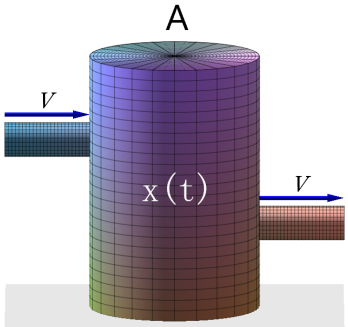
Theme Exercise 4

Coupled liquid containers

In all engineering sciences, there are real life problems that are modeled using systems of differential equations. Quite obvious examples are electrical networks or oscillations in building structures. In all cases, the mathematical challenge is the same: first it is about being able to model the given problem, thinking out of the box, so to speak, and then using the mathematical solution methods that are available. In this thematic exercise, we have chosen to describe the development of salt concentrations in systems of coupled liquid containers, and we confine ourselves to systems of linear first-order differential equation with constant coefficients. As an introduction to the modeling work, we first analyze a single (uncoupled) liquid container.



In the figure we see a container A with volume 100, which is full of water with added salt. At time $t = 0$ it opens up for an inflow into the container of saline water (same kind of salt) with constant concentration k . The inflow rate (which is measured in Liter / Sec, for example) is denoted v . An outflow occurs from the container at the same flow rate v .

The salt concentration in the container is denoted $x(t)$, it is the same everywhere in the container due to stirring. We want a functional expression for $x(t)$ so that at all times we know the salt concentration. But we can not extract that from the information given.

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We need to proceed indirectly. It turns out that it is possible to think of an expression for the velocity $x'(t)$ with which the salt concentration in the container is changing. It is the mathematical model of the problem, and once we have set it up, we can move on and hope that we can find solutions, ie. find $x(t)$. In the following initial problem, the model must be set up from scratch.

Problem 1 Exercise with setting up of mathematical model and solution

Let δt denote a small time interval, and answer the following questions:

a) How much water flows in / out of the container during δt ?

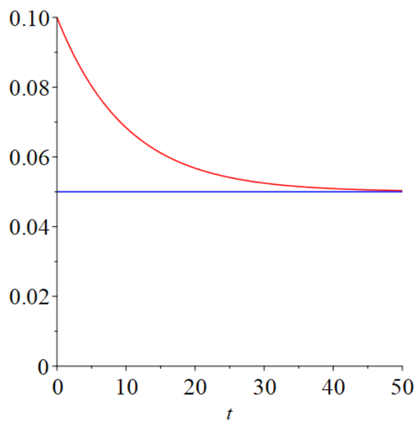
b) How much salt flows into the container during δt ?

c) How much salt flows out of the container during δt , assuming that the concentration of the outflowing water is $x(t)$?

d) How big is the change δx in the salt concentration of the container during δt (the volume of the container is set to 100 as mentioned above)?

e) On the basis of questions a) to d) find an expression for $x'(t)$, it is a first-order linear differential equation. Find the general solution to the differential equation as we set $k = \frac{1}{20}$ and $v = 10$.

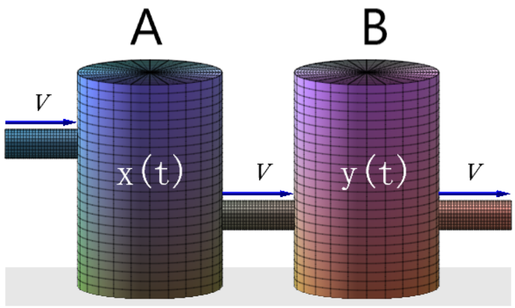
f) The solution to the differential equation that satisfies the initial value condition that the salt concentration at time $t = 0$ is $\frac{1}{10}$, is here plotted using Maple with the color red:



Find the functional expression for the conditional solution and describe in words how the salt concentration develops!

We now consider a system of two containers A and B, see the figure.

