Theme Exercise 1

Taylor Approximation and Error Estimation

The Theme Exercise consists of a number of examples of using Taylor approximation and estimation of the error made by using the approximations. We present here some tasks that can be used for preparation. On the Short Day in semester week 4, the work continues in Möbius with the problems, and there may also be new problems within the same area.

Problem 1 Determination of the Number e

The number e is one of the most important in mathematical analysis. It is a *trancendental* number, so it is not easy to decide its value. But what is its value? How can we compare it to other numbers? We must expand it as a decimal! We do this in the present exercise using the Taylor approximation.

Intro: The number e is the base for the natural exponential function exp. Exp is often introduced as the exponential increasing function whose slope at 0 is 1. It is (relatively) easy to show that an arbitrary exponential function $f(x) = a^x$, $a > 0$ has the derivative $f'(x) = f'(0) \cdot a^x$. From this it follows that

$$\exp'(x) = \exp'(0) \cdot \exp(x) = 1 \cdot \exp(x) = \exp(x)\,.$$

In short: the slope is always equal to the value of the function. When the graph for $y = \exp(x)$ reaches the height $y = 2$, its tangent has the slope 2, etc. This can be used to sketch a graph for $\exp$ from its possible tangents as shown in the figure below.

Further: The number e is the value of $\exp(x)$ at $x = 1$, since $\exp(1) = \mathrm{e}^1 = \mathrm{e}$. From the figure we allow ourselves to conclude that

$$\mathrm{e} < 3\,.$$

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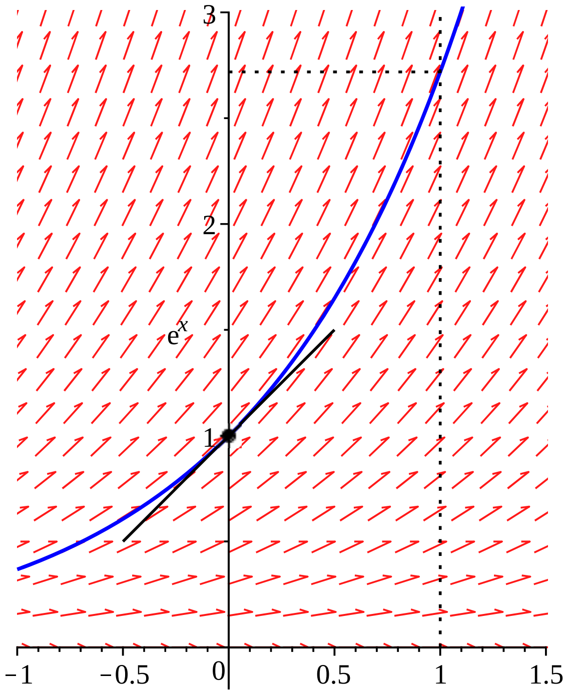
We will use this result in what follows, where we will determine e more precisely.



Let $P_n(x)$ denote the approximating polynomial of degree n for $\exp(x)$ expanded about the point $x_0 = 0$.

a) State the approximating polynomial $P_3(x)$. Show by estimating the remainder function $R_3(x)$, corresponding to $P_3(x)$, that the error you risk to incur, if you use the approximation $\mathrm{e} \approx P_3(1)$, is less than $\dfrac{1}{8} = 0.1250$.

b) Show that, in general, the error you risk to incur if you use the approximation $\mathrm{e} \approx P_n(1)$, is less than $\dfrac{3}{(n+1)!}$.

|||| Problem 2 Limits for Approximation

For the function $\ln(x)$ one cannot, of course, use the development point $x_0 = 0$ in the Taylor approximation.

a) Why is the development point $x_0 = 1$ the only reasonable for $\ln(x)$?

b) Determine, using elementary methods, the approximating polynomial of degree 4 for $\ln(x)$ with the development point $x_0 = 1$.

