \alpha)^k R(\alpha) = 0$$

which is contradictory to the assumption that α is not a root of S.

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|||| **Example 2.9**

In Example 2.7 we found that

$$P(z) = (z-3)\,(2z^3 - 6z^2 + z - 3)$$

where 3 is a root. But 3 is also a root of the factor $2z^3 - 6z^2 + z - 3$. By using the theorem of descent, Theorem 2.6, on this polynomial we get

$$P(z) = (z-3)\;\;(z-3)(2z^2+1)\;\; = (z-3)^2\,(2z^2+1)\,.$$

Since 3 is not a root of $2z^2+1$, the root 3 in P has the multiplicity 2.

Now we have started a process of descent! How far can we get along this way? To continue this investigation we will need a fundamental result, viz the *Fundamental Theorem*.

2.2.1 The Fundamental Theorem of Algebra

A decisive reason for the introduction of complex numbers is that every (complex) polynomial has a root in the set of complex numbers. This result was proven by the mathematician Gauss in his ph.d.-dissertation from 1799. The proof of the theorem is demanding, and Gauss strove all his life to refine his proof more. Four versions of the proof by Gauss exist, so there is no doubt that he put a lot of emphasis on this theorem. Here we take the liberty to state Gauss' result without proof:

|||| **Theorem 2.10 The Fundamental Theorem of Algebra**

Every polynomial of degree $n \geq 1$ has at least one root within the set of complex numbers.

The polynomial $P(z) = z^2 + 1$ has no roots within the set of real numbers. But within the set of complex numbers it has two roots i and $-\mathrm{i}$ because

$$P(\mathrm{i}) = \mathrm{i}^2 + 1 = -1 + 1 = 0 \;\text{ and }\; P(-\mathrm{i}) = (-\mathrm{i})^2 + 1 = \mathrm{i}^2 + 1 = 0\,.$$

The road from the fundamental theorem of algebra until full knowledge of the number of roots is not long. We only have to develop the ideas put forward in the theorem of descent futher.

We consider a polynomial P of degree n with leading coefficient a_n. If $n \geq 1$, P has according to the fundamental theorem of algebra a root α_1 and therefore by the use of method of coefficients, cf. Theorem 2.6 it can be written as

$$P(z) = (z - \alpha_1) Q_1(z) \tag{2-7}$$

where Q_1 is a polynomial of degree n-1 with leading coefficient a_n. If $n \geq 2$, then Q_1 has a root α_2 and can be written as

$$Q_1(z) = (z - \alpha_2) Q_2(z)$$

where Q_2 is a polynomial of degree n-2 also with a leading coefficient a_n. By substitution we now get

$$P(z) = (z - \alpha_1)(z - \alpha_2) Q_2(z) .$$

In this way the construction of polynomials of descent Q_k of degree $n - k$ for $k = n - 1, \ldots, 0$ continues until we reach the polynomial Q_n of degree n-$n = 0$, which in accordance with Example 2.2, is equal to its leading coefficient a_n. Hereafter P can be written in its *completely factorized form*:

$$P(z) = a_n (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n) . \tag{2-8}$$

In this expression we should note three things:

- First all the n numbers $\alpha_1, \ldots, \alpha_n$ that are listed in (2-8), are roots of P since substitution into the formula gives the value 0.
- The second thing we notice is that P cannot have other roots than the n given ones. That there cannot be more roots is easily seen as follows: If an arbitrary number $\alpha \neq \alpha_k, k = 1, \ldots, n$, is inserted in place of z in (2-8), all factors on the right-hand side of (2-8) will be different from zero. Hence their product will also be different form zero. Therefore $P(\alpha) \neq 0$, and α is not a root of P.
- The last thing we notice in (2-8), is that the roots are not necessarily different. If $z_1, z_2, \ldots, z_p$ are the p *different* roots of P, and m_k is the multiplicity of z_k, $k = 1, \ldots, p$, then the completely factorized form (2-8) can be simplified as follows

  $$P(z) = a_n (z - z_1)^{m_1} (z - z_2)^{m_2} \cdots (z - z_p)^{m_p} \tag{2-9}$$

  where the following applies:

  $$m_1 + m_2 \cdots + m_p = n .$$

According to the preceding arguments we can now present the fundamental theorem of algebra in the extended form.

|||| Theorem 2.11 The Fundamental Theorem of Algebra — Version 2

Every polynomial of degree $n \geq 1$ has within the set of complex numbers exactly n roots, when the roots are counted with multiplicity.