The Theme Exercise Continues⟶



A has volume R_1 and B volume R_2, and they are filled with the same saline solution but with different concentrations. At time $t = 0$ they are connected by a pipe so that liquid flows from A to B at the flow rate v. There is also an inflow from the outside to A with clean water and a drain from B in both cases with the flow rate v. We want to find out how the salt concentration in each of the containers develops, ie. find an expression of the salt concentration $x(t)$ in A and the salt concentration $y(t)$ in B.

Problem 2 Mathematical model and solution for two containers

a) How big is the change δx in A's salt concentration over time δt? And how big is the change δy in B's salt concentration over time δt?

b) Now set up the mathematical model by, on the basis of question a) finding expressions for $x'(t)$ and $y'(t)$. Hint: The model is a system of two coupled first-order linear differential equations with constant coefficients, where the system matrix is a triangular matrix.

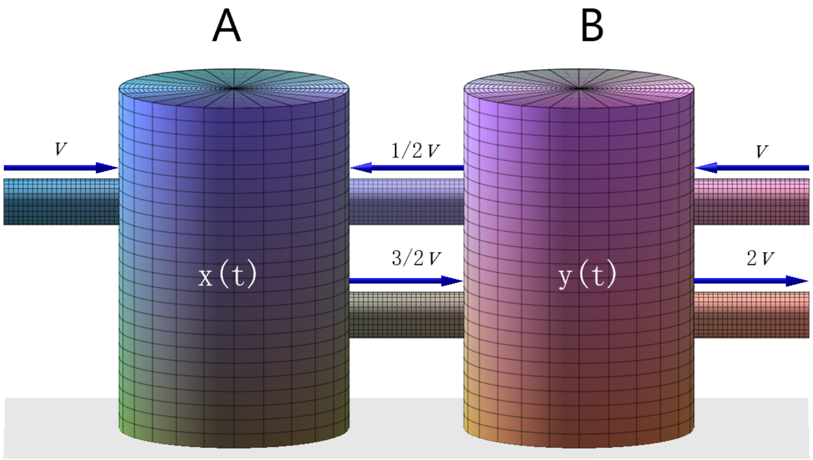
Now we fix $R_1 = 100, R_2 = 120$ and $v = 5$.

c) Set up the equation system in matrix form, and find eigenvalues and associated eigenspaces for the system matrix. Determine the general solution for the system in matrix form.

d) In the same plot, illustrate the solutions $x(t)$ and $y(t)$ to the system that fulfills the initial value condition $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 25 \\ 10 \end{bmatrix}$.

Compared to the previous scenario, a certain return from B to A is now added, and there is also a supply of clean water from the outside to tank B. See the relationship between the flow rates in the figure.

The Theme Exercise Continues⟶



Problem 3 Homogeneous and inhomogeneous system

a) How big is the change δx in A's salt concentration over time δt ? And how big is the change δy in B's salt concentration over time δt ?

b) Set up the mathematical model by, on the basis of question a) finding expressions for $x'(t)$ and $y'(t)$. Hint: The model is a homogeneous system of two coupled first-order linear differential equations with constant coefficients.

Now se fix $R_1 = R_2 = 100$ and $v = 10$.

c) Find (preferably with Maple's dsolve), and illustrate in the same plot the solutions $x(t)$ and $y(t)$ to the system that satisfy the initial value condition $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. Express in words what happens to the saline solutions in A and B.

We now further assume that the liquid supplied from the outside to A and B is not pure water, but a saline solution. For A with concentration 1 and for B concentration 4. This results in an inhomogeneous system.

d) Find a particular solution for the system, by guessing a solution of the form

$$\mathbf{x}(t) = \begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{where } a, b \in \mathbb{R}.$$

Then determine the general solution to the inhomogeneous system using the structural theorem.

e) Find and illustrate in the same plot the solutions $x(t)$ and $y(t)$ to the system that fulfills the initial value condition $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. Express in words what happens to the saline solutions in A and B.

The Theme Exercise Ends