Theme Exercise 5

Description of a Rocket Explosion

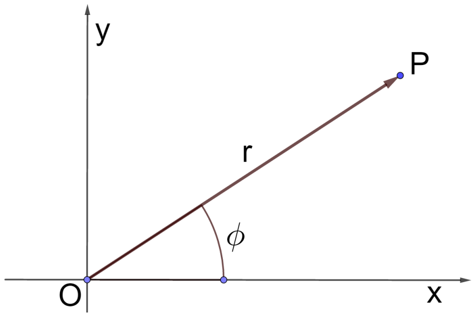
Antares CRS-3 shortly after the launch 2014, October 28

In this theme exercise we give a geometrical description of a rocket explosion. Where do the parts of the rocket move using simple model assumptions? We shall both consider motion of individual parts and an overall picture of where the rockets parts are. We shall need elementary calculus, spherical coordinates and parametric representations of curves and other geometrical objects. The first three problems introduce these subjects, so we are ready to solve the problem in the final exercise. We start cautiously with polar coordinates in the xy-plane.

The Theme Exercise Continues⟼

|||| Problem 1 Polar Coordinates in the (x, y)-plane

We consider an arbitrary point P in the (x, y)-plane that is not the origin.



In polar coordinates the first coordinate r is defined as the length of the position vector $\vec{OP}$, while the second coordinate ϕ is the angle from the first axis to the position vector. The angle is stated with sign in accordance with the ordinary orientation of the coordinate system and is chosen in the interval $]-\pi, \pi]$.

a) Draw the points $\left(5, \frac{\pi}{4}\right)$ and $\left(2, -\frac{\pi}{3}\right)$ in the (x, y)-plane.

b) Sketch the two curves $(5, \phi)$, $\phi \in \left[\frac{\pi}{4}, \pi\right]$ and $\left(r, -\frac{\pi}{3}\right)$, $r \in \left[\frac{1}{2}, 3\right]$.

c) Determine the area of the region (r, ϕ) , $r \in [2, 7]$, $\phi \in \left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$.

d) A point has the set of polar coordinates (r, ϕ). Explain that the point's (x, y)-coordinates are given by the formula

$$\begin{bmatrix} x \\ y \end{bmatrix} = r \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}. \tag{1}$$

e) A curve K is given by the parametric representation

$$\begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \ \phi \in]-\pi, \pi]. \tag{2}$$

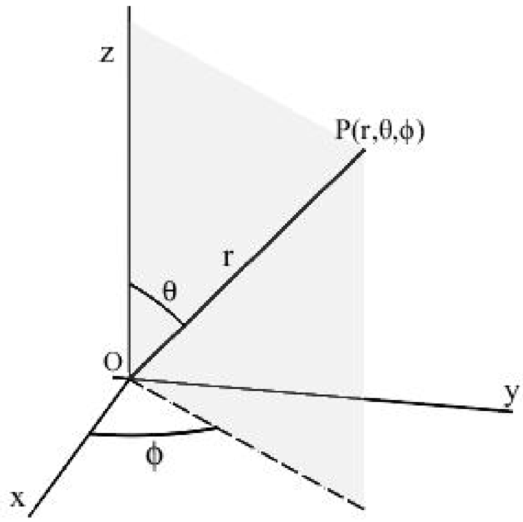
Explain that K is a circle and make a description of this using the ordinary equation for a circle

$$(x-a)^2 + (x-b)^2 = c^2 .$$

The Theme Exercise Continues$\longmapsto$

Problem 2 Spherical Coordinates in (x, y, z)-Space

Now we consider an arbitrary point P in (x, y, z)-space (not the origin).



Spherical coordinates are defined like this: The first coordinate r is the length of the position vector $\overrightarrow{OP}$. The second coordinate θ is the angle between the z-axis and the position vector, and is chosen in the interval $[0, \pi]$. Finally a third coordinate ϕ is the angle between the (x, z)-plane and the plane that contains both the z-axis and $\overrightarrow{OP}$. ϕ is stated with a sign chosen in accordance with the ordinary orientation of the (x, y)-plane, and it is chosen in the interval $]-\pi, \pi]$.