|||| Example 2.12 Quadratic Polynomial in Completely Factorized Form

An arbitrary quadratic polynomial $P(z) = az^2 + bz + c$ can be written in the form

$$P(z) = a(z-\alpha)(z-\beta)$$

where α and β are roots of P. If $\alpha \neq \beta$, P has two different roots, both with algebraic multiplicity 1. If $\alpha = \beta$, P has only one root with the algebraic multiplicity 2. The roots are then denoted a *double root*.

|||| Example 2.13 Algebraic Multiplicity

A polynomial P is given in complete factorized form as:

$$P(z) = 7(z-1)^2(z+4)^3(z-5)\,.$$

We see that P has three different roots: $1, -4$ and 5 with the algebraic multiplicities 2, 3 and 1, respectively.

We notice that the sum of the algebraic multiplicities is 6 which equals the degree of P in concordance with the Fundamental Theorem of Algebra — Version 2.

|||| Example 2.14 Algebraic Multiplicity

State the number of roots of $P(z) = z^3$.

P has only one root $z = 0$. The algebraic multiplicity of this root is 3. One says that 0 is a *triple root* in the polynomial.

2.3 Identical Polynomials

Two polynomials P and Q are equal (as functions of z) if $P(z) = Q(z)$ for all z. But what does it *take* for two polynomials to be equal? Is it possible that a fourth-degree and a fifth-degree polynomial take on the same value for all variables as long as you choose the right coefficients? This is not the case as is seen from the following theorem.

|||| Theorem 2.15 The Identity Theorem for Polynomials

Two polynomials are identical if and only if they are of the same degree, and all coefficients for corresponding terms of the same degree from the two polynomials are equal.

|||| Proof

We consider two arbitrary polynomials P og Q. If they are of the same degree, and all the coefficients for terms of the same degree are equal, they must have the same value for all variables and hence they are identical. This proves the first direction of the theorem of identity.

Assume hereafter that P og Q are identical as functions of z, but that not all coefficients for terms of the same degree from the two polynomials are equal. We assume further that P has the degree n and Q the degree m where $n \geq m$. Let a_k be the coefficients for P and let b_k be the coefficients for Q, and consider the difference polynomial

$$\begin{aligned} R(z) =& P(z) - Q(z) \\ =& (a_n - b_n)z^n + (a_{n-1} - b_{n-1})z^{n-1} + \cdots + (a_1 - b_1)z + (a_0 - b_0) \end{aligned} \tag{2-10}$$

where we for the case $n > m$ put $b_k = 0$ for $m < k \leq n$. We note that the 0-degree coefficient $(a_0 - b_0)$ cannot be the only coefficient of $R(z)$ that is different from 0, since this would make $P(0) - Q(0) = (a_0 - b_0) \neq 0$ which contradicts that P and Q are identical as functions. Therefore the degree of R is greater than or equal to 1. On the other hand (2-10) shows that the degree of R at the most is n. Now let z_k, $k = 1, \ldots, n+1$, be $n+1$ different numbers. They are all roots of R since

$$R(z_k) = P(z_k) - Q(z_k) = 0\,,\; k = 1\ldots n+1\,.$$

This contradicts the fundamental theorem of algebra – version 2, Theorem 2.11: R cannot have a number of roots that is higher than its degree. The assumption, that not all coefficients of terms of the same degree from P and Q are equal, must therefore be wrong. From this it also follows that P and Q have the same degree. By this the second part of the identity theorem is proven.

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|||| Example 2.16 Two Identical Polynomials

The equation

$$3z^2 - z + 4 = az^2 + bz + c$$

is satisfied for all z exactly when $a = 3$, $b = -1$ og $c = 4$.

|||| Exercise 2.17 To Identical Polynomials

Determine the numbers a, b and c such that

$$(z-2)(az^2 + bz + c) = z^3 - 5z + 2 \quad \text{for} \quad \text{all} \quad z\,.$$

In the following section we treat methods of finding roots of certain types of polynomials.

2.4 Polynomial Equations

From the fundamental theorem of algebra, Theorem 2.10, we know that every polynomial of degree greater than or equal to 1 has roots. Moreover, in the extended version, Theorem 2.11, it is maintained that for every polynomial the degree is equal to the number of roots if the roots are counted with multiplicity. But the theorem is a theoretical theorem of existence that does not help in *finding* the roots.

In the following methods for finding the roots of simple polynomials are introduced.