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c) State using Maples mtaylor the approximating polynomials $P_5(x)$ and $P_6(x)$ of degree 5 and 6, respectively, for f with the development point $x_0 = 1$.
Plot the graphs of $f(x)$, $P_4(x), P_5(x)$ and $P_6(x)$ in the same coordinate system. Compare the approximate values for $\ln(\frac{9}{5})$ you get using $P_4(x), P_5(x)$ and $P_6(x)$ with the value given by Maple. How large a degree is necessary for the difference between the Maple value and the value of the approximating polynomial to be less than $\frac{1}{100}$?

d) In the same way as in the previous question, try to determine approximate values for $\ln(\frac{11}{5})$. Comment.

|||| Problem 3 Estimation of a Remainder Function

Consider the function

$$f(x) = x \cdot \mathrm{e}^{-x^2}, \; x \in \mathbb{R}.$$

a) Find the global maximum point and the global maximum value for f in the interval $[-1;1]$. Do the same for the global minimum point and the global minimum value. Hint: $f'(x) = (1 - 2x^2)\mathrm{e}^{-x^2}$.

Consider now the function:

$$g(x) = \int_0^x \mathrm{e}^{-t^2}\, dt$$

b) An estimation of the magnitude of the difference between $g(x)$ and the function's approximating first degree polynomial $P_{1,x_0=0}(x)$ with development point $x_0 = 0$ is wanted. Thus the task is about the largest absolute value that the remainder function $|R_{1,x_0=0}(x)|$ can attain in the interval $[-1, 1]$. Hint: You will need the derivative $g'(0)$. Possibly use Maple to get started.

|||| Problem 4 Classical Physics as an Approximation to the Theory of Relativity

If we use the art of *estimating* the error using the remainder function, we can here develop an interesting approximation to the theory of relativity.

a) Let $x \in [0;1[$. Show using Taylor's formula that

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{x}{2} + \frac{3}{8}(1-\xi)^{-\frac{5}{2}}x^2$$

for an ξ between 0 and x.

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According to Albert Einstein the kinetic energy of a particle is given by

$$E_{kin}(v) = m_0 \cdot c^2 \left(\frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - 1 \right), \quad 0 \le v < c$$

where m_0 is the resting mass of the particle, c is the speed of light $(3 \cdot 10^5 \frac{km}{s})$ and v is the speed of the particle. The classical kinetic energy is, as is well known,

$$T(v) = \frac{1}{2} m_0 \cdot v^2.$$

The relative error by replacing $E_{kin}(v)$ with $T(v)$ is defined as

$$F = \frac{E_{kin}(v) - T(v)}{E_{kin}(v)}.$$

b) Show by using the approximating polynomial with the development point $x_0 = 0$ of $(1-x)^{-\frac{1}{2}}$ that

$$F < \frac{3 \left(\frac{v}{c}\right)^2}{4 \left(1 - \left(\frac{v}{c}\right)^2\right)^{\frac{5}{2}}} .$$

Hint: Replace$(\frac{v}{c})^2$ with x. Then

$$E_{kin}(v) = \frac{1}{2} m_0 \cdot v^2 + \frac{3}{8} \frac{m_0 \cdot v^4}{c^2 (1 - \xi)^{\frac{5}{2}}}$$

where $0 < \xi < (\frac{v}{c})^2$.

Hint: What consequences does this have for the relative error F? Substitute into the inequality and reduce! When you continue to estimate the fraction on the right-hand side of the inequality, then remember the basic rules: Some times you simplify the numerator by making it slightly larger. Other times you simplify the denominator by making it slightly smaller. Both actions are allowed because they make the fraction larger. The error estimation then becomes more crude, but this is the price you are willing to pay in order to obtain simple expressions for the further work.

c) Show, by using the now proven estimation of F, that $F < 10^{-2}$ for $v \le 3 \cdot 10^4 \frac{\text{km}}{\text{s}}$.

The Theme Exercise Ends