a) Determine the (x, y, z)-coordinates for the two points that have the spherical coordinates

$$\left(5, \pi, \frac{\pi}{2}\right) \text{ og } \left(2, \frac{\pi}{3}, -\frac{\pi}{4}\right) .$$

b) Explain that the following expression in spherical coordinates

$$(5, \theta, \phi) \ , \ \theta \in \left[0, \frac{\pi}{2}\right] \ , \ \phi \in \left[0, \frac{\pi}{4}\right]$$

describes part of a spherical surface, and state the area of this spherical surface segment.

c) A solid region in (x, y, z)-space is described by spherical coordinates like this:

$$(r, \theta, \phi) \ , \ r \in [3, 5] \ , \ \theta \in \left[0, \frac{\pi}{2}\right] \ , \ \phi \in \left[0, \frac{\pi}{4}\right] .$$

Determine the volume of the solid region.

The Theme Exercise Continues ⟼

d) A point has the set of spherical coordinates (r, θ, ϕ). Explain that the (x, y, z)-coordinates of the point is given by the formula

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix} . \tag{3}$$

e) A spherical surface F in (x, y, z)-space is given by the equation

$$(x+2)^2 + (x-3)^2 + (z+1)^2 = 4 .$$

State a parametric representation for F that builds upon the formula in the preceding question. Remember to state the intervals for the two parameters, θ and ϕ.

f) Explain that the expression

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \ \theta \in [0, \pi], \ \phi \in [0, 2\pi] \tag{4}$$

represents an entire spherical surface with radius r and centre at (c_1, c_2, c_3). Plot using Maple the spherical surface in Question e) using its parametric representation.

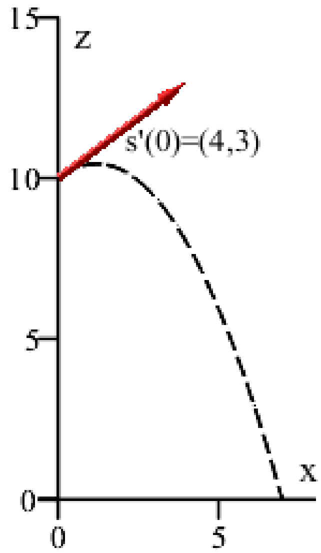
|||| Problem 3 Parametric Representations of Projectile Motion

In the following we consider a projectile that in the point $(0, 10)$ in the (x, z)-plane is ejected with the initial velocity 4 in the x-axis' direction and 3 in the z-axis' direction, see the figure.

We wish to find a parametric representation of the trajectory of the projectile. We imagine a simple model where the mass of the projectile is 1 and where the projectile is only acted upon by the acceleration of gravity which has the magnitude g.

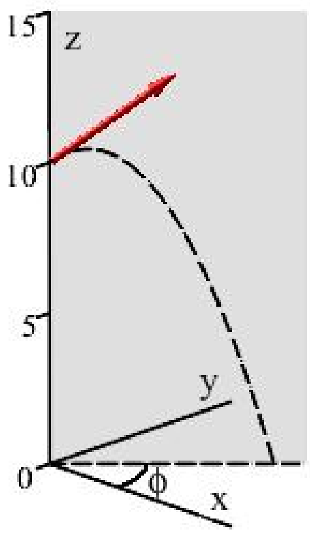
Thus we have that if $\begin{bmatrix} x \\ z \end{bmatrix} = \mathbf{s}(t)$, $t > 0$ is a parametric representation for the trajectory of the projectile, then $\mathbf{s}''(t) = \begin{bmatrix} 0 \\ -g \end{bmatrix}$.

The Theme Exercise Continues ⟼



a) Put $g = 10$ and determine a parametric representation first for $\mathbf{s}'(t)$ and then for $\mathbf{s}(t)$. Plot the trajectory and plot the spherical surface.

Now we rotate the plane of the trajectory of the projectile through an arbitrary angle ϕ around the z-axis. ϕ is stated with a sign in accordance with the ordinary orientation of the (x, y)-plane. All other conditions are as before.

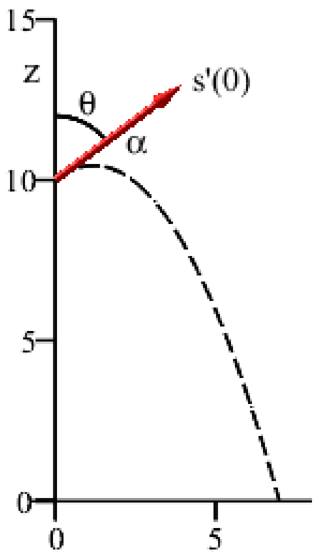


b) Determine the point in which the projectile now reaches its maximum height, and the point in which the projectile hits the surface of the ground (i.e. the (x, y)-plane).