But let us keep the level of ambition (safely) low, because in the beginning of the $17'$th century the Norwegian algebraicist Abel showed that one *cannot* establish general methods for finding the roots of arbitrary polynomials of degree larger than four!

For polynomials of higher degree than four a number of smart tricks exist by which one can successfully find a single root. Hereafter one descends to a polynomial of lower degree — and successively decends to a polynomial of fourth degree or lower for which one can find the remaining roots.

Let us at the outset maintain that when you want to find the roots of a polynomial $P(z)$, you should solve the corresponding *polynomial equation* $P(z) = 0$. As a simple illustration we can look at the root of an arbitrary first-degree polynomial:

$$P(z) = az + b\,.$$

To find this we shall solve the equation

$$az + b = 0\,.$$

this is not difficult. It has the solution $z_0 = -\dfrac{b}{a}$ which therefore is a root of $P(z)$.

Finding the *roots* of a polynomial P, is tantamount to finding the *solutions* to the polynomial equation $P(z) = 0$.

|||| Example 2.18 The Root of a Linear Polynomial

Find the root of a linear polynomial P given by

$$P(z) = (1-\mathrm{i})\,z - (5+2\mathrm{i})\,.$$

We shall solve the following equation

$$(1-\mathrm{i})\,z - (5+2\mathrm{i}) = 0 \Leftrightarrow (1-\mathrm{i})\,z = (5+2\mathrm{i})\,.$$

We isolate z on the left-hand side:

$$z = \frac{5+2\mathrm{i}}{1-\mathrm{i}} = \frac{(5+2\mathrm{i})(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})} = \frac{3+7\mathrm{i}}{2} = \frac{3}{2} + \frac{7}{2}\mathrm{i}\,.$$

Hence the equation has the solution $z_0 = \dfrac{3}{2} + \dfrac{7}{2}\mathrm{i}$ that also is the root of P.

2.4.1 Binomial Equations

A binomial equation is an equation of the degree n in which only the coefficients a_n (the term of highest degree) and a_0 (the constant term) are different from 0. A given binomial equation can only be simplified to the following form:

|||| Definition 2.19 Binomial Equation

A binomial equation has the form $z^n = w$ where $w \in \mathbb{C}$ and $n \in \mathbb{N}$.

For binomial equations an explicit solution formula exists, which we present in the following theorem.

|||| Theorem 2.20 Binomial Equations Solved by Use of the Exponential Form

Let $w \neq 0$ be a complex number with the exponential form

$$w = |w|\,\mathrm{e}^{\mathrm{i}v}\,.$$

The binomial equation

$$z^n = w \tag{2-11}$$

has n different solutions given by the formula

$$z_p = \sqrt[n]{|w|}\;\mathrm{e}^{\mathrm{i}(\frac{v}{n}+p\frac{2\pi}{n})}\;\text{ where }\; p = 0\,,\,1\,,\,\ldots\,,\,n-1\,. \tag{2-12}$$

|||| Proof

For every $p \in \{0,1,\ldots,n-1\}$ $z_p = \sqrt[n]{|w|}\;\mathrm{e}^{\mathrm{i}(\frac{v}{n}+p\frac{2\pi}{n})}$ is a solution to (2-11), since

$$(z_p)^n = \left(\sqrt[n]{|w|}\mathrm{e}^{\mathrm{i}(\frac{v}{n}+p\frac{2\pi}{n})}\right)^n = |w|\;\mathrm{e}^{\mathrm{i}(v+p\,2\pi)} = |w|\;\mathrm{e}^{\mathrm{i}v} = w.$$

It is seen that the n solutions viewed as points in the complex plane all lie on a circle with centre at $z = 0$, radius $\sqrt[n]{|w|}$ and a consecutive angular distance of $\frac{2\pi}{n}$. In other words the connecting lines between $z = 0$ and the solutions divide the circle in n angles of the same size.

From this it follows that all n solutions are mutually different. That there are no more solutions is a consequence of the fundamental theorem of algebra – version 2, Theorem 2.11. By this the theorem is proven.

