eNote 1

Complex Numbers

In this eNote we introduce and investigate the set of numbers $\mathbb{C}$ *, the complex numbers. Since* $\mathbb{C}$ *is considered to be an extension of* $\mathbb{R}$ *, the eNote presumes general knowledge of the real numbers, including the elementary real functions such as the trigonometric functions and the natural exponential function. Finally elementary knowledge of vectors in the plane is taken for granted.*

Updated 22.09.21. *by David Brander. Version* 29.05.16. *by Karsten Schmidt.*

1.1 Introduction

A simple quadratic equation such as $x^2 = 25$ has two real solutions, viz.

$$x = 5 \text{ and } x = -5\,,$$

since $5^2 = 25$ and $(-5)^2 = 25$. Likewise the equation $x^2 = 2$ has two solutions, viz.

$$x = \sqrt{2} \text{ and } x = -\sqrt{2}\,,$$

since $\sqrt{2}^{\,2} = 2$ and $(-\sqrt{2})^{\,2} = 2$.

In the two examples above the right-hand sides were *positive*. When considering the equation

$$x^2 = k\ ,\ k \in \mathbb{R}$$

we must be more careful; here everything depends on the sign of k. If $k \geq 0$, the equation has the solutions

$$x = \sqrt{k} \text{ and } x = -\sqrt{k}\,,$$

since $\sqrt{k}^{\,2} = k$ and $(-\sqrt{k})^{\,2} = k$. But if $k < 0$ the equation has no solutions, since real numbers with negative squares do not exist.

But now we ask ourselves the question, is it possible to imagine a set of numbers larger than the set of real numbers; a set that includes all the real numbers but in addition *also* includes solutions to an equation like

$$x^2 = -1?$$

The equation should then in analogy to the equations above have two solutions

$$x = \sqrt{-1} \text{ and } x = -\sqrt{-1}\,.$$

Let us be bold and assume that this is in fact possible. We then choose to call this number $\mathrm{i} = \sqrt{-1}$. The equation $x^2 = -1$ then has two solutions, viz.

$$x = \mathrm{i} \text{ and } x = -\mathrm{i}$$

since, if we assume that the usual rules of algebra hold,

$$\mathrm{i}^2 = \sqrt{-1}^{\,2} = -1 \text{ and } (-\mathrm{i})^2 = (-1 \cdot \sqrt{-1}\,)^{\,2} = (-1)^2(-\sqrt{-1})^2 = -1\,.$$

As we just mentioned, we make the further demand on the hypothetical number i, that one must be able to use the same algebraic rules that apply to the real numbers. We must e.g. be able to multiply i by a real number b and add this to another real number a. In this way a new kind of number z of the type

$$z = a + \mathrm{i}b\,, \quad (a, b) \in \mathbb{R}^2$$

emerges.

Below we describe how these ambitions about a larger set of numbers can be fulfilled. We look at how the structure of the set of numbers should be and at which rules apply. We call this set of numbers *the complex numbers* and use the symbol $\mathbb{C}$. $\mathbb{R}$ must be a proper subset of $\mathbb{C}$ — that is, $\mathbb{C}$ contains all of $\mathbb{R}$ *together with* the new numbers which fulfill the above ambitions that are impossible in $\mathbb{R}$. As we have already hinted $\mathbb{C}$ must be *two-dimensional* in the sense that a complex number contains *two* real numbers, a and b.

1.2 Complex Numbers Introduced as Pairs of Real Numbers

The common way of writing a complex number z is

$$z = a + \mathrm{i}b\,, \tag{1-1}$$

where a and b are real numbers and i is the new *imaginary* number that satisfies $\mathrm{i}^2 = -1$. This form is very practical in computation with complex numbers. But we have not really clarified the meaning of the expression (1-1). For what is the meaning of a product like $\mathrm{i}b$, and what does the addition $a + \mathrm{i}b$ mean?

A satisfactory way of introducing the complex numbers is as the *set of pairs of real numbers* (a, b). In this section we will show how in this set we can define arithmetic operations (addition, subtraction, multiplication and division) that fulfill the ordinary arithmetic rules for real numbers. This will turn out to give full credit to the form (1-1).

|||| Definition 1.1 The Complex Numbers

The complex numbers $\mathbb{C}$ are defined as the set of ordered pairs of real numbers:

$$\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\} \tag{1-2}$$

equipt with the arithmetic rules described below.

As the symbol for an arbitrary complex number we will use the letter z .

|||| Example 1.2

Here we show five different complex numbers:

$$z_1 = (2,7)\,,\ z_2 = (7,2)\,,\ z_3 = (0,1)\,,\ z_4 = (-5,0)\,,\ z_5 = (0,0)\,.$$

First we introduce the arithmetic rule for the addition of complex numbers. Then subtraction as a special form of addition.

|||| Definition 1.3 Addition of Complex Numbers

Let $z_1 = (a, b)$ and $z_2 = (c, d)$ be two complex numbers.

The sum $z_1 + z_2$ is defined as

$$z_1 + z_2 = (a, b) + (c, d) = (a + c, b + d)\,. \tag{1-3}$$

|||| Example 1.4 Addition

For the two complex numbers $z_1 = (2, 7)$ and $z_2 = (4, -3)$ we have:

$$z_1 + z_2 = (2, 7) + (4, -3) = (2 + 4, 7 + (-3)) = (6, 4)\,.$$

The complex number $(0, 0)$ is *neutral* with respect to addition, since for every complex number $z = (a, b)$ we have:

$$z + (0, 0) = (a, b) + (0, 0) = (a + 0, b + 0) = (a, b) = z\,.$$

It is evident that $(0, 0)$ is the only complex number that is neutral with respect to addition.

For every complex number z there exists an *additive inverse* (also calld *opposite number*) denoted $-z$, which, when added to z, gives $(0, 0)$. The complex number $z = (a, b)$ has the additive inverse $-z = (-a, -b)$, since

$$(a, b) + (-a, -b) = (a + (-a), b + (-b)) = (a - a, b - b) = (0, 0)\,.$$

It is clear that $(-a, -b)$ is the only additive inverse for $z = (a, b)$, so the notation $-z$ is well-defined. By use of this, subtraction of complex numbers can be introduced as a special form of addition.

|||| Definition 1.5 Subtraction of Complex Numbers

For the two complex numbers z_1 and z_2 the difference $z_1 - z_2$ is defined as the sum of z_1 and *the additive inverse* for z_2 :

$$z_1 - z_2 = z_1 + (-z_2)\,. \tag{1-4}$$

Let us for two arbitrary complex numbers $z_1 = (a, b)$ and $z_2 = (c, d)$ calculate the difference $z_1 - z_2$ using definition 1.5:

$$z_1 - z_2 = (a, b) + (-c, -d) = (a + (-c), b + (-d)) = (a - c, b - d)\,.$$

This gives the simple formula

$$z_1 - z_2 = (a - c, b - d)\,. \tag{1-5}$$

|||| Example 1.6 Subtraction of Complex Numbers

For the two complex numbers $z_1 = (5, 2)$ and $z_2 = (4, -3)$ we have:

$$z_1 - z_2 = (5 - 4, 2 - (-3)) = (1, 5)\,.$$

While addition and subtraction appear to be simple and natural, multiplication and division of complex numbers appear to be more odd. Later we shall see that all the four arithmetic rules have geometrical equivalents in the so-called *complex plane* that constitutes the graphical representation of the complex numbers. But first we must accept the definitions at their face value. First we give the definition of multiplication. Then follows the definition of division as a special form of multiplication.

|||| Definition 1.7 Multiplication of Complex Numbers

Let $z_1 = (a, b)$ and $z_2 = (c, d)$ be two complex numbers.

The product $z_1\, z_2$ is defined as

$$z_1\, z_2 = z_1 \cdot z_2 = (ac - bd, ad + bc)\,. \tag{1-6}$$

|||| Example 1.8 Multiplication of Complex Numbers

For the two complex numbers $z_1 = (2,3)$ and $z_2 = (1,-4)$ we have:

$$z_1 \, z_2 = (2,3) \cdot (1,-4) = (2 \cdot 1 - (3 \cdot (-4)), 2 \cdot (-4) + 3 \cdot 1) = (14,-5) \, .$$

The complex number $(1,0)$ is *neutral* with respect to multiplication, since for every complex number $z = (a,b)$ we have that:

$$z \cdot (1,0) = (a,b) \cdot (1,0) = (a \cdot 1 - b \cdot 0 \, , a \cdot 0 + b \cdot 1) = (a,b) = z \, .$$

It is clear that $(1,0)$ is the only complex number that is neutral with respect to multiplication.

For every complex number z apart from $(0,0)$ there exists a unique reciprocal number that when multiplied by the given number gives $(1,0)$. It is denoted $\frac{1}{z}$. The complex number (a,b) has the reciprocal number

$$\frac{1}{z} = \left(\frac{a}{a^2+b^2} \, , \, - \frac{b}{a^2+b^2} \right) \, , \tag{1-7}$$

since

$$(a,b) \cdot \left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2} \right) = \left(\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} \, , \, - \frac{ab}{a^2+b^2} + \frac{ba}{a^2+b^2} \right) = (1,0) \, .$$

|||| Exercise 1.9

Show that every complex number $z \neq (0,0)$ has exactly one reciprocal number.

By the use of reciprocal numbers we can now introduce division as a special form of multiplication.

|||| Definition 1.10 Division of Complex Numbers

Let z_1 and z_2 be arbitrary complex numbers, where $z_2 \neq (0,0)$.