The Theme Exercise Continues⟼

c) Determine a parametric representation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{s}(t)\,,\ t > 0$ for the trajectory of the projectile.

Until now we have worked under the assumption that the initial velocity vector has the length 4 in the horizontal direction and 3 in the vertical direction. More generally we now let the initial velocity vector $\mathbf{s}'(0)$ of the trajectory be given by $|\mathbf{s}'(0)| = \alpha > 0$, and the angle from the z-axis to $\mathbf{s}'(0)$ as an arbitrary angle $\theta \in [0, \pi]$, see the figure (where the trajectory continues to be rotated through the angle ϕ with respect to the (x, z)-plane).



d) How does the parametric representation of the trajectory of the projectile look now?

|||| Problem 4 Exploding Rocket

Now we are ready to solve the problem. A rocket explodes in (x, y, z)-space at the time $t = 0$. For convenience we imagine that the rocket is at point $(0, 0, h)$. Due to technical problems the rocket has lost its speed in the vertical direction and it is at rest at the moment of explosion. Now it is torn into pieces that are ejected in all directions at the same speed which we denote α. All pieces have the mass = 1 and the acceleration of gravity is denoted g.

a) Explain that all parts of the rocket at every moment in time until they start to hit the ground level, is situated on a common spherical surface. Determine the (timedependent) radius and the center of the sphere.

Problem 5 Ekstremum Investigation

We now set $\alpha = 10$, $g = 10$ and $h = 100$. For convenience, we rename the Greek parameters θ to u and ϕ to v.A rocket fragment is then determined by a unique set of parameters $u \in [0, \pi]$ and $v \in [0, 2\pi]$.

a) Explain that the the spherical surface hereafter can be parametrized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{r}(u, v, t) = 10 \cdot t \cdot \begin{bmatrix} \sin(u) \cdot \cos(v) \\ \sin(u) \cdot \sin(v) \\ \cos(u) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 100 - 5 \cdot t^2 \end{bmatrix}$$

where $t \geq 0$ and $u \in [0, \pi]$ and $v \in [0, 2\pi]$. Hint: When creating the parameter representation in Maple, it is advantageous to use unapply rather than the arrow procedure.

Plot using Maple the spherical surface as it appears at $t = 1$ and at $t = 4$.Hint: plot3d(p-representation, parameter1=?..?,parameter2=?..?,scaling=constrained,view=?..?)

The fallout area is a circular disk, determine its radius.

Two drones has already been sent up to film and collect data on the rocket launch. One is hit by the explosion, while the other barely escapes. In the following, we shall examine the details of the fate of the two drones.

b) Drone A is described as a point that moves according to the parametric representation

$$\mathbf{a}(t) = \begin{bmatrix} 36 \\ 3 \\ 46 \end{bmatrix} + t \cdot \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$$

where t is the same time parameter that applies to the spherical surface.

Determine the time and place where the collision takes place. Hint: There is only one collision. The drone is smashed and no longer exists.

Make a plot using Maple, that shows the spherical surface and the trajectories for A and for the rocket fragment that hits A. Hint: spacecurve(p-representation, t=?..?,color=?,thickness=?)

c) Drone B is described as a point that moves according to the parametric representation

$$\mathbf{b}(t) = \begin{bmatrix} 30 \\ 16 \\ 70 \end{bmatrix} + t \cdot \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}.$$

The Theme Exercise Continues ⟼

State a differentiable function dist(u, v, t) which indicates the distance from the points of the spherical surface (the rocket fragments) to B.

Find the stationary points for dist and investigate using the Hessian matrix if they are locations for local extrema. Hints: 1) Explain that the stationary points for dist is equal to the stationary points for dist squared (it is easier to use dist squared, we get rid of the square root). 2) Use Maple's `fsolve` to find the stationary points from the partial derivatives with respect to u, v and t. Note that you can specify the intervals in which you want `fsolve` to search, see Maple's help.

Determine the minimum distance from the drone to a passing rocket fragment, specify which rocket fragment C it is and state the position of C at the minimum moment.

Make a plot that contains the speherical surface, B, C and the trajectories for B and C at the minimum moment.

The Theme Exercise Ends