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In the next examples we will consider some important special cases of binomial equations.

|||| Example 2.21 Binomial Equation of the Second Degree

We consider a complex number in the exponential form $w = |w|\,\mathrm{e}^{\mathrm{i}v}$. It follows from (2-12) that the quadratic equation

$$z^2 = w$$

has two solutions

$$z_0 = \sqrt{|w|}\;\mathrm{e}^{\mathrm{i}\,\frac{v}{2}} \quad \text{and} \quad z_1 = -\sqrt{|w|}\;\mathrm{e}^{\mathrm{i}\,\frac{v}{2}}\,.$$

|||| Example 2.22 Binomial Equation of the Second Degree with a Negative Right-Hand Side

Let r be an arbitrary positive real number. By putting $v = \mathrm{Arg}(-r) = \pi$ in Example 2.21 it is seen that the binomial of the second degree

$$z^2 = -r$$

has two solutions

$$z_0 = \mathrm{i}\,\sqrt{r} \;\; \text{og} \;\; z_1 = -\mathrm{i}\,\sqrt{r}\,.$$

As a concrete example the equation $z^2 = -16$ has the solutions $z = \pm \mathrm{i}\,4$.

Sometimes the method used in Example 2.21 can be hard to carry out. In the following example we show an alternative method.

|||| Example 2.23 Binomial Equation of the Second Degree, Method 2

Solve the equation

$$z^2 = 8 - 6i\,. \tag{2-13}$$

Since we expect the solution to be complex we put $z = x + \mathrm{i}y$ where x and y are real numbers. If we can find x and y, then we have found the solutions for z. Therefore we have $z^2 = (x + \mathrm{i}y)^2 = x^2 - y^2 + 2xy\mathrm{i}$ and we see that (2-13) is equivalent to

$$x^2 - y^2 + 2xy\mathrm{i} = 8 - 6\mathrm{i}\,.$$

Since a complex equation is true exactly when both the real parts and the imaginary parts of the right-hand and the left-hand sides of the equation are identical, (2-13) is equivalent to

$$x^2 - y^2 = 8 \text{ and } 2xy = -6\,. \tag{2-14}$$

If we put $y = \frac{-6}{2x} = -\frac{3}{x}$ in $x^2 - y^2 = 8$, and put $x^2 = u$, we get a quadratic equation that can be solved:

$$x^2 - \left(-\frac{3}{x}\right)^2 = 8 \Leftrightarrow x^2 - \frac{9}{x^2} = 8 \Leftrightarrow$$
$$\left(x^2 - \frac{9}{x^2}\right)x^2 = 8x^2 \Leftrightarrow x^4 - 9 = 8x^2 \Leftrightarrow x^4 - 8x^2 - 9 = 0 \Leftrightarrow$$
$$u^2 - 8u - 9 = 0 \Leftrightarrow u = 9 \text{ or } u = -1\,.$$

The equation $x^2 = u = 9$ has the solutions $x_1 = 3$ and $x_2 = -3$, while the equation $x^2 = u = -1$ has no solution, since x and y are real numbers. If we put $x_1 = 3$ respective $x_2 = -3$ in (2-14), we get the corresponding y-values $y_1 = -1$ and $y_2 = 1$.

From this we conclude that the given equation (2-13) has the roots

$$z_1 = x_1 + \mathrm{i}y_1 = 3 - \mathrm{i} \text{ and } z_2 = x_2 + \mathrm{i}y_2 = -3 + \mathrm{i}\,.$$

2.4.2 Quadratic Equations

For the solution of quadratic equations we state below the formula that corresponds to the well-known solution formula for real quadratic equations. There is a single deviation, viz. we do not compute the square-root of the discriminant since we in this theorem do not presuppose knowledge of square-roots of complex numbers.

|||| Theorem 2.24 Solution Formula for Quadratic Equation

For the quadratic equation

$$az^2 + bz + c = 0 \,, \; a \neq 0 \tag{2-15}$$

we introduce the *discriminant* D by $D = b^2 - 4ac$. The equations has two solutions

$$z_1 = \frac{-b - w_0}{2a} \;\text{ og }\; z_2 = \frac{-b + w_0}{2a} \tag{2-16}$$

where w_0 is a solution to the binomial equation of the second degree $w^2 = D$.

If in particular $D = 0$, we have that $z_1 = z_2 = \dfrac{-b}{2a}$.

|||| Proof

Let w_0 be an arbitrary solution to the binomial equation $w^2 = D$. We then have:

$$
\begin{aligned}
az^2 + bz + c &= a\left(z^2 + \frac{b}{a}z + \frac{c}{a}\right) \\
&= a\left(\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right) \\
&= a\left(\left(z + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) \\
&= a\left(\left(z + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right) \\
&= a\left(\left(z + \frac{b}{2a}\right)^2 - \frac{w_0^2}{4a^2}\right) \\
&= a\left(\left(z + \frac{b}{2a}\right) + \frac{w_0}{2a}\right)\left(\left(z + \frac{b}{2a}\right) - \frac{w_0}{2a}\right) \\
&= a\left(z + \frac{b + w_0}{2a}\right)\left(z + \frac{b - w_0}{2a}\right) = 0 \\
\Leftrightarrow z &= \frac{-b - w_0}{2a} \quad \text{or} \quad z = \frac{-b + w_0}{2a} \; .
\end{aligned}
$$

By this the solution formula (2-16) is derived.