The quotient $\dfrac{z_1}{z_2}$ is defined as the product of z_1 and *the reciprocal number* $\dfrac{1}{z_2}$ for z_2 :

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} \ . \tag{1-8}$$

Let us for two arbitrary complex numbers $z_1 = (a,b)$ and $z_2 = (c,d) \neq (0,0)$ compute the quotient $\dfrac{z_1}{z_2}$ from the Definition 1.10:

$$z_1 \cdot \frac{1}{z_2} = (a,b)\left(\frac{c}{c^2+d^2}, -\frac{d}{c^2+d^2}\right) = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right).$$

From this we get the following formula for division:

$$\frac{z_1}{z_2} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right) . \tag{1-9}$$

|||| Example 1.11 Division of Complex Numbers

Consider two complex numbers $z_1 = (1,2)$ and $z_2 = (3,4)$.

$$\frac{z_1}{z_2} = \left(\frac{1\cdot 3 + 2\cdot 4}{3^2+4^2}, \frac{2\cdot 3 - 1\cdot 4}{3^2+4^2}\right) = \left(\frac{11}{25}, \frac{2}{25}\right) .$$

We end this section by showing that the complex numbers, with the above arithmetic operations, fulfill the computational rules known from the real numbers.

|||| Theorem 1.12 Properties of Complex Numbers

The complex numbers fulfill the following computational rules:

1. Commutative rule for addition: $z_1 + z_2 = z_2 + z_1$
2. Associative rule for addition: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
3. The number $(0, 0)$ is neutral with respect to addition
4. Every z has an opposite number $-z$ where $z + (-z) = (0, 0)$
5. Commutative rule for multiplication: $z_1\, z_2 = z_2\, z_1$
6. Associative rule for multiplication: $(z_1\, z_2)\, z_3 = z_1\, (z_2\, z_3)$
7. The number $(1, 0)$ is neutral with respect to multiplication
8. Every $z \neq (0, 0)$ has a reciprocal number $\dfrac{1}{z}$ where $z \cdot \dfrac{1}{z} = (1, 0)$
9. Distributive rule: $z_1\, (z_2 + z_3) = z_1\, z_2 + z_1\, z_3$

|||| Proof

Let us look at property 1, the commutative rule. Given two complex numbers $z_1 = (a, b)$ and $z_2 = (c, d)$. We see that

$$z_1 + z_2 = (a + c, b + d) = (c + a, d + b) = z_2 + z_1 .$$

To establish the second equality sign we have used that for both the first and the second coordinates the commutative rule for addition of real numbers applies. By this it is seen that the commutative rule also applies to complex numbers.

In the proof of the properties $2, 5, 6$ and 9 we similarly use the fact that the corresponding rules apply to the real numbers. The details are left to the reader. For the properties $3, 4, 7$ and 8 we refer to treatment above in this section.

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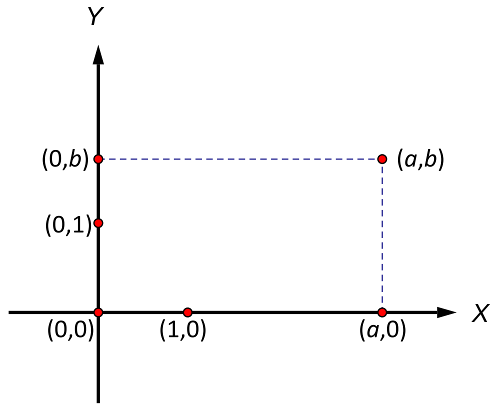

Figure 1.1: Six complex numbers in the (x, y)-plane

1.3 Complex Numbers in Rectangular Form

Since to every ordered pair of real numbers corresponds a unique point in the (x, y)-plane and vice versa, $\mathbb{C}$ can be considered to be the set of points in the (x, y)-plane. Figure 1.1 shows six points in the (x, y)-plane, i.e. six complex numbers.

In the following we will change our manner of writing complex numbers.

First we identify all complex numbers of the type $(a, 0)$, i.e. the numbers that lie on the x-axis, with the corresponding real number a . In particular the number $(0, 0)$ is written as 0 and the number $(1, 0)$ as 1 . Note that this will not be in conflict with the arithmetic rules for complex numbers and the ordinary rules for real numbers, since

$$(a, 0) + (b, 0) = (a + b, 0 + 0) = (a + b, 0)$$

and

$$(a, 0) \cdot (b, 0) = (a \cdot b - 0 \cdot 0\,, a \cdot 0 + 0 \cdot b) = (ab, 0)\,.$$

In this way the x-axis can be seen as an ordinary real number axis and is called the *real axis*. In this way the real numbers can be seen as a subset of the complex numbers. That the y-axis is called the *imaginary axis* is connected to the extraordinary properties of the complex number i which we now introduce and investigate.

|||| Definition 1.13 The Number i

By the complex number i we understand the number $(0,1)$.

A decisive motivation for the introduction of complex numbers was the wish for a set of numbers that contained the solution to the equation

$$x^2 = -1\,.$$

With the number i we have got such a solution because:

$$\mathrm{i}^2 = \mathrm{i}\cdot\mathrm{i} = (0,1)\cdot(0,1) = (0\cdot 0 - 1\cdot 1\,, 0\cdot 1 + 1\cdot 0) = (-1,0) = -1\,.$$

|||| Theorem 1.14 Complex Numbers in Rectangular Form

Every complex number $z = (a,b)$ can be written in the form

$$z = a + \mathrm{i}\cdot b = a + \mathrm{i}b\,. \tag{1-10}$$

This way of writing the complex number is called *the rectangular form* of z .

|||| Proof

The proof consists of simple manipulations in which we use the new way of writing numbers of this type.

$$(a,b) = (a,0) + (0,b) = (a,0) + (0,1)\cdot(b,0) = a + \mathrm{i}\,b\,.$$

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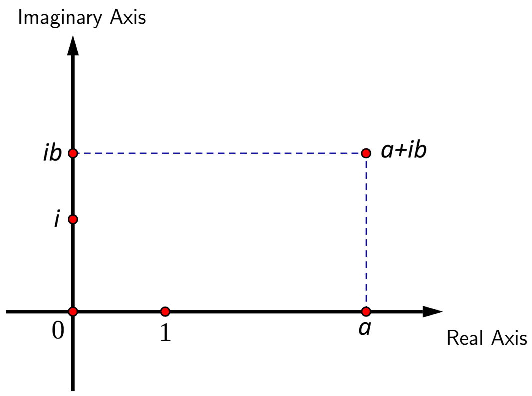

Figure 1.2: Six complex numbers in rectangular form in the complex number plane

Since $0 = (0,0)$ is neutral with respect to addition, and $1 = (1,0)$ is neutral with respect to multiplication, the following identities apply:

$$0 + z = z \text{ and } 1z = z\,.$$

Furthermore it is easily seen that

$$0z = 0\,.$$

Let us now consider all complex numbers of the type $(0, b)$. Since

$$(0, b) = 0 + \mathrm{i}b = \mathrm{i}b\,,$$

i can be understood as the unit of the y-axis, and therefore we refer to i as the *imaginary unit*. From this comes the name the *imaginary axis* for the y-axis.

In Figure 1.2 we see an update of the situation from Figure 1.1, where numbers are given in their rectangular form.

All real numbers are complex but not all complex numbers are real!

|||| Method 1.15 Computation Using the Rectangular form

A decisive advantage arising from the rectangular form of complex numbers is that one does not have to remember the formulas for the arithmetic rules for addition, subtraction, multiplication and division given in the definitions 1.3, 1.5, 1.7 and 1.10. All computations can be carried out by following the usual arithmetic rules for real numbers and treating the number i as one would treat a real variable — with the difference, though, that we replace i^2 by -1 .

In the following example it is shown how multiplication can be carried out through ordinary computation with the rectangular form of the factors.

|||| Example 1.16 Multiplication Using the Rectangular Form

We compute the product of two complex numbers given in rectangular form $z_1 = a + ib$ and $z_2 = c + \mathrm{i}d$:

$$\begin{aligned} z_1 z_2 &= (a + \mathrm{i}b)(c + \mathrm{i}d) = ac + \mathrm{i}ad + \mathrm{i}bc + \mathrm{i}^2 bd = ac + \mathrm{i}ad + \mathrm{i}bc - bd \\ &= (ac - bd) + \mathrm{i}(ad + bc) . \end{aligned}$$

The result corresponds to the definition, see Definition 1.7!

|||| Exercise 1.17

Prove that the following rule for real numbers — the so-called *zero rule* — also applies to complex numbers: "‘A product is 0 if and only if at least one of factors is 0 ."’

|||| Remark 1.18 Powers of Complex Numbers

The property 6 in Theorem 1.12 gives us the possibility to introduce integer powers of complex numbers, corresponding to integer powers of real numbers. In the following let n be a natural number.

1. $z^1 = z\,,\ z^2 = z \cdot z\,,\ z^3 = z \cdot z \cdot z$ etc.

2. By definition $z^0 = 1\,.$

3. Finally we put $z^{-n} = \dfrac{1}{z^n}\,.$

It is easily shown that the usual rules for computations with integer powers of real numbers also apply for integer powers of complex numbers:

$$z^n\,z^m = z^{n+m} \ \text{ and } \ (z^n)^m = z^{n\,m}\,.$$

We end this section by introducing the concepts *real part* and *imaginary part* of complex numbers.

|||| Definition 1.19 Real Part and Imaginary Part

Given a complex number z in rectangular form $z = a + ib$. By the *real part* of z we understand the real number

$$\text{Re}(z) = \text{Re}(a + ib) = a\,, \tag{1-11}$$

and by the *imaginary part* of z we understand the real number

$$\text{Im}(z) = \text{Im}(a + ib) = b\,. \tag{1-12}$$

The expression *rectangular form* refers to the position of the number in the complex number plane, where $\mathrm{Re}(z)$ is the number's perpendicular drop point on the real axis, and $\mathrm{Im}(z)$ its perpendicular drop point on the imaginary axis. In short the real part is the first coordinate of the number while the imaginary part is the second coordinate of the number.