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|||| Example 2.25 Real Quadratic Equation with a Positive Value of the Discriminant

Solve the following quadratic equation with real coefficients:

$$2z^2 + 5z - 3 = 0 \, .$$

We identify the coefficients: $a = 2, b = 5, c = -3$, and find the discriminant as:

$$D = 5^2 - 4 \cdot 2 \cdot (-3) = 49 \, .$$

It is seen that $w_0 = 7$ is a solution to the binomial equation of the second degree $w^2 = D =$

49 . Now the solutions can be computed as:

$$z_1 = \frac{-5+7}{2\cdot 2} = \frac{1}{2} \text{ and } z_2 = \frac{-5-7}{2\cdot 2} = -3\,. \tag{2-17}$$

|||| Example 2.26 Real Quadratic Equation with a Negative Discriminant

Solve the following quadratic equation with real coefficients:

$$z^2 - 2z + 5 = 0\,.$$

We identify the coefficients: $a = 1, b = -2, c = 5$, and find the discriminant as:

$$D = (-2)^2 - 4\cdot 1\cdot 5 = -16\,.$$

According to Example 2.22 the solution to the binomial equation of the second degree $w^2 = D = -16$ is given by $w_0 = 4\mathrm{i}$. Now the solutions can be computed as:

$$z_1 = \frac{-(-2)+4\mathrm{i}}{2\cdot 1} = 1+2\mathrm{i} \text{ and } z_2 = \frac{-(-2)-4\mathrm{i}}{2\cdot 1} = 1-2\mathrm{i}\,. \tag{2-18}$$

|||| Example 2.27 A Quadratic Equation with Complex Coefficients

Solve the quadratic equation

$$z^2 - (1+\mathrm{i})z - 2 + 2\mathrm{i} = 0\,. \tag{2-19}$$

First we identify the coefficients: $a = 1, b = -(1+\mathrm{i}), c = -2+2\mathrm{i}$, and we find the discriminant:

$$D = (-(1+\mathrm{i}))^2 - 4\cdot 1\cdot(-2+2\mathrm{i}) = 8-6\mathrm{i}\,.$$

From Example 2.23 we know that the solution to the binomial equation $w^2 = D = 8-6\mathrm{i}$ is $w_0 = 3-\mathrm{i}$. From this we find the solution to (2-19) as

$$z_1 = \frac{-(-(1+\mathrm{i}))+(3-\mathrm{i})}{2\cdot 1} = 2 \text{ and } z_2 = \frac{-(-(1+\mathrm{i}))-(3-\mathrm{i})}{2\cdot 1} = -1+\mathrm{i}\,. \tag{2-20}$$

2.4.3 Equations of the Third and Fourth Degree

From antiquity geometrical methods for the solution of (real) quadratic equations are known. But not until A.D. 800 did algebraic solution formulae became known, through the work (in Arabic) of the Persian mathematician Muhammad ibn Musa al-Khwarismes famous book al-Jabr. In the West the name al-Khwarisme became the well-known word *algorithm*, while the book title became *algebra*.

Three centuries later history repeated itself. Around A.D. 1100 another Persian mathematician (and poet) Omar Khayyám gave exact methods on how to find solutions to real equations of the third and fourth degree by use of advanced geometrical methods. As an example he solved the equation $x^3 + 200x = 20x^2 + 2000$ by intersecting a circle with a hyperbola the equations of which he could derive from the equation of third degree.

Omar Khayyám did not think it possible to draw up algebraic formulae for solutions to equations of degree greater than two. He was proven wrong by the Italian Gerolamo Cardano who in the 16th century published formulae for the solution of Equations of the third and fourth degree.

Khayyáms methods and Cardanos formulae are beyond the scope of this eNote. Here we only give — see the previous Example 2.9 and the following Example 2.28 — a few examples by use of the "method of descent", Theorem 2.6, on how one can find all solutions to equations of degree greater that two if one in advance knows or can guess a sufficient number of the solutions.

|||| Example 2.28 An Equation of the Third Degree with an Initial Guess

Solve the equation of third degree

$$z^3 - 3z^2 + 7z - 5 = 0 \,.$$

It is easily guessed that 1 is a solution. By use of the algorithm of descent one easily gets the factorization:

$$z^3 - 3z^2 + 7z - 5 = (z-1)(z^2 - 2z + 5) = 0 \,.$$

We know that 1 is a solution, the remaining solutions are found by solving the quadratic equation

$$z^2 - 2z + 5 = 0 \,,$$

which, according to Example 2.26, has the solutions $1 + 2\mathrm{i}$ and $1 - 2\mathrm{i}$.

Collectively the equation of the third degree has the solutions $1, 1+2\mathrm{i}$ og $1-2\mathrm{i}$.

2.5 Real Polynomials

The theory that has been unfolded in the previous section applies to all polynomials with complex coefficients. In this section we present two theorems that *only* apply to polynomials with real coefficients — that is the subset called *the real polynomials*. The first theorem shows that non-real roots always appear in pairs.

|||| Theorem 2.29 Roots in Real Polynomials

If the number $a+ib$ is a root of the polynomial that only has real coefficients, then also the conjugate number $a-ib$ is a root of the polynomial.

|||| Proof

Let

$$P(z) = a_n\, z^n + a_{n-1}\, z^{n-1} + \cdots + a_1\, z + a_0$$

be a real polynomial. By use of the arithmetic rules for conjugation of the sum and product of complex numbers (see eNote 1 about complex numbers) with the condition that all coefficients are real, we get

$$\begin{aligned} \overline{P(z)} &= \overline{a_n\, z^n + a_{n-1}\, z^{n-1} + \cdots + a_1\, z + a_0} \\ &= a_n\, \overline{z}^n + a_{n-1}\, \overline{z}^{n-1} + \cdots + a_1\, \overline{z} + a_0 \\ &= P(\overline{z})\ . \end{aligned}$$

If z_0 is a root of P, we get

$$\overline{P(z_0)} = \overline{0} = 0 = P(\overline{z_0})$$

from which it is seen that $\overline{z_0}$ is also a root. Thus the theorem is proven.

∎

|||| Example 2.30 Conjugated Roots

Given that the polynomial

$$P(z) = 3z^2 - 12z + 39 \tag{2-21}$$

has the root $2 - 3\mathrm{i}$. Determine all roots of P, and write P in a complete factorized form.

We see that all the three coefficients in P are real. Therefore the conjugate of the given root $2 + 3\mathrm{i}$ is also a root of P. Since P is a quadratic polynomial, there are no more roots.

According to Example 2.12 the complete factorized form for P : is

$$P(z) = 3\,(z - (2 - 3\mathrm{i}))(z - (2 + 3\mathrm{i}))\,.$$

In the complete factorized form of a polynomial it is always possible to multiply the two factors that correspond to a pair of conjugated roots such that the product forms a *real quadratic polynomial* in this way:

$$\begin{aligned}(z - (a + \mathrm{i}b))(z - (a - \mathrm{i}b)) &= ((z - a) + \mathrm{i}b))((z - a) - \mathrm{i}b) \\ &= (z - a)^2 - (\mathrm{i}b)^2 \\ &= z^2 - 2az + (a^2 + b^2)\,.\end{aligned}$$

From Theorem 2.29 we know that complex roots always are present in conjugated pairs. This leads to the following theorem:

|||| Theorem 2.31 Real Factorization

A real polynomial can be written as a product of real polynomials of the first degree and real quadratic polynomials without any real roots.

|||| Example 2.32 Real Factorization

Given that a real polynomial of seventh degree P has the roots $1, \mathrm{i}, 1+2\mathrm{i}$ as well as the double root -2, and that the coefficient to its term of the highest degree is $a_7 = 5$. Write P as a product of real linear and real quadratic polynomials without real roots.

We use the fact that the conjugates of the complex roots are also roots and write P in its complete factorized form:

$$P(z) = 5\,(z-1)(z-\mathrm{i})(z+\mathrm{i})(z-(1+2\mathrm{i}))((z-(1-2\mathrm{i}))(z-2)^2\,.$$

Two pairs of factors correspond to conjugated roots. When we multiply these we obtain the form we wanted:

$$P(z) = 5\,(z-1)(z^2+1)(z^2-2z+5)(z-2)^2\,.$$

By this we end the treatment of polynomials in one variable.