Note that every complex number z can be written in rectangular form like this:

$$z = \mathrm{Re}(z) + \mathrm{i}\,\mathrm{Im}(z)\,.$$

|||| Example 1.20 Real Part and Imaginary Part

Three complex numbers are given by

$$z_1 = 3 - 2\mathrm{i}\,,\; z_2 = \mathrm{i}5\,,\; z_3 = 25 + \mathrm{i}\,.$$

Find the real part and the imaginary part of each number.

$$\begin{aligned}
&\mathrm{Re}(z_1) = 3\,,\ \mathrm{Im}(z_1) = -2\\
&\mathrm{Re}(z_2) = 0\,,\ \mathrm{Im}(z_2) = 5\\
&\mathrm{Re}(z_3) = 25\,,\ \mathrm{Im}(z_3) = 1
\end{aligned}$$

Two complex numbers in rectangular form are *equal* if and only if both their real parts and imaginary parts are equal.

1.4 Conjugation of Complex Numbers

|||| Definition 1.21 Conjugation

Let z be a complex number with the rectangular form $z = a + ib$. By the conjugated number corresponding to z we understand the complex number $\bar{z}$ given by

$$\bar{z} = a - ib\,. \tag{1-13}$$

Conjugating a complex number corresponds to reflecting the number in the real axis as shown in Figure 1.3.

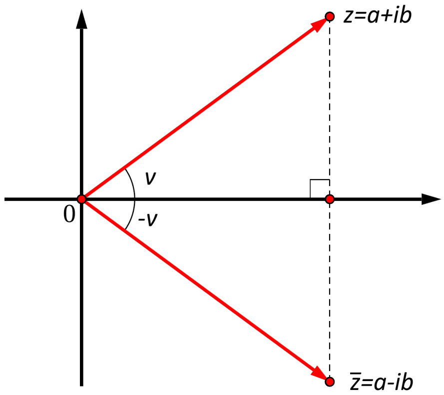

Figure 1.3: Reflection in the real axis

It is obvious that the conjugate number of a conjugate number is the original number:

$$\bar{\bar{z}} = z\,. \tag{1-14}$$

Furthermore the following useful formula for the product of complex number and its conjugate applies:

$$z \cdot \bar{z} = |\,z\,|^2 \tag{1-15}$$

which is shown by simple calculation.

In the following method we show a smart way of finding the rectangular form of a fraction when the denominator is not real: we use the fact that the product of a number $z = a + ib$ and its conjugate $z = a - ib$ is *always* a real number, cf. (1-15).

|||| Method 1.22 Finding the rectangular form of a complex fraction

The way to remember: *Multiply the numerator and the denominator by the conjugate of the denominator*. Here the denominator is written in its rectangular form:

$$\frac{z}{a+ib} = \frac{z(a-ib)}{(a+ib)(a-ib)} = \frac{z(a-ib)}{a^2+b^2}.$$

An example:

$$\frac{2-\mathrm{i}}{1+\mathrm{i}} = \frac{(2-\mathrm{i})(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})} = \frac{1-3\mathrm{i}}{1^2+1^2} = \frac{1-3\mathrm{i}}{2} = \frac{1}{2} - \frac{3}{2}\mathrm{i}\,.$$

In conjugation in connection with the four ordinary arithmetic operations the following rules apply.

|||| Theorem 1.23 Arithmetic Rules for Conjugation

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
2. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
3. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
4. $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2} \quad , z_2 \neq 0\,.$

|||| Proof

The proof is carried out by simple transformation using the rectangular form of the numbers. As an example we show the first formula. Suppose that $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Then:

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{(a_1 + \mathrm{i}b_1) + (a_2 + \mathrm{i}b_2)} = \overline{(a_1 + a_2) + \mathrm{i}(b_1 + b_2)}\\ &= (a_1 + a_2) - \mathrm{i}(b_1 + b_2) = (a_1 - \mathrm{i}b_1) + (a_2 - \mathrm{i}b_2)\\ &= \overline{z_1} + \overline{z_2}\,.\end{aligned}$$

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Finally we note that all complex numbers on the real axis are identical with their conjugate number and that they are the only complex numbers that fulfill this condition. Therefore we can state a criterion for whether a given number in a set of complex numbers is real:

|||| Theorem 1.24 The Real Criterion

Let A be a subset of $\mathbb{C}$, and let $A_{\mathbb{R}}$ denote the subset of A that consists of real numbers. Then:

$$A_{\mathbb{R}} = \{z \in A \mid \overline{z} = z\}\ .$$

|||| Proof

Let z be an arbitrary number in $A \subseteq \mathbb{C}$ with rectangular form $z = a + ib$. Then:

$$\overline{z} = z \Leftrightarrow a - \mathrm{i}b = a + \mathrm{i}b \Leftrightarrow 2\mathrm{i}b = 0 \Leftrightarrow b = 0 \Leftrightarrow z \in A_{\mathbb{R}}\,.$$

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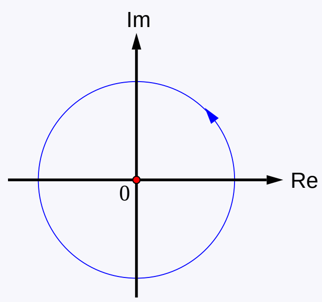
1.5 Polar Coordinates

The obvious way of stating a point (or a position vector) in an ordinary (x, y)-coordinate system is by the point's *rectangular*, i.e. *orthogonal*, coordinates (a, b). In many situations it is, however, useful to be able to determine a point by its *polar coordinates*, consisting of its *distance* to $(0, 0)$ together with its *direction angle* from the x-axis to its position vector. The direction angle is then positive if it is measured counter-clockwise and negative if measured clockwise.

Analogously, we now introduce polar coordinates for complex numbers. Let us first be absolutely clear about the orientation of the complex number plane.

|||| Definition 1.25 Orientation of the Complex Number Plane

The orientation of the complex number plane is determined by a circle with its centre at the origen being traversed *counter-clockwise*.



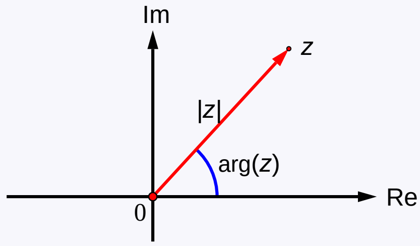
The ingredients in the polar coordinates of complex numbers are (as mentioned above) its distance to $(0, 0)$ called the *absolute value*, and its direction angle called the *argument*. We now introduce these two quantities.

|||| Definition 1.26 Absolute Value and Argument

Given a complex number z .

By the *absolute value* of z we understand the length of the corresponding position vector. The absolute value is written $|\,z\,|$ and is also called the *modulus* or *numerical value*.

Suppose $z \neq 0$. Every angle from the positive part of the real axis to the position vector for z is called an *argument* for z and is denoted $\arg(z)$. The angle is positive or negative relative to the orientation of the complex number plane.



The pair

$$(\,|\,z\,|\,, \arg(z)\,)$$

of the absolute value of z and an argument for z will collectively be called the *polar coordinates* of the number.

Note that the argument for a number z is not unique. If you add 2π to an arbitrary argument for z, you get a new valid direction angle for z and therefore a valid argument. Therefore a complex number has infinitely many arguments corresponding to turning an integer number of times extra clockwise or counter-clockwise in order to reach the same point again.

You can always choose an argument for z that lies in the interval from $-\pi$ to π. Traditionally this argument is given a preferential position. It is called the *principal value* of the argument.

|||| Definition 1.27 Principal Value

Given a complex number z that is not 0. By the *principal argument* $\operatorname{Arg}(z)$ for z we understand the uniquely determined argument for z that satisfies:

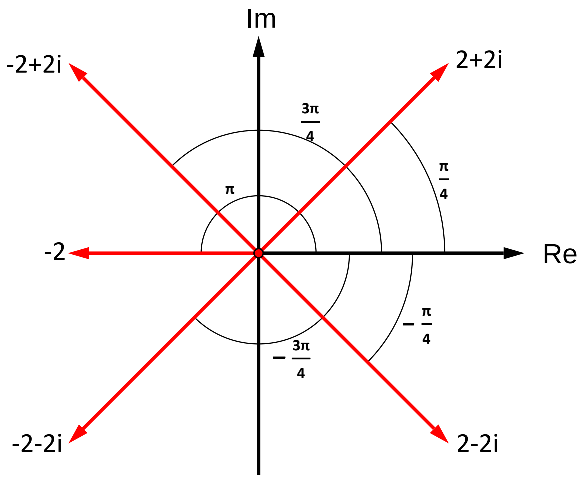
$$\arg(z) \in \,]-\pi, \pi]\,.$$

We denoted the *principal value* with a capital initial $\operatorname{Arg}(z)$ as compared to $\arg(z)$ that denotes an *arbitrary* argument. All arguments for a complex number z are then given by

$$\arg(z) = \operatorname{Arg}(z) + p \cdot 2\pi \ , \ p \in \mathbb{Z}\,. \tag{1-16}$$

Two complex numbers are *equal* if and only if both their absolute values and the principal arguments are equal.

|||| Example 1.28 Principal Arguments



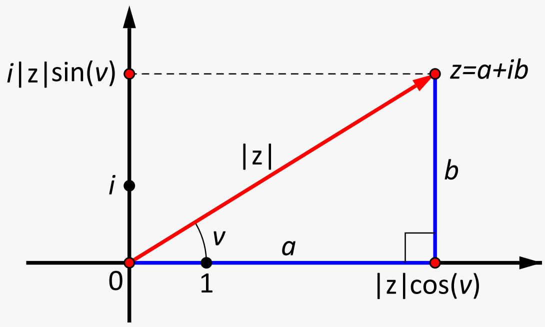
The figure shows five complex numbers, four of which lie on the lines through $(0, 0)$ bisecting the four quadrants. We read:

- $2+2\mathrm{i}$ has the principal argument $\frac{\pi}{4}$,
- $2-2\mathrm{i}$ has the principal argument $-\frac{\pi}{4}$,
- $-2+2\mathrm{i}$ has the principal argument $\frac{3\pi}{4}$,
- $-2-2\mathrm{i}$ has the principal argument $-\frac{3\pi}{4}$, and
- -2 has the principal argument π .

Whether it is advantageous to use the rectangular format of the complex numbers or their polar form depends on the situation at hand. In Method (1.29) it is demonstrated how one can shift between the two forms.

|||| Method 1.29 Rectangular and Polar Coordinates

We consider a complex number $z \neq 0$ that has the rectangular form $z = a + ib$ and an argument v :



1. The rectangular form is computed from the polar coordinates like this:

$$a = |z|\cos(v) \quad \text{and} \quad b = |z|\sin(v)\,. \tag{1-17}$$

2. The absolute value is computed from the rectangular form like this :

$$|z| = \sqrt{a^2 + b^2}\,. \tag{1-18}$$

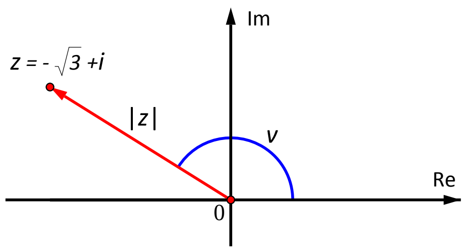
3. An argument is computed from the rectangular form by finding an angle v that satisfies *both* of the following equations:

$$\cos(v) = \frac{a}{|z|} \quad \text{and} \quad \sin(v) = \frac{b}{|z|}\,. \tag{1-19}$$

When z is drawn in the first quadrant it is evident that the computational rules (1-17) and (1-19) are derived from well-known formulas for cosine and sine to acute angles in right-angled triangles and (1-18) from the theorem of Pythagoras. By using the same formulas it can be shown that the introduced methods are valid regardless of the quadrant in which z lies.

|||| Example 1.30 From Rectangular to Polar Form

Find the polar coordinates for the number $z=-\sqrt{3}+\mathrm{i}$.



We use the rules in Method 1.29. Initially we identify the real and the imaginary parts of z as

$$a=-\sqrt{3} \quad \text{and} \quad b=1.$$

First we determine the absolute value:

$$|z|=\sqrt{a^2+b^2}=\sqrt{(-\sqrt{3})^2+1^2}=\sqrt{3+1}=2\,.$$

Then the argument is determined. From the equation

$$\cos(v)=\frac{a}{|z|}=-\frac{\sqrt{3}}{2}$$

we get two possible principal arguments for z, viz.

$$v=\frac{5\pi}{6} \;\text{ and }\; v=-\frac{5\pi}{6}\,.$$

From the figure it is seen that z lies in the second quadrant, and the correct principal argument must therefore be the first of these possibilities. But this can also be determined without inspection of the figure, since also the equation

$$\sin(v)=\frac{1}{2}$$

must be fulfilled. From this we also get two possible principal arguments for z, viz.

$$v=\frac{\pi}{6} \;\text{ and }\; v=\frac{5\pi}{6}\,.$$

Since only $v=\frac{5\pi}{6}$ satisfies both equations, we see that $\operatorname{Arg}(z)=\frac{5\pi}{6}$.

Thus we have found the set of polar coordinates for z :

$$(|z|, \operatorname{Arg}(z)) = \left(2, \frac{5\pi}{6}\right).$$

We end this section with the important product rules for absolute values and arguments.

|||| Theorem 1.31 The Product Rule for Absolute Values

The absolute value of the product of two complex numbers z_1 and z_2 is found by

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|. \tag{1-20}$$

From Theorem 1.31 we get the corollary

|||| Corollary 1.32

The absolute value for the quotient of two complex numbers z_1 and z_2 where $z_2 \neq 0$ is found by

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}. \tag{1-21}$$

The absolute value of the nth power of a complex number z is for every $n \in \mathbb{Z}$ given by

$$|{z_1}^n| = |z_1|^n. \tag{1-22}$$

|||| Exercise 1.33

Write down in words what the formulas (1-20), (1-21) and (1-22) say and prove them.

|||| Theorem 1.34 The Product Rule for Arguments

Given two complex numbers $z_1 \neq 0$ and $z_2 \neq 0$ (which also means $z_1 z_2 \neq 0$). Then if v_1 is an argument for z_1 and v_2 is an argument for z_2, then $v_1 + v_2$ is an argument for the product $z_1 z_2$.

|||| Corollary 1.35

Given two complex numbers $z_1 \neq 0$ and $z_2 \neq 0$.Then:

1. If v_1 is an argument for z_1 and v_2 is an argument for z_2, then $v_1 - v_2$ is an argument for the fraction $\frac{z_1}{z_2}$.
2. If v is an argument for z, then $n \cdot v$ is an argument for the power z^n.

|||| Exercise 1.36

Prove Theorem 1.34 and Corollary 1.35.

1.6 Geometric Understanding of the Four Computational Operations

We started by introducing addition, subtraction, multiplication and division of complex numbers as algebraic operations carried out on pairs of real numbers (a, b), see definitions 1.3, 1.5, 1.7 and 1.10. Then we showed that the rectangular form of the complex numbers $a + ib$ leads to a more practical way of computation: One can compute with complex numbers just as with real numbers, as long as the number i is treated as a real parameter and it is understood that $\mathrm{i}^2 = -1$. In this section we shall see that the computational operations can also be viewed as geometrical constructs.

The first exact description of the complex numbers was given by the Norwegian surveyor Caspar Wessel in 1796. Wessel introduced complex numbers as line segments with given lengths and directions, that is what we now call vectors in the plane. Therefore computations with complex numbers were geometric operations carried out on

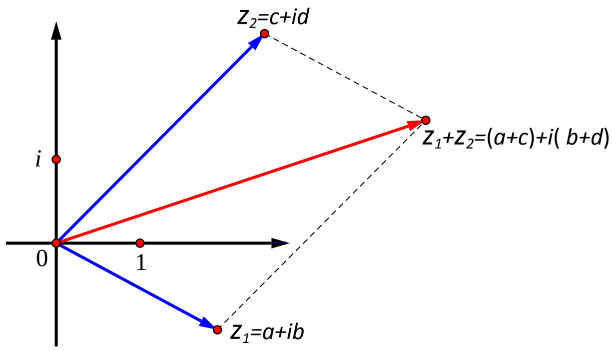


Figure 1.4: Addition by the method of parallelograms

vectors. In the following we recollect the ideas in the definition of Wessel. It is easy to see the equivalence between the algebraic and geometric representations of addition and subtraction — it is more demanding to understand the equivalence when it comes to multiplication and division.

|||| Theorem 1.37 Geometric Addition

Addition of two complex numbers z_1 and z_2 can be obtained geometrically in the following way:

The position vector for $z_1 + z_2$ is the sum of the position vectors for z_1 and z_2 . (See Figure 1.4).

|||| Proof

Suppose that z_1 and z_2 are given in rectangular form as $z_1 = a + ib$ and $z_2 = c + id$. Then the position vector for z_1 has the coordinates (a, b) and the position vector for z_2 has the coordinates (c, d) . The sum of the two position vectors is then $(a + c, b + d)$, being the coordinates of the position vector for the complex number $(a + c) + \mathrm{i}(b + d)$. Since we have that $z_1 + z_2 = (a + ib) + (c + \mathrm{i}d) = (a + c) + \mathrm{i}(b + d)$, we have proven the theorem.

■

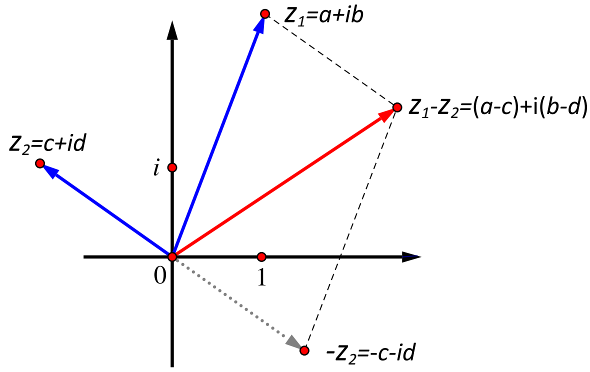


Figure 1.5: Subtraction by the method of parallelograms

Geometric subtraction is given as a special form of geometric addition: The position vector for $z_1 - z_2$ is the sum of the position vectors for z_1 and the opposite vector to the position vector for z_2 . This is illustrated in Figure 1.5

While in the investigation of geometrical addition (and subtraction) we have used the rectangular form of complex numbers, in the treatment of geometric multiplication (and division) we shall need their polar coordinates.

|||| Theorem 1.38 Geometrical Multiplication

Given two complex numbers z_1 and z_2 that are both different from 0 (which also means that $z_1 z_2 \neq 0$) . Multiplication of z_1 and z_2 can be obtained geometrically in the following way:

1. The absolute value of the product $z_1 z_2$ is found by multiplication of the absolute value of z_1 by the absolute value of z_2 .
2. An argument for the product $z_1 z_2$ is found by adding an argument for z_1 and an argument for z_2 .

|||| Proof

First part of the theorem appears from Theorem 1.31 while the second part is evident from Theorem 1.34.

■

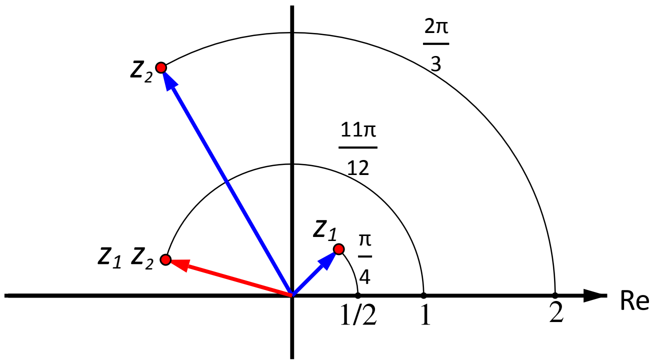


Figure 1.6: Multiplication

|||| Example 1.39 Multiplication by Use of Polar Coordinates

Two complex numbers z_1 and z_2 are given by the polar coordinates $\left(\frac{1}{2}, \frac{\pi}{4}\right)$ and $\left(2, \frac{2\pi}{3}\right)$, respectively. (Figure 1.6,)

We compute the product of z_1 and z_2 by the use of their absolute values and arguments:

$$|z_1 z_2| = |z_1|\,|z_2| = \frac{1}{2} \cdot 2 = 1$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \frac{\pi}{4} + \frac{2\pi}{3} = \frac{11\pi}{12}\,.$$

Thus the product $z_1 z_2$ is the complex number that has the absolute value 1 and the argument $\frac{11\pi}{12}$.

Note that it is important to observe whether a set of coordinates is given in *rectangular* or in *polar* form.

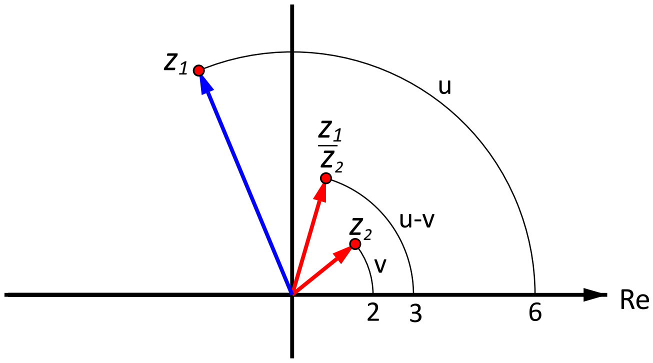


Figure 1.7: Division

|||| Example 1.40 Division by Use of Polar Coordinates

The numbers z_1 and z_2 are given by $|z_1| = 6$ with $\arg(z_1) = u$ and $|z_2| = 2$ with $\arg(z_2) = v$ respectively. Then $\frac{z_1}{z_2}$ can be determined as

$$\left|\frac{z_1}{z_2}\right| = \frac{6}{2} = 3 \text{ and } \arg\left(\frac{z_1}{z_2}\right) = u - v\,.$$

1.7 The Complex Exponential Function

The ordinary exponential function $x \mapsto e^x$, $x \in \mathbb{R}$ has, as is well known, the characteristic properties,

1. $e^0 = 1$,
2. $e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}$ for all $x_1, x_2 \in \mathbb{R}$, and
3. $(e^x)^n = e^{nx}$ for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$.

In this section we will introduce a particularly useful extension of the real exponential function to a complex exponential function, that turns out to follow the same rules of computation as its real counterpart.

|||| Definition 1.41 Complex Exponential Function

By the complex exponential function $\exp_{\mathbb{C}}$ we understand a function that to each number $z \in \mathbb{C}$ with the rectangular form $z = x + \mathrm{i}y$ attaches the number

$$\exp_{\mathbb{C}}(z) = \exp_{\mathbb{C}}(x + \mathrm{i}y) = \mathrm{e}^x \cdot (\cos(y) + \mathrm{i}\sin(y)) \,, \tag{1-23}$$

where e (about 2.7182818 . . .) is base for the real natural exponential function.

Since we for every *real* number x get

$$\exp_{\mathbb{C}}(x) = \exp_{\mathbb{C}}(x + \mathrm{i} \cdot 0) = \mathrm{e}^x \, (\cos(0) + \mathrm{i}\sin(0)) = \mathrm{e}^x \,,$$

we see that the complex exponential function is everywhere on the real axis identical to the real exponential function. Therefore we do not risk a contradiction when we in the following allow (and often use) the way of writing

$$\exp_{\mathbb{C}}(z) = \mathrm{e}^z \quad \text{for} \quad z \in \mathbb{C} \,. \tag{1-24}$$

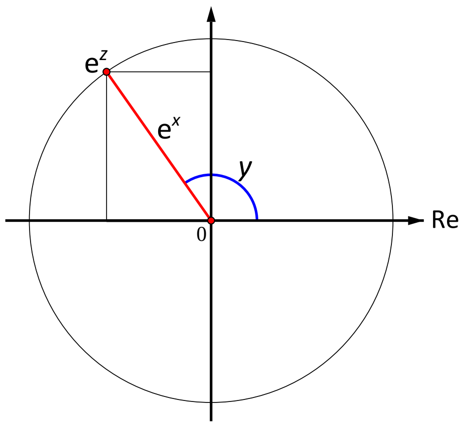


Figure 1.8: Geometric Interpretation of e^z

We now consider the complex number e^z where z is an arbitrary complex number with the rectangular form $z = x + \mathrm{i}y$. Then (by use of Theorem 1.31) we see that

$$|\mathrm{e}^z| = |\mathrm{e}^x \, (\cos(y) + \mathrm{i}\sin(y))| = |\mathrm{e}^x| \, |(\cos(y) + \mathrm{i}\sin(y))| = |\mathrm{e}^x| = \mathrm{e}^x \,. \tag{1-25}$$

Furthermore (by use of Theorem 1.34) we see that

$$\arg\left(\mathrm{e}^z\right) = \arg\left(\mathrm{e}^x\right) + \arg\left(\cos(y) + \mathrm{i}\sin(y)\right) = 0 + y = y \,. \tag{1-26}$$

The polar coordinates for $z = x + \mathrm{i}y$ are then (e^x, y), which is illustrated in Figure 1.8.

For the trigonometric functions $\cos(x)$ and $\sin(x)$ we know that for every integer p $\cos(x + p2\pi) = \cos(x)$ and $\sin(x + p2\pi) = \sin(x)$. If the graph for $\cos(x)$ or $\sin(x)$ is displaced by an arbitrary multiple of 2π, it will be mapped onto itself. Therefore the functions are called *periodic* having a *period* of 2π.

A similar phenomenon is seen for the complex exponential function. It has the *imaginary* period $\mathrm{i}\,2\pi$. This is closely connected to the periodicity of the trigonometric functions as can be seen in the proof of the following theorem.

|||| Theorem 1.42 Periodicity of e^z

For every complex number z and every integer p:

$$\mathrm{e}^{z+\mathrm{i}p2\pi} = \mathrm{e}^z . \tag{1-27}$$

|||| Proof

Suppose that z has the rectangular form $z = x + \mathrm{i}y$ and $p \in \mathbb{Z}$.

Then:

$$\begin{aligned} \mathrm{e}^{z+\mathrm{i}p2\pi} &= \mathrm{e}^{x+\mathrm{i}(y+p2\pi)} \\ &= \mathrm{e}^x\big(\cos(y+p2\pi) + \mathrm{i}\sin(y+p2\pi)\big) = \mathrm{e}^x\big(\cos(y) + \mathrm{i}\sin(y)\big) \\ &= \mathrm{e}^z . \end{aligned}$$

By this the theorem is proved.

■

In the following example the periodicity of the complex exponential function is illustrated.

|||| Example 1.43 Exponential Equation

Determine all solutions to the equation

$$e^z = -\sqrt{3} + \mathrm{i}\,. \tag{1-28}$$

First we write z in rectangular form: $z = x + \mathrm{i}y$. In Example 1.30 we found that the right-hand side in (1-28) has the absolute value $|z| = 2$ and the principal argument $v = \frac{5\pi}{6}$. Since the left-hand and the right-hand sides must have the same absolute value and the same argument, apart from an arbitrary multiple of 2π , we get

$$|\,e^z\,| = |\,-\sqrt{3} + \mathrm{i}\,| \Leftrightarrow e^x = 2 \;\Leftrightarrow\; x = \ln(2)$$

$$\arg(e^z) = \arg(-\sqrt{3} + \mathrm{i}) \Leftrightarrow y = v + p2\pi = \frac{5\pi}{6} + p2\pi\;,\; p \in \mathbb{Z}\,.$$

All solutions for (1-28) are then

$$z = x + \mathrm{i}y = \ln(2) + \mathrm{i}\left(\frac{5\pi}{6} + p2\pi\right)\;,\; p \in \mathbb{Z}\,.$$

We end this section by stating and proving the rule of computations mentioned in the introduction and known from the real exponential function.

|||| Theorem 1.44 Complex Exponential Function Computation Rules

1. $e^0 = 1$
2. $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$ for all $z_1, z_2 \in \mathbb{C}$
3. $(e^z)^n = e^{nz}$ for all $n \in \mathbb{Z}$ og $z \in \mathbb{C}$

|||| Proof

Point 1 in the theorem that $e^0 = 1$, follows from the fact that the complex exponential function is identical with the real exponential function on the real axis, cf. (1-24).

In point 2 we set $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. From the set of polar coordinates and 1.38 we get:

$$\begin{aligned} e^{z_1} \cdot e^{z_2} &= (e^{x_1}, y_1) \cdot (e^{x_2}, y_2) = (e^{x_1} \cdot e^{x_1}, y_1 + y_2) = (e^{x_1+x_2}, y_1 + y_2) \\ &= e^{(x_1+x_2)+i(y_1+y_2)} = e^{(x_1+iy_1)+(x_2+iy_2)} \\ &= e^{z_1+z_2}. \end{aligned}$$

In point 3 we set $z = x + iy$ and with the use of sets of polar coordinates and the repeated use of Theorem 1.38 we get:

$$\begin{aligned} (e^z)^n &= ((e^x)^n, n \cdot y) = (e^{n \cdot x}, n \cdot y) = e^{n \cdot x + i \cdot n \cdot y} = e^{n(x+i \cdot y)} \\ &= e^{n \cdot z}. \end{aligned}$$

By this the Theorem is proved.

■

|||| Exercise 1.45

Show that for every $z \in \mathbb{C}$ $e^z \neq 0$.

1.8 The Exponential Form of Complex Numbers

Let v be an arbitrary real number. If we substitute the pure imaginary number iv into the complex exponential function we get from the Definition 1.41:

$$e^{iv} = e^{0+iv} = e^0 \left(\cos(v) + i\sin(v)\right),$$

which yields the famous *Euler's formula*.

|||| Theorem 1.46 Euler's Formula

For every $v \in \mathbb{R}$:

$$e^{iv} = \cos(v) + i\sin(v). \tag{1-29}$$

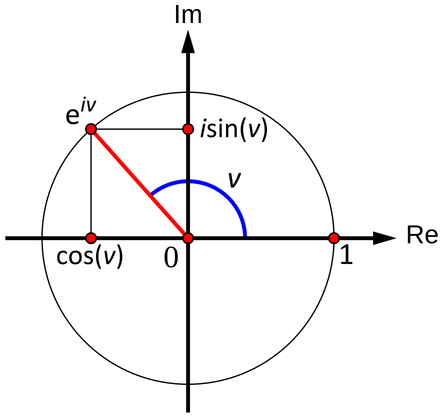


Figure 1.9: The number $\mathrm{e}^{\mathrm{i}v}$ in the complex number plane

By use of the definition of the complex exponential function, see Definition 1.41, we derived Euler's formula. In return we can now use Euler's formula to write the complex exponential function in the convenient form

$$\mathrm{e}^{z} = \mathrm{e}^{x}\big(\cos(y) + \mathrm{i}\sin(y)\big) = \mathrm{e}^{x}e^{\mathrm{i}y}\,. \tag{1-30}$$

The two most-used ways of writing complex numbers both in pure and applied mathematics are the *rectangular* form (as is frequently used above) and the *exponential* form. In the exponential form the polar coordinates of the number (absolute value and argument), in connection with the complex exponential function. Since the polar coordinates appear explicitly in this form, it is also called the *polar form*.

|||| Theorem 1.47 The Exponential Form of Complex Numbers

Every complex number $z \neq 0$ can be written in the form

$$z = |z|\,\mathrm{e}^{\mathrm{i}v}\,, \tag{1-31}$$

where v is an argument for z. This way of writing is called the *exponential form* (or the *polar form*) of the number.

|||| Proof

Let v be an argument for the complex number $z \neq 0$, and put $r = |z|$. We show that $r\mathrm{e}^{\mathrm{i}v}$ has the same absolute value and argument as z, and thus the two numbers are identical:

1. $$\left|r\mathrm{e}^{\mathrm{i}v}\right| = |r|\left|\mathrm{e}^{\mathrm{i}v}\right| = r\,.$$

2. Since 0 is an argument for r, and v is an argument for $\mathrm{e}^{\mathrm{i}v}$, we have that $0 + v = v$ is an argument for the product $r\mathrm{e}^{\mathrm{i}v}$.

■

|||| Method 1.48 Computations Using the Exponential Form

A decisive advantage of the exponential form of complex numbers is that one does not have to think about the rule of computations for multiplication, division and powers when the polar coordinates are used, see Theorem 1.31, Corollary 1.32 and Theorem 1.34. All computations can be carried out using the ordinary rules of computation on the exponential form of the numbers.

We now give an example of multiplication following Method 1.48; cf. Example 1.39.

|||| Example 1.49 Multiplication in Exponential Form

Two complex numbers are given in exponential form,

$$z_1 = \frac{1}{2}\,\mathrm{e}^{\frac{\pi}{4}\mathrm{i}} \text{ and } z_2 = 2\,\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}\,.$$

The product of the numbers is found in exponential form as

$$z_1 z_2 = \Big(\frac{1}{2}\,\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}\Big)\big(2\,\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}\big) = \big(\frac{1}{2}\cdot 2\big)\mathrm{e}^{\frac{\pi}{4}\mathrm{i}+\frac{3\pi}{2}\mathrm{i}} = 1\mathrm{e}^{\mathrm{i}(\frac{\pi}{4}+\frac{3\pi}{2})} = \mathrm{e}^{\frac{7\pi}{4}\mathrm{i}}\,.$$

|||| Exercise 1.50

Show that Method 1.48 is correct.

In the following we will show how so-called *binomial equations* can be solved by the use of the exponential form. A binomial equation is an equation *with two terms* in the form

$$z^n = w\,, \tag{1-32}$$

where $w \in \mathbb{C}$ and $n \in \mathbb{N}$. Binomial equations are described in more detail in eNote 2 about polynomials.

First we show an example of the solution of a binomial equation by use of the exponential form and then we formulate the general method.

|||| Example 1.51 Binomial Equation in Exponential Form

Find all solutions to the binomial equation

$$z^4 = -8 + 8\sqrt{3}\,\mathrm{i}\,. \tag{1-33}$$

The idea is that we write both z and the right-hand side in exponential form.

If z has the exponential form $z = s e^{\mathrm{i}u}$, then the equation's left-hand side can be computed as

$$z^4 = (s e^{\mathrm{i}u})^4 = s^4\,(e^{\mathrm{i}u})^4 = s^4\,e^{\mathrm{i}4u}\,. \tag{1-34}$$

The right-hand side is also written in exponential form. The absolute value r of the right-hand side is found by

$$r = |-8 + 8\sqrt{3}\,\mathrm{i}\,| = \sqrt{(-8)^2 + (8\sqrt{3})^2} = 16\,.$$

The argument v of the right-hand side satisfies

$$\cos(v) = \frac{-8}{16} = -\frac{1}{2} \quad \text{and} \quad \sin(v) = \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2}\,.$$

By use of the two equations the principal argument of the right-hand side can be determined to be

$$v = \arg(-8 + 8\sqrt{3}\mathrm{i}) = \frac{2\pi}{3}\,,$$

and so the exponential form of the right-hand side is

$$r\mathrm{e}^{\mathrm{i}v} = 16\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}} . \tag{1-35}$$

We now substitute (1-34) and (1-35) into (1-33) in order to replace the right- and left-hand side with the exponential counterparts

$$s^4\, e^{\mathrm{i}4u} = 16\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}} .$$

Since the absolute value of the left-hand side must be equal to absolute value of the right-hand side we get

$$s^4 = 16 \;\Leftrightarrow\; s = \sqrt[4]{16} = 2 .$$

The argument of the left-hand side $4u$ and the argument of the right-hand side $\frac{2\pi}{3}$ must be equal apart from a multiple of 2π . Thus

$$4u = \frac{2\pi}{3} + p2\pi \;\Leftrightarrow\; u = \frac{\pi}{6} + p\frac{\pi}{2} \;,\; p \in \mathbb{Z} .$$

These infinitely many arguments correspond, as we have seen earlier, to only *four* half-lines from $(0,0)$ determined by the arguments obtained by putting $p = 0, p = 1, p = 2$ and $p = 3$. For any other value of p the corresponding half-line will be identical to one of the four mentioned above. E.g. the half-line corresponding to $p = 4$ has the argument

$$u = \frac{\pi}{6} + 4\,\frac{\pi}{2} = \frac{\pi}{6} + 2\pi ,$$

i.e. the same half-line that corresponds to $p = 0$, since the difference in argument is a whole revolution, that is 2π .

Therefore the given equation (1-33) has exactly four solutions that lie on the four mentioned half-lines and that are separated the distance $s = 2$ from 0 . Stated in exponential form:

$$z = 2\,\mathrm{e}^{\mathrm{i}(\frac{\pi}{6}+p\frac{\pi}{2})} \;,\; p = 0\,,1\,,2\,,3\,.$$

Or each recomputed to rectangular form by means of Euler's formula (1-29):

$$z_0 = \sqrt{3} + \mathrm{i}\,,\; z_1 = -1 + \mathrm{i}\sqrt{3}\,,\; z_2 = -\sqrt{3} - \mathrm{i}\,,\; z_3 = 1 - \mathrm{i}\sqrt{3}\,.$$

All solutions to a binomial equation lie on a circle with the centre at 0 and radius equal to the absolute value of the right-hand side. The connecting lines between 0 and the solutions divide the circle into equal angles. This is illustrated in Figure 1.10 which shows the solutions to the equation of the fourth degree from Example 1.51.

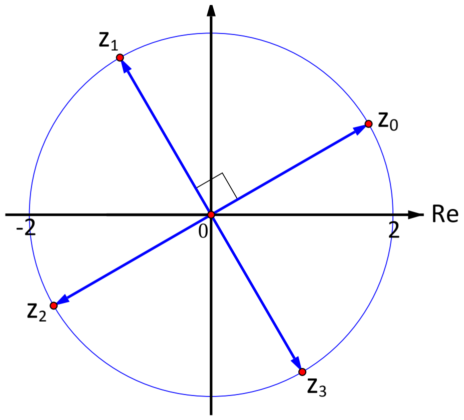


Figure 1.10: The four solutions for $z^4 = -8 + 8\sqrt{3}i$

The method in Example 1.51 we now generalize in the following theorem. The theorem is proved in eNote 2 about polynomials.

|||| Theorem 1.52 Binomial Equation Solved Using the Exponential Form

Given a complex number w that is different from 0 and that has the exponential form

$$w = |w|\,\mathrm{e}^{\mathrm{i}v}\,.$$

The binomial equation

$$z^n = w\;,\; n \in \mathbb{N} \tag{1-36}$$

has n solutions that can be found with the formula

$$z = \sqrt[n]{|w|}\;\mathrm{e}^{\mathrm{i}\left(\frac{v}{n}+p\frac{2\pi}{n}\right)}\;,\;\; \text{where}\;\; p = 0\,,1\,,\ldots\,,n-1\,. \tag{1-37}$$

|||| Exercise 1.53 Binomial Equation with a Negative Right-Hand Side

Let r be an arbitrary positive real number. Show by use of Theorem 1.52 that the binomial quadratic equation

$$z^2 = -r$$

has the two solutions

$$z_0 = \mathrm{i}\sqrt{r}\;\;\text{and}\;\; z_1 = -\mathrm{i}\sqrt{r}\,.$$

1.9 Linear and Quadratic Equations

Let a and b be complex numbers with $a \neq 0$. A *complex linear equation* of the form

$$az = b$$

in analogy with the corresponding real linear equation has exactly one solution

$$z = \frac{b}{a}\,.$$

With a and b in rectangular form, the solution is easily found in rectangular form, as shown in the following example.

|||| Example 1.54 Solution of a Linear Equation

The equation

$$(1-\mathrm{i})\,z = (5+2\mathrm{i})$$

has the solution

$$z = \frac{5+2\mathrm{i}}{1-\mathrm{i}} = \frac{(5+2\mathrm{i})(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})} = \frac{3+7\mathrm{i}}{2} = \frac{3}{2} + \frac{7}{2}\,\mathrm{i}\,.$$

Also in the solution of *complex quadratic equations* we use a formula that corresponds to the well-known solution formula for real quadratic equations. This is given in the following theorem that is proved in eNote 2 about polynomials.

|||| Theorem 1.55 Solution Formula for Complex Quadratic Equations

Let a, b and c be arbitrary complex numbers with $a \neq 0$. We define the *discriminant* by $D = b^2 - 4ac$. The quadratic equation

$$az^2 + bz + c = 0 \tag{1-38}$$

has two solutions

$$z_0 = \frac{-b - w_0}{2a} \quad \text{and} \quad z_1 = \frac{-b + w_0}{2a}, \tag{1-39}$$

where w_0 is a solution to the binomial quadratic equation $w^2 = D$.

If in particular $D = 0$, we find $z_0 = z_1 = \dfrac{-b}{2a}$.

In this eNote we do not introduce square roots of complex numbers. Therefore the complex solution formula above differs in one detail from the ordinary real solution formula.

Concrete examples of the application of the theorem can be found in Section 30.5.2 in eNote 2 about polynomials.

1.10 Complex Functions of a Real Variable

In this section we use the theory of the so-called epsilon functions for the introduction of differentiability. The material is a bit more advanced than previously and knowledge about epsilon functions from eNote 3 (see Section 3.4) may prove advantageous. Furthermore the reader should be familiar with the rules of differentiation of ordinary real functions.

We will make a special note of functions of the type

$$f : t \mapsto e^{ct}, t \in \mathbb{R}, \tag{1-40}$$

where c is a given complex number. This type of function has many uses in pure and applied mathematics. A main purpose of this section is to give a closer description of these. They are examples of the so-called *complex functions of a real variable*. Our investigation starts off in a wider sense with this broader class of functions. I.a. we show how concepts such as differentiability and derivatives can be introduced. Then we give a fuller treatment of functions of the type in (1-40).

|||| Definition 1.56 Complex Functions of a Real Variable

By a *complex function of a real variable* we understand a function f that for every $t \in \mathbb{R}$ attaches exactly one complex number that is denoted $f(t)$. A short way of writing a function f of this type is

$$f : \mathbb{R} \mapsto \mathbb{C} .$$

The notation $f : \mathbb{R} \mapsto \mathbb{C}$ tells us the function f uses a variable in the real number space, but ends up with a result in the complex number space. Consider e.g. the function $f(t) = \mathrm{e}^{\mathrm{i}t}$. At the real number $t = \frac{\pi}{4}$ we get the complex function value

$$f\left(\frac{\pi}{4}\right) = \mathrm{e}^{\frac{\pi}{4}\mathrm{i}} = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i} .$$

Let us consider a function $f : \mathbb{R} \mapsto \mathbb{C}$. We introduce two real functions g and h by

$$g(t) = \mathrm{Re}\big(f(t)\big) \text{ and } h(t) = \mathrm{Im}\big(f(t)\big)$$

for all $t \in \mathbb{R}$. By this f can be stated in *rectangular form*:

$$f(t) = g(t) + \mathrm{i} \cdot h(t) \, , \; t \in \mathbb{R} . \tag{1-41}$$

When in the following we introduce differentiability of complex functions of one real variable, we shall need a special kind of complex function, viz. the so-called *epsilon functions*. Similar to *real* epsilon functions they are auxiliary functions, whose functional expression is of no interest. The two decisive properties for a real epsilon function $\epsilon : \mathbb{R} \mapsto \mathbb{R}$ are that it satisfies $\epsilon(0) = 0$, and that $\epsilon(t) \to 0$ when $t \to 0$. The complex epsilon function is introduced in a similar way.

|||| Definition 1.57 Epsilon Function

By a *complex epsilon function of a real variable* we understand a function $\epsilon : \mathbb{R} \mapsto \mathbb{C}$, that satisfies:

1. $\epsilon(0) = 0$, and

2. $|\epsilon(t)| \to 0$ for $t \to 0$.

Note that if ϵ is an epsilon function, then it follows directly from the Definition 1.57 that for every $t_0 \in \mathbb{R}$:

$$|\,\epsilon(t - t_0)\,| \to 0 \;\text{ for }\; t \to t_0\,.$$

In the following example a pair of complex epsilon functions of a real variable are shown.

|||| Example 1.58 Epsilon Functions

The function

$$t \mapsto \mathrm{i}\sin(t)\,,\; t \in \mathbb{R}$$

is an epsilon function. This is true because requirement 1 in definition 1.57 is fulfilled by

$$\mathrm{i}\sin(0) = \mathrm{i}\cdot 0 = 0$$

and requirement 2 by

$$|\mathrm{i}\sin(t)| = |\mathrm{i}|\,|\sin(t)| = |\sin(t)| \to 0 \;\text{ for }\; t \to 0\,.$$

Also the function

$$t \mapsto t + \mathrm{i}t\,,\; t \in \mathbb{R}$$

is an epsilon function, since

$$0 + \mathrm{i}\cdot 0 = 0$$

and

$$|t + \mathrm{i}t| = \sqrt{t^2 + t^2} = \sqrt{2}\,|t| \to 0 \;\text{ for }\; t \to 0\,.$$

We are now ready to introduce the concept of *differentiability* for complex functions of a real variable.

|||| Definition 1.59 Derivative of a $\mathbb{C}$-valued Function of a Real Variable

A function $f : \mathbb{R} \mapsto \mathbb{C}$ is called *differentiable* at $t_0 \in \mathbb{R}$, if a constant $c \in \mathbb{C}$ and an epsilon function $\epsilon : \mathbb{R} \mapsto \mathbb{C}$ exist such that

$$f(t) = f(t_0) + c(t - t_0) + \epsilon(t - t_0)(t - t_0)\,,\; t \in \mathbb{R}\,. \tag{1-42}$$

If f is differentiable at t_0 then c is called the *derivative* for f at t_0.

If f is differentiable at every t_0 in an open interval I then f is said to be *differentiable* on I.

Differentiability for a complex function of a real variable is tightly connected to the differentiability of the two real parts of the rectangular form. We now show this.

|||| Theorem 1.60

For a function $f : \mathbb{R} \mapsto \mathbb{C}$ with the rectangular form $f(t) = g(t) + ih(t)$ and a complex number c with the rectangular form $c = a + ib$:

f is differentiable at $t_0 \in \mathbb{R}$ with

$$f'(t_0) = c\,,$$

if and only if g and h are differentiable at t_0 with

$$g'(t_0) = a \quad \text{and} \quad h'(t_0) = b\,.$$

|||| Proof

First suppose that f is differentiable at t_0 and $f'(t_0) = a + ib$, where $a, b \in \mathbb{R}$. Then there exists an epsilon function ϵ such that f for every t can be written in the form

$$f(t) = f(t_0) + (a + ib)(t - t_0) + \epsilon(t - t_0)(t - t_0)\,.$$

We rewrite both the left- and the right-hand side into their rectangular form:

$$\begin{aligned}
&g(t) + \mathrm{i}h(t) = \\
&g(t_0) + \mathrm{i}h(t_0) + a(t - t_0) + \mathrm{i}b(t - t_0) + \mathrm{Re}(\epsilon(t - t_0)(t - t_0)) + \mathrm{i}\mathrm{Im}(\epsilon(t - t_0)(t - t_0)) = \\
&(g(t_0) + a(t - t_0) + \mathrm{Re}(\epsilon(t - t_0))(t - t_0)) + \mathrm{i}(h(t_0) + b(t - t_0) + \mathrm{Im}(\epsilon(t - t_0))(t - t_0))\,.
\end{aligned}$$

From this we get

$$g(t) = g(t_0) + a(t - t_0) + \mathrm{Re}(\epsilon(t - t_0))(t - t_0) \text{ and } h(t) = h(t_0) + b(t - t_0) + \mathrm{Im}(\epsilon(t - t_0))(t - t_0)\,.$$

In order to conclude that g and h are differentiable at t_0 with $g'(t_0) = a$ and $h'(t_0) = b$, it only remains for us to show that $\mathrm{Re}(\epsilon)$ and $\mathrm{Im}(\epsilon)$ are real epsilon functions. This follows from

1. $\epsilon(0) = \mathrm{Re}(\epsilon(0)) + \mathrm{i}\mathrm{Im}(\epsilon(0)) = 0$ yields $\mathrm{Re}(\epsilon(0)) = 0$ and $\mathrm{Im}(\epsilon(0)) = 0$, and
2. $|\epsilon(t)| = \sqrt{|\mathrm{Re}(\epsilon(t))|^2 + |\mathrm{Im}(\epsilon(t))|^2} \to 0$ for $t \to 0$ yields that $\mathrm{Re}(\epsilon(t)) \to 0$ for $t \to 0$ and $\mathrm{Im}(\epsilon(t)) \to 0$ for $t \to 0$.

The converse statement in the theorem is similarly proved.

■

|||| Example 1.61 Derivative of a Complex Function

By the expression

$$f(t) = t + \mathrm{i}t^2$$

a function $f : \mathbb{R} \mapsto \mathbb{C}$ is defined. Since the real part of f has the derivative 1 and the imaginary part of f the derivative $2t$ we obtain from Theorem 1.60:

$$f'(t) = 1 + \mathrm{i}2t\,,\; t \in \mathbb{R}\,.$$

|||| Example 1.62 Derivative of a Complex-valued Function

Consider the function $f : \mathbb{R} \mapsto \mathbb{C}$ given by

$$f(t) = \mathrm{e}^{\mathrm{i}t} = \cos(t) + \mathrm{i}\sin(t)\ ,\ t \in \mathbb{R}\,.$$

Since $\cos'(t) = -\sin(t)$ and $\sin'(t) = \cos(t)$, it is seen from Theorem 1.60 that

$$f'(t) = -\sin(t) + \mathrm{i}\cos(t)\,,\ t \in \mathbb{R}\,.$$

In the following theorem we consider the so-called *linear* properties of differentiation. These are well known from real functions.

|||| Theorem 1.63 Computational Rules for Derivatives

Let f_1 and f_2 be differentiable complex functions of a real variable, and let c be an arbitrary complex number. Then:

1. The function $f_1 + f_2$ is differentiable with the derivative

$$(f_1 + f_2)'(t) = {f_1}'(t) + {f_2}'(t)\,. \tag{1-43}$$

2. The function $c \cdot f_1$ is differentiable with the derivative

$$(c \cdot f_1)'(t) = c \cdot {f_1}'(t)\,. \tag{1-44}$$

|||| Proof

Let $f_1(t) = g_1(t) + \mathrm{i}\,h_1(t)$ and $f_2(t) = g_2(t) + \mathrm{i}\,h_2(t)$, where g_1, h_1, g_2 and h_2 are differentiable real functions. Furthermore let $c = a + ib$ be an arbitrary complex number in rectangular form.

First part of the theorem:

$$\begin{aligned}(f_1 + f_2)(t) &= f_1(t) + f_2(t) = g_1(t) + \mathrm{i}\,h_1(t) + g_2(t) + \mathrm{i}\,h_2(t)\\ &= (g_1(t) + g_2(t)) + \mathrm{i}\,(h_1(t) + h_2(t))\ .\end{aligned}$$

We then get from Theorem 1.60 and by the use of computational rules for derivatives for real functions:

$$\begin{aligned}(f_1+f_2)'(t) &= (g_1+g_2)'(t) + \mathrm{i}\,(h_1+h_2)'\,(t)\\ &= \left(g_1'(t)+g_2'(t)\right) + \mathrm{i}\left(h_1'(t)+h_2'(t)\right)\\ &= \left(g_1'(t)+\mathrm{i}\,h_1'(t)\right) + g_2'(t) + \mathrm{i}\,h_2'(t)\\ &= f_1'(t)+f_2'(t)\,.\end{aligned}$$

By this the first part of the theorem is proved.

Second part of the theorem:

$$\begin{aligned}c\cdot f_1(t) &= (a+\mathrm{i}b)\cdot\left(g_1(t)+\mathrm{i}\,h_1(t)\right)\\ &= \left(a\,g_1(t)-b\,h_1(t)\right)+\mathrm{i}\left(a\,h_1(t)+b\,g_1(t)\right)\,.\end{aligned}$$

We get from Theorem 1.60 and by the use of computational rules for derivatives for real functions:

$$\begin{aligned}(c\cdot f_1)'(t) &= (a\,g_1-b\,h_1)'\,(t)+\mathrm{i}\,(a\,h_1+b\,g_1)'\,(t)\\ &= \left(a\,g_1'(t)-b\,h_1'(t)\right)+\mathrm{i}\left(a\,h_1'(t)+b\,g_1'(t)\right)\\ &= (a+\mathrm{i}b)\left(g_1'(t)+\mathrm{i}\,h_1'(t)\right)\\ &= c\cdot f_1'(t)\,.\end{aligned}$$

By this the second part of the theorem is proved.

■

|||| Exercise 1.64

Show that if f_1 and f_2 are differentiable complex functions of a real variable, then the function f_1-f_2 is differentiable with the derivative

$$(f_1-f_2)'(t)=f_1'(t)-f_2'(t)\,. \qquad (1\text{-}45)$$

We now return to functions of the type (1-40). First we give a useful theorem about their conjugation.

|||| Theorem 1.65

For an arbitrary complex number c and every real number t:

$$\overline{\mathrm{e}^{ct}} = \mathrm{e}^{\bar{c}\,t}\,. \tag{1-46}$$

|||| Proof

Let $c = a + ib$ be the rectangular form of c . We then get by the use of Definition 1.41 and the rules of computation for conjugation in Theorem 1.23:

$$\begin{aligned}
\overline{\mathrm{e}^{ct}} &= \overline{\mathrm{e}^{at+ibt}} \\
&= \overline{e^{at}\left(\cos(bt) + \mathrm{i}\sin(bt)\right)} \\
&= \overline{e^{at}}\ \overline{\left(\cos(bt) + \mathrm{i}\sin(bt)\right)} \\
&= e^{at}\left(\cos(bt) - \mathrm{i}\sin(bt)\right) \\
&= e^{at}\left(\cos(-bt) + \mathrm{i}\sin(-bt)\right) \\
&= e^{at-ibt} \\
&= e^{\bar{c}t}\,.
\end{aligned}$$

Thus the theorem is proved.

■

For ordinary real exponential functions of the type

$$f : x \mapsto \mathrm{e}^{kx}\ ,\ x \in \mathbb{R}\,,$$

where k is a real constant we have the well-known derivative

$$f'(x) = kf(x) = k\mathrm{e}^{kx}\,. \tag{1-47}$$

We end this eNote by showing that the complex exponential function of a real variable satisfies a quite similar rule of differentiation.

|||| Theorem 1.66 Differentiation of e^{ct}

Consider an arbitrary number $c \in \mathbb{C}$. The function $f : \mathbb{R} \mapsto \mathbb{C}$ given by

$$f(t) = e^{ct} \, , \, t \in \mathbb{R} \tag{1-48}$$

is differentiable and its derivative is determined by

$$f'(t) = c f(t) = c e^{ct} \, . \tag{1-49}$$

|||| Proof

Let the rectangular form of c be $c = a + ib$. We then get

$$\begin{aligned} e^{ct} &= e^{at+ibt} \\ &= e^{at} \left(\cos(bt) + i \sin(bt) \right) \\ &= e^{at} \cos(bt) + i \left(e^{at} \sin(bt) \right) \, . \end{aligned}$$

Thus we have

$$f(t) = g(t) + ih(t) \, , \text{ where } g(t) = e^{at} \cos(bt) \text{ and } h(t) = e^{at} \sin(bt) \, .$$

Since g and h are differentiable, f is also differentiable . Furthermore since

$$g'(t) = ae^{at} \cos(bt) - e^{at} b \sin(bt) \text{ and } h'(t) = ae^{at} \sin(bt) + e^{at} b \cos(bt) \, ,$$

we now get

$$\begin{aligned} f'(t) &= ae^{at} \cos(bt) - e^{at} b \sin(bt) + i \left(ae^{at} \sin(bt) + e^{at} b \cos(bt) \right) \\ &= (a + ib) e^{at} \left(\cos(bt) + i \sin(bt) \right) \\ &= (a + ib) e^{at+ibt} \\ &= c \, e^{ct} \, . \end{aligned}$$

Thus the theorem is proved.

■

If c in Theorem 1.66 is real, (1-49) naturally only expresses the ordinary differentiation of the real exponential function as in (1-47), as